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QCD Theory and Monte Carlo Tools

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Fermilab-CERN HCPSS18
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References

- Useful resources for perturbative QCD — additional background and further reading for more advanced topics — are:

- QCD and Collider Physics
  R. K. Ellis, W. J. Stirling and B. R. Webber
  Cambridge Monographs on Particle Physics, Nuclear Physics and Cosmology

- The Black Book of Quantum Chromodynamics: A Primer for the LHC Era
  JC, J. Huston, F. Krauss
  Oxford University Press

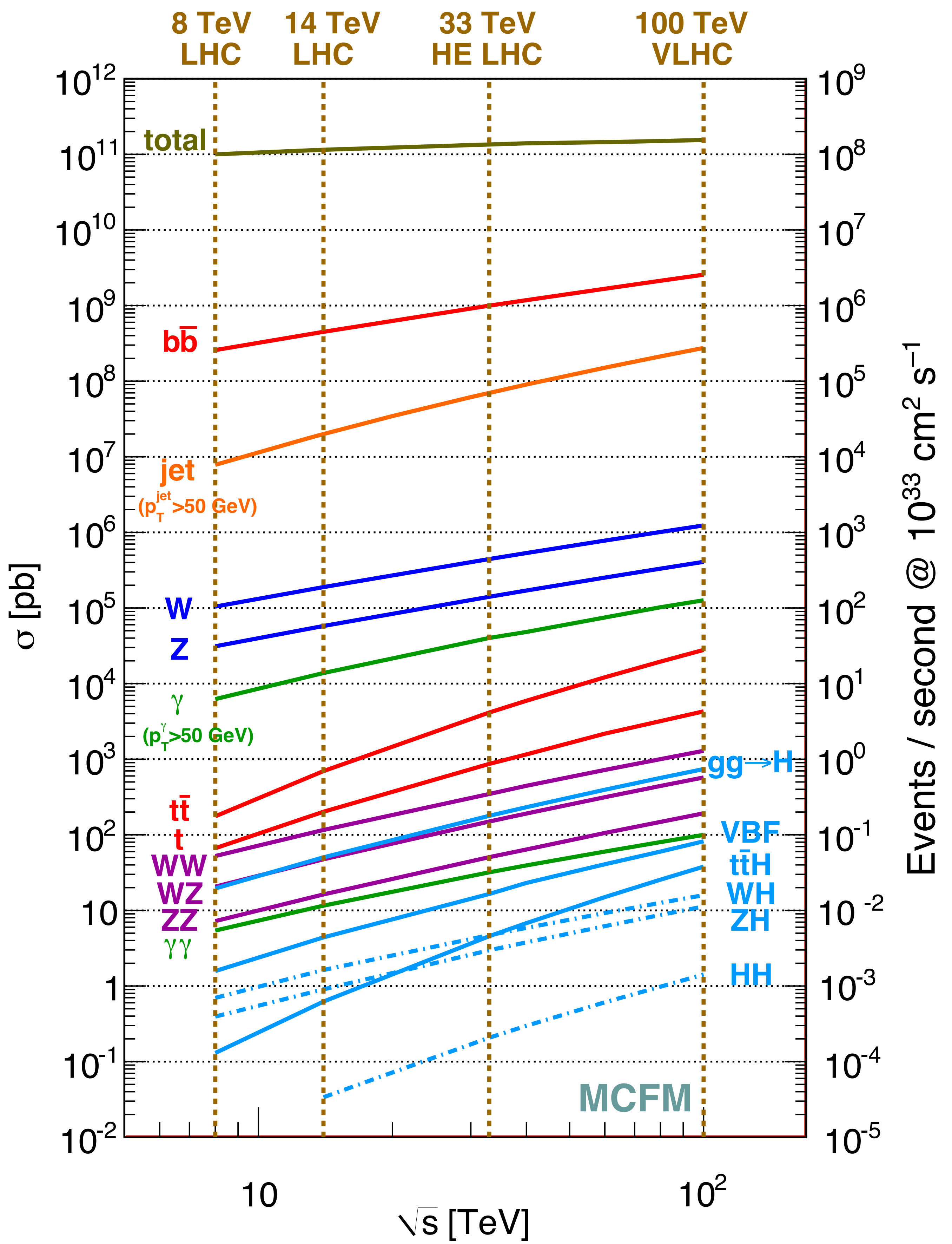
- Resource Letter: Quantum Chromodynamics
  A. S. Kronfeld and C. Quigg
  arXiv: 1002.5032 [hep-ph], prepared for the American Journal of Physics

Outline

- Introduction to the theory of QCD
  - Lagrangian, color, Feynman rules, strong coupling
- QCD for hadron colliders
  - factorization, parton distribution functions, hard scattering
- Structure of QCD matrix elements
  - infrared singularities, real and virtual radiation
- Beyond leading order
  - techniques for NLO, NNLO and beyond
- Parton shower techniques
  - Sudakov factors, resolvable emissions, hadronization
- Modern event generators
  - merging, matching, hybrid schemes, precision matching

QCD: why we care

- It is no surprise that hadron colliders require an understanding of QCD.
- However, the level of sophistication we require is demonstrated by the inclusive cross-sections for final states that we are typically interested in.
- In order to test the SM (and models of new physics), we require a quantitative understanding of QCD and precise theoretical predictions.
- These lectures will describe the ways in which we reach this goal.



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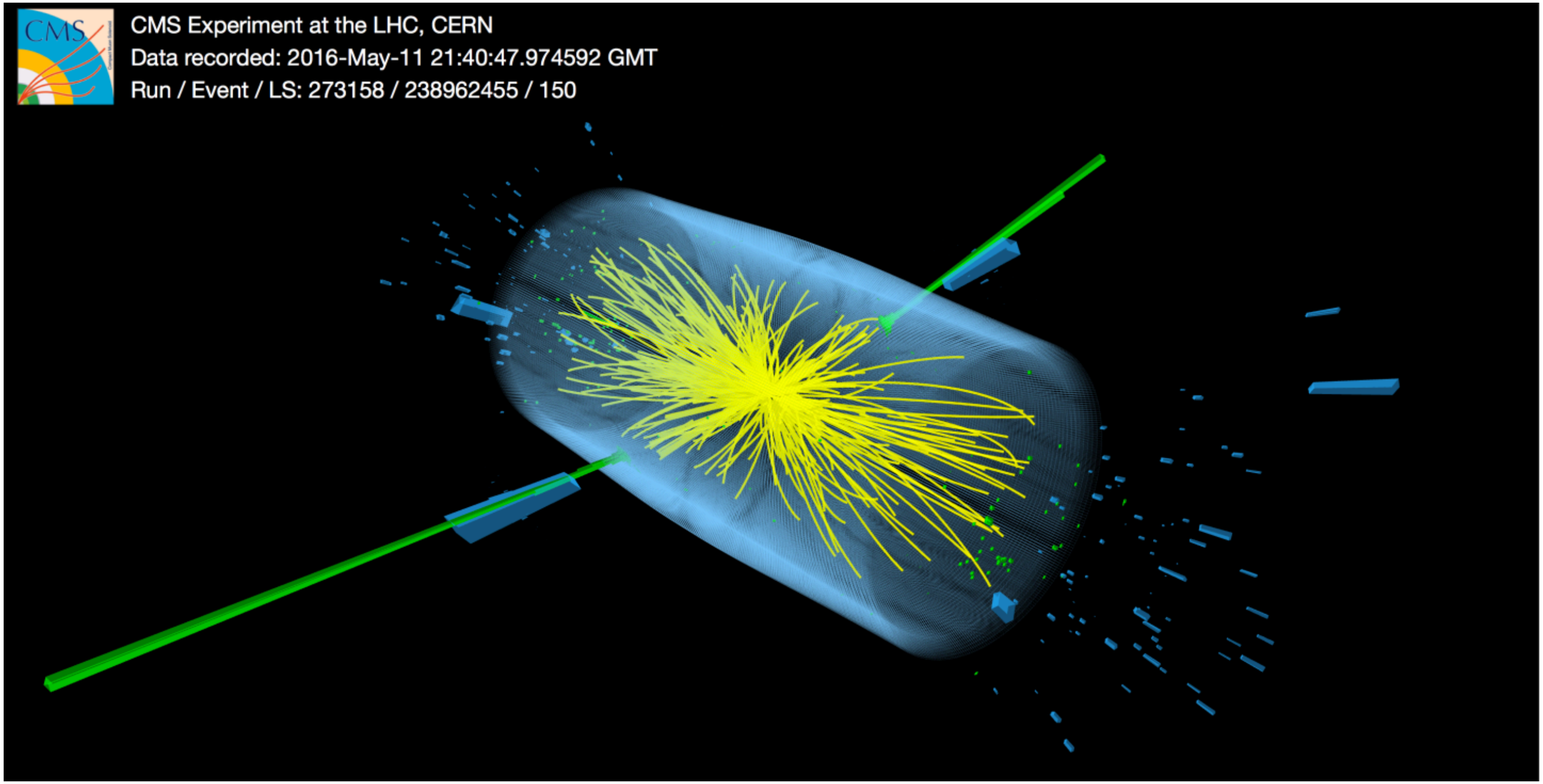
The challenge of QCD

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^{a}_{\mu\nu} F^{\mu\nu}_{a} + \sum_{\text{flavors}} \bar{q}_i \left(i D_\mu \gamma^\mu - m \right)_{ij} q_j$$



QCD and color

- The Lagrangian looks a lot like the one for QED: a field strength term representing the gluon field and a Dirac term for the quarks.
- However, it has one important difference: color.

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^{a}{}_{\mu\nu} F^{\mu\nu}_{a} + \sum_{\text{flavors}} \bar{q}_i \left(i D_\mu \gamma^\mu - m \right)_{ij} q_j$$

- Within the quark model, the additional quantum number of color was initially introduced to accommodate the existence of the Δ^{++} baryon.
  - antisymmetry to satisfy Pauli exclusion principle carried by color
  - quarks and gluons carry color but observed hadrons are colorless
- The color degrees of freedom can also be directly probed in electron-positron collisions, by comparing the production of hadrons and muons.

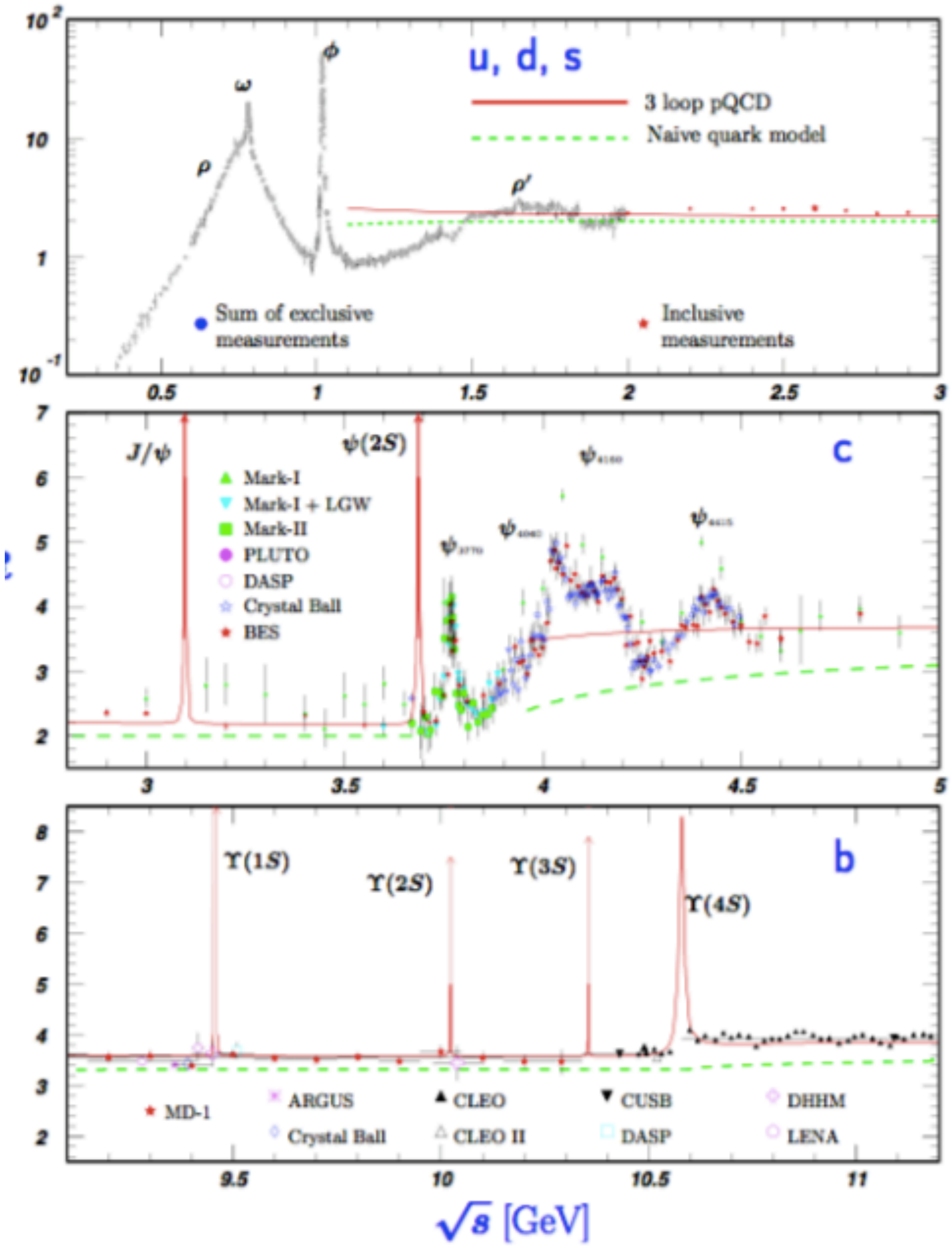
$$R = \frac{\sigma\left(e^+e^- \rightarrow \text{hadrons}\right)}{\sigma\left(e^+e^- \rightarrow \mu^+\mu^-\right)} = N_c \sum_f Q_f^2$$

quark charge

assume N_c colors of quark

sum over active quarks

"R-ratio" measurements



$$R_{u,d,s} = 3 \times \left[\left(\frac{2}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 + \left(-\frac{1}{3}\right)^2\right] = 2$$

$$R_{u,d,s,c} = R_{u,d,s} + 3 \times \left(\frac{2}{3}\right)^2 = \frac{10}{3}$$

$$R_{u,d,s,c,b} = R_{u,d,s,c} + 3 \times \left(-\frac{1}{3}\right)^2 = \frac{11}{3}$$

Color in the QCD Lagrangian: quarks

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^{a}{}_{\mu\nu} F^{\mu\nu}_{a} + \sum_{\text{flavors}} \bar{q}_i \left(i D_\mu \gamma^\mu - m \right)_{ij} q_j$$

- The gauge principle — invariance under local gauge transformations — requires the introduction of the gauge-invariant derivative:

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} + i g_s \left(t^a A^a_\mu \right)_{ij}$$

- When inserted in the Lagrangian this introduces interactions between quarks of color *i* and *j*, mediated by the gluon field A^a_{ij}.
- Strength of the interaction depends on the strong coupling (g_s) and a matrix in color space (t^a) that is related to a Gell-Mann matrix (λ^a) by $t^a = \lambda^a/2$.
- These are Hermitian, traceless & satisfy commutation relation:

$$\left[\lambda^a, \lambda^b\right] = 2i f_{abc} \lambda^c$$

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

SU(3)

- The matrices t^a are the generators of the group SU(3) in the fundamental representation: eight 3x3 matrices that satisfy:

$$[t^a, t^b] = i f_{abc} t^c$$

where f_{abc} form a set called the SU(3) structure constants. They're real numbers and are completely antisymmetric in the indices.

$$f_{123} = 1$$

$$f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}$$

$$f_{458} = f_{678} = \frac{\sqrt{3}}{2}.$$

- The matrices also obey a normalization condition:

$$\mathrm{Tr}\left(t^a t^b\right) = T_R\, \delta^{ab} \quad \text{with } T_R = 1/2$$

- By inspection, we can also see that:

$$\sum_a t^a t^a = C_F\, \mathbf{1}$$

$$\text{with } C_F = \frac{4}{3} \equiv \frac{N_c^2 - 1}{2N_c}$$

- The quantity C_F is called a Casimir.

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Color in the QCD Lagrangian: gluons

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a{}_{\mu\nu} F_a^{\mu\nu} + \sum_{\text{flavors}} \bar{q}_i \left(i D_\mu \gamma^\mu - m \right)_{ij} q_j$$

- The field strength tensor in the first term is fundamentally different from the QED case:

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu$$

- The final term involves the SU(3) structure constants and two gluon fields.
- When inserted into the Lagrangian this leads to self-interactions between gluons, involving three or four fields.

- To handle these gluon interactions we will need a second Casimir:

$$\sum_{c,d} f^{acd} f^{bcd} = C_A \delta^{ab} \quad \text{with } C_A = 3 \equiv N_c$$

(can again check this by inspection)

From the Lagrangian to calculations

- To use this Lagrangian we need to be able to calculate scattering amplitudes and ultimately cross-sections.
- The main toolbox for collider physics is perturbative QCD:
  - expand the Lagrangian about the free (non-interacting) case in powers of the coupling
  - interactions correspond to at least one power of the coupling
  - represent amplitudes as Feynman diagrams, with rules read off from the Lagrangian

- In the free case ($g_s \to 0$) there are only propagators.
- These are easily read off from two-point interactions in the Lagrangian, that give the inverse propagator, after making the momentum-space replacement (c.f. Fourier expansion): $\partial_\mu \to -ip_\mu$

- E.g. quarks: $\bar{q}_i \left(i\partial_\mu \gamma^\mu - m \right) \delta_{ij} q_j \to \bar{q}_i \left(p_\mu \gamma^\mu - m \right) \delta_{ij} q_j$

$$\frac{(p_\mu \gamma^\mu + m)}{p^2 - m^2} \delta_{ij}$$

Note: inverse of gluon propagator requires extra gauge-fixing term

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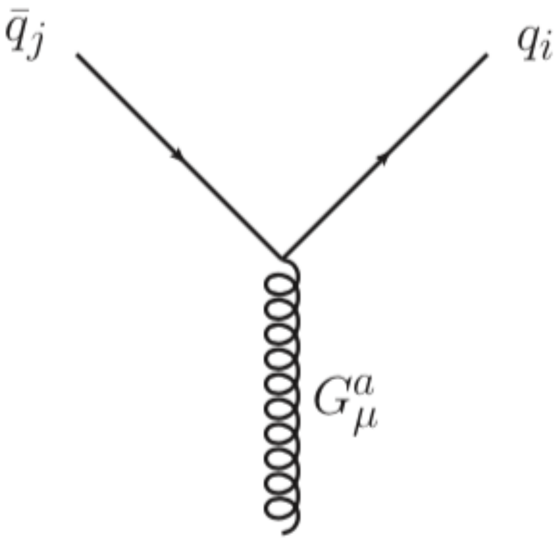
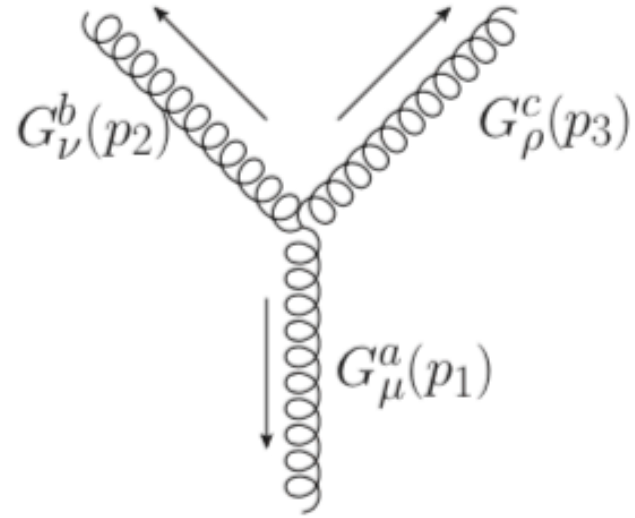
QCD interactions



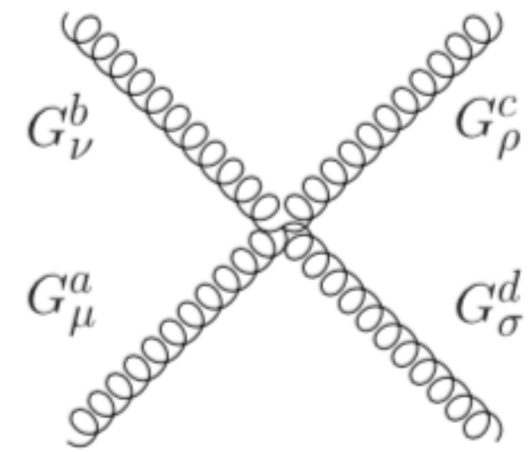
$$-ig_s T^a_{ij}\,\gamma_\mu$$

Quark-gluon interaction (gauge-invariant derivative)



$$g_s f^{abc}[(p_1-p_2)_\rho\, g_{\mu\nu} + (p_2-p_3)_\mu\, g_{\nu\rho} + (p_3-p_1)_\nu\, g_{\rho\mu}]$$

Triple-gluon vertex (field-strength tensor, one derivative)



$$ig_s^2[f^{eac}f^{ebd}(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\sigma}g_{\nu\rho}) + f^{ead}f^{ebc}(g_{\mu\nu}g_{\rho\sigma}-g_{\mu\rho}g_{\nu\sigma}) + f^{eab}f^{ecd}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho})]$$

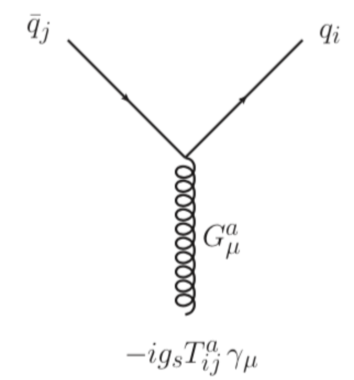
Four-gluon vertex (field-strength tensor, no derivatives)

$$(D_\mu)_{ij} = \partial_\mu \delta_{ij} + ig_s\left(t^a A^a_\mu\right)_{ij} \qquad F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g_s f_{abc} A^b_\mu A^c_\nu$$

Color factors in action

- As an example of how these work, consider additional gluon emission from a hard quark or gluon.
- Looking at just the color matrices (i.e. ignoring kinematics) we can find the effect on gluon emission probabilities from just the Feynman rules so far:

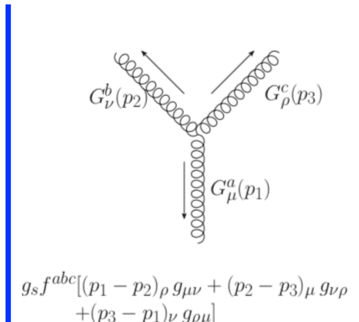
$$P_{\text{emit}}(q) \sim \left| \quad \right|^2$$



$$\sim \sum_{a,j} t^a_{ij} \left(t^a_{ij}\right)^* = \sum_{a,j} t^a_{ij}\, t^a_{ji} = C_F$$

(sum over colors) (Hermitian) (Casimir)

$$P_{\text{emit}}(g) \sim \left| \quad \right|^2$$



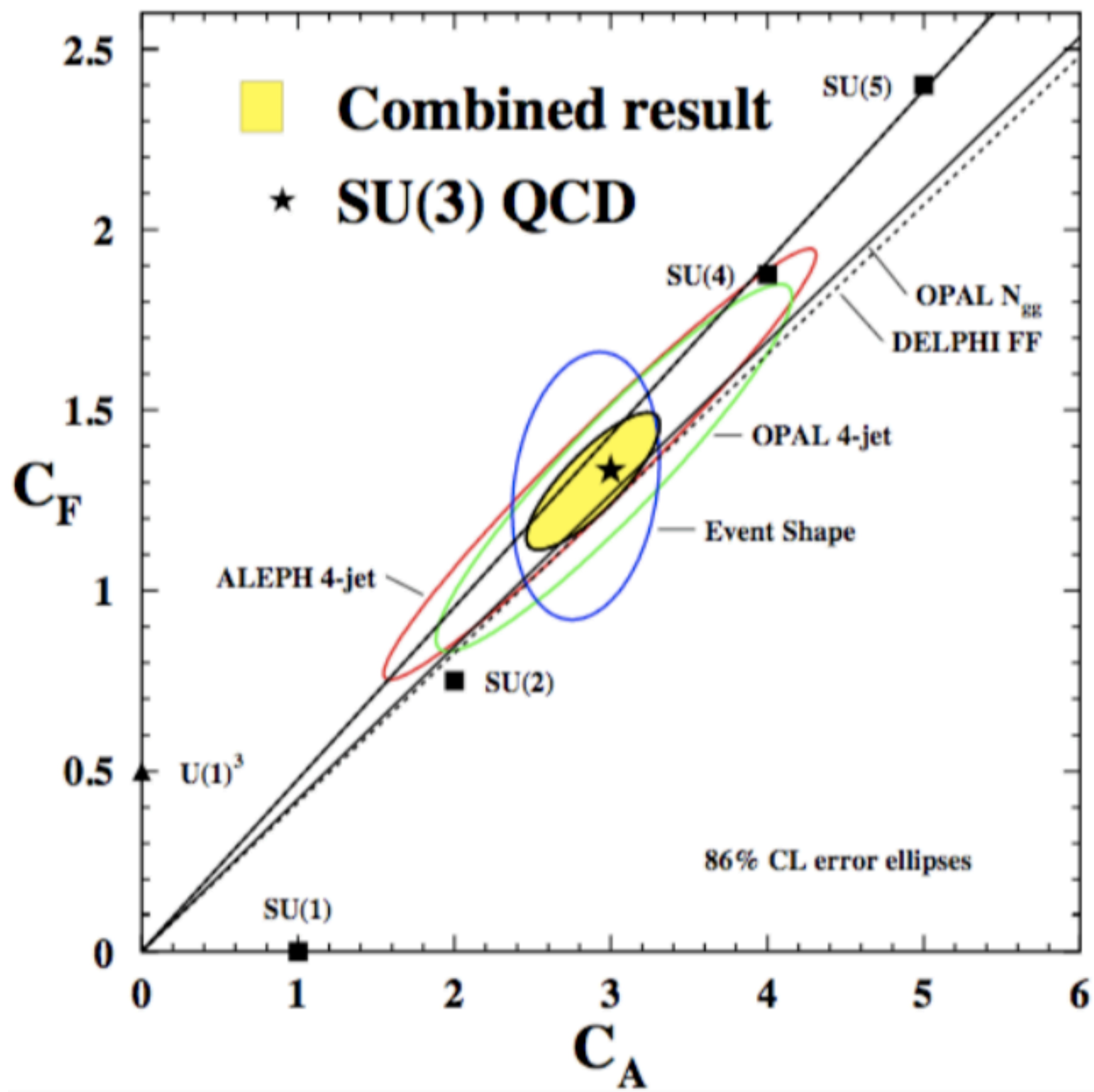
$$\sim \sum_{b,c} f_{abc}\, f_{abc} = C_A$$

(sum over colors) (Casimir)

- This gives rise to the expectation that gluon jets radiate more copiously than quark jets.

Color factors and cross-sections

- In calculations of cross-sections these sums over color factors are ubiquitous (and can be arduous to handle, for very many colored particles).
- The Casimirs of SU(3), C_A and C_F, are intrinsic to the theory of QCD.
- Their values have been tested, e.g. in measurements of jet event shapes at LEP.
  - A neat demonstration of the manifestation of group theory in physical observables.



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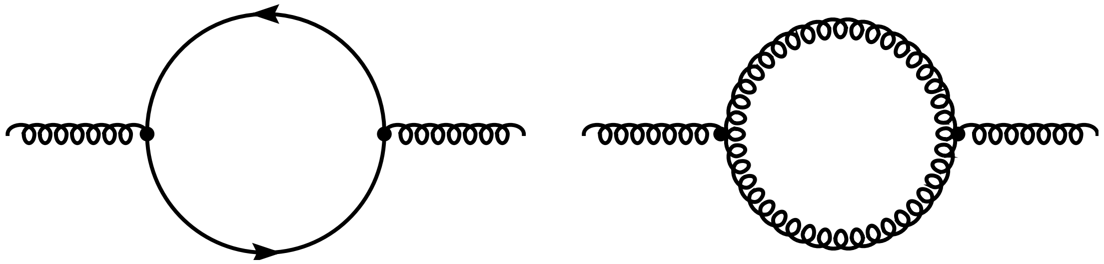
The strong coupling

- The effects of the quantum field theory vacuum — populated by short-lived, virtual quark and gluon pairs — affects the strength of the coupling.
- Trying to measure the coupling of an individual quark by probing with a gluon is only possible at sufficiently high energy.
- At lower energies the probe will instead only resolve a cloud of virtual particles that partially screen the coupling.
- This dependence on the energy scale μ is encoded in the beta function, which governs the running of the strong coupling:

$$\beta(\alpha_s) = \mu \frac{\partial \alpha_s(\mu)}{\partial \mu} = \frac{\partial \alpha_s(\mu)}{\partial (\log \mu)} \qquad \text{with } \alpha_s = \frac{g_s^2}{4\pi}$$

- It can be computed perturbatively by considering exactly these vacuum fluctuation diagrams, e.g. one power of α_s:



Running the strong coupling

- At this order the result is: $\beta(\alpha_s) = -b_0 \alpha_s^2 + \ldots$ $\qquad b_0 = \frac{11 C_A - 2 n_f}{6\pi}$
- With quark loops alone (c.f. QED) the result would be a positive beta-function; in contrast, in QCD the gluon loops make it negative.
- Can now solve for the coupling relative to some other scale *Q*:

$$\frac{\partial \alpha_s(\mu)}{\partial(\log \mu)} = -b_0 \alpha_s(\mu)^2 \quad \Longrightarrow \quad \left[\frac{1}{\alpha_s(\mu)}\right]_{\mu=Q} = b_0 \left[\log \mu)\right]_{\mu=Q}$$

$$\Longrightarrow \quad \alpha_s(\mu) = \frac{\alpha_s(Q)}{1 + \alpha_s(Q) b_0 \log(\mu/Q)}$$

- Note that this diverges at the scale $\Lambda \ll Q$ when the denominator vanishes.
- This condition gives an alternate expression for the running:

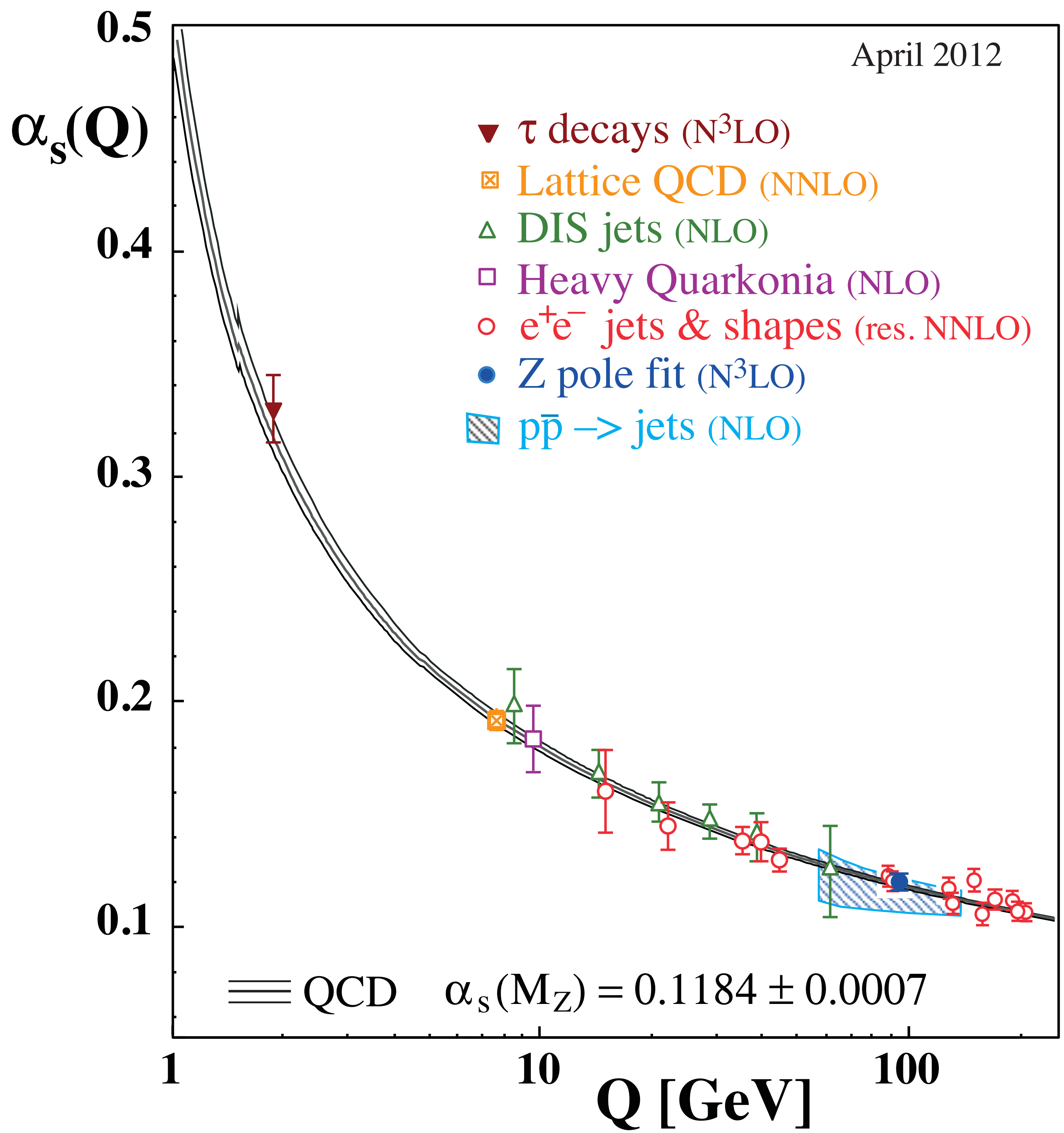
$$\alpha_s(\mu) = \frac{1}{b_0 \log(\mu/\Lambda)}$$

Λ is the QCD scale; it represents the position of the Landau pole in QCD

Consequences and tests

$$\alpha_s(\mu) = \frac{1}{b_0 \log(\mu/\Lambda)}$$

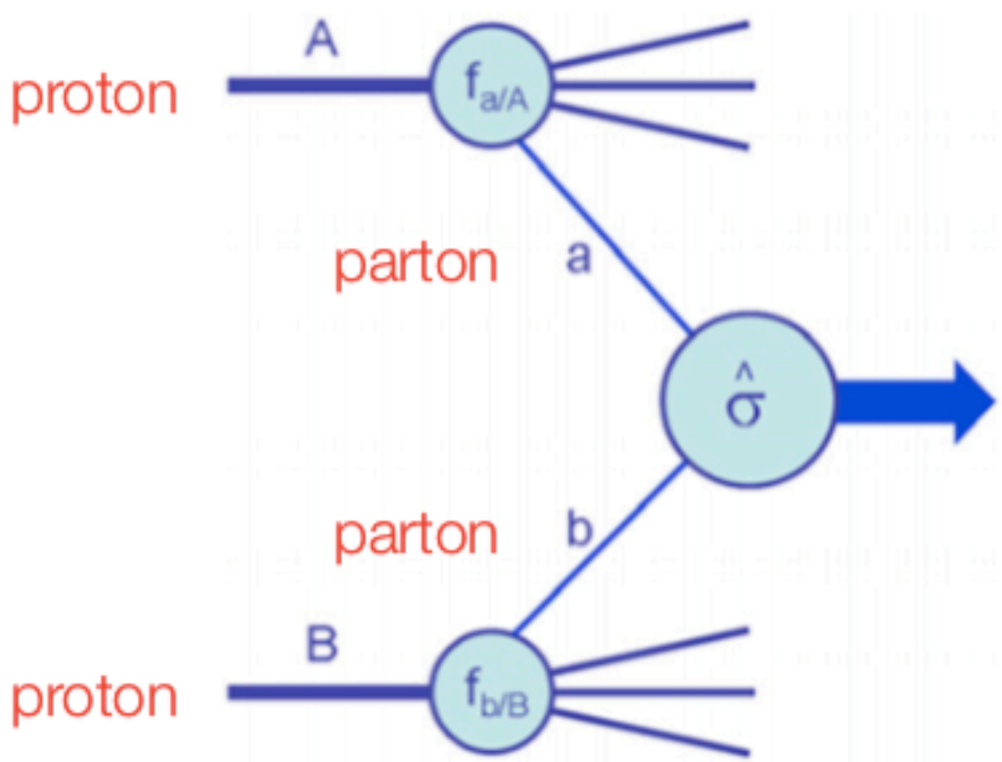
- Strong coupling decreases at high energy: asymptotic freedom.
- Perturbation theory requires sufficiently high energy, unreliable close to Λ.
- Measured value of the strong coupling $\Longleftrightarrow$ values of Λ around 250 MeV.



Protons and partons

- We now have to understand how to apply QCD in the era of hadron colliders.
- To do so, we have to understand how to apply a theory of quarks and gluons to the protons found in the beams.

$$\sigma_{2\to n} = \sum_{a,b} \int_0^1 \mathrm{d}x_a \mathrm{d}x_b \, f_{a/h_1}(x_a,\mu_F)\, f_{b/h_2}(x_b,\mu_F)\, \hat{\sigma}_{ab\to n}(\mu_F,\mu_R)$$



- The appropriate formalism is called collinear factorization.
- It divides the problem into:
  - soft physics, corresponding to the probability of finding, within a proton, a parton with a given momentum fraction *x*.
  - hard physics, the subsequent scattering between the incident quarks and gluons.
- Strictly only proven in special cases: Drell-Yan and deep inelastic scattering (DIS).

Parton distribution functions (pdfs)

- Depend on the momentum fraction (x_a) and the factorization scale (μ_F), that is implicit in the separation into soft and hard scales: $f_{a/h_1}(x_a, \mu_F)$
- Interpret as a probability ⇒ must integrate over fraction x_a (and x_b)
- In the simplest, non-interacting, picture one might assume the proton consists of just the three valence quarks. With no quark preferred above others one would get:

$$f_{u/p}(x,\, \mu^2) = 2\delta\left(x - \frac{1}{3}\right)$$
$$f_{d/p}(x,\, \mu^2) = \delta\left(x - \frac{1}{3}\right)$$

- By construction, these satisfy the momentum sum rule:

$$\int_0^1 \mathrm{d}x\, x \sum_i f_{i/h}(x,\, \mu^2) = 1 \;\; \forall \mu^2 \text{ and for all hadrons } h$$

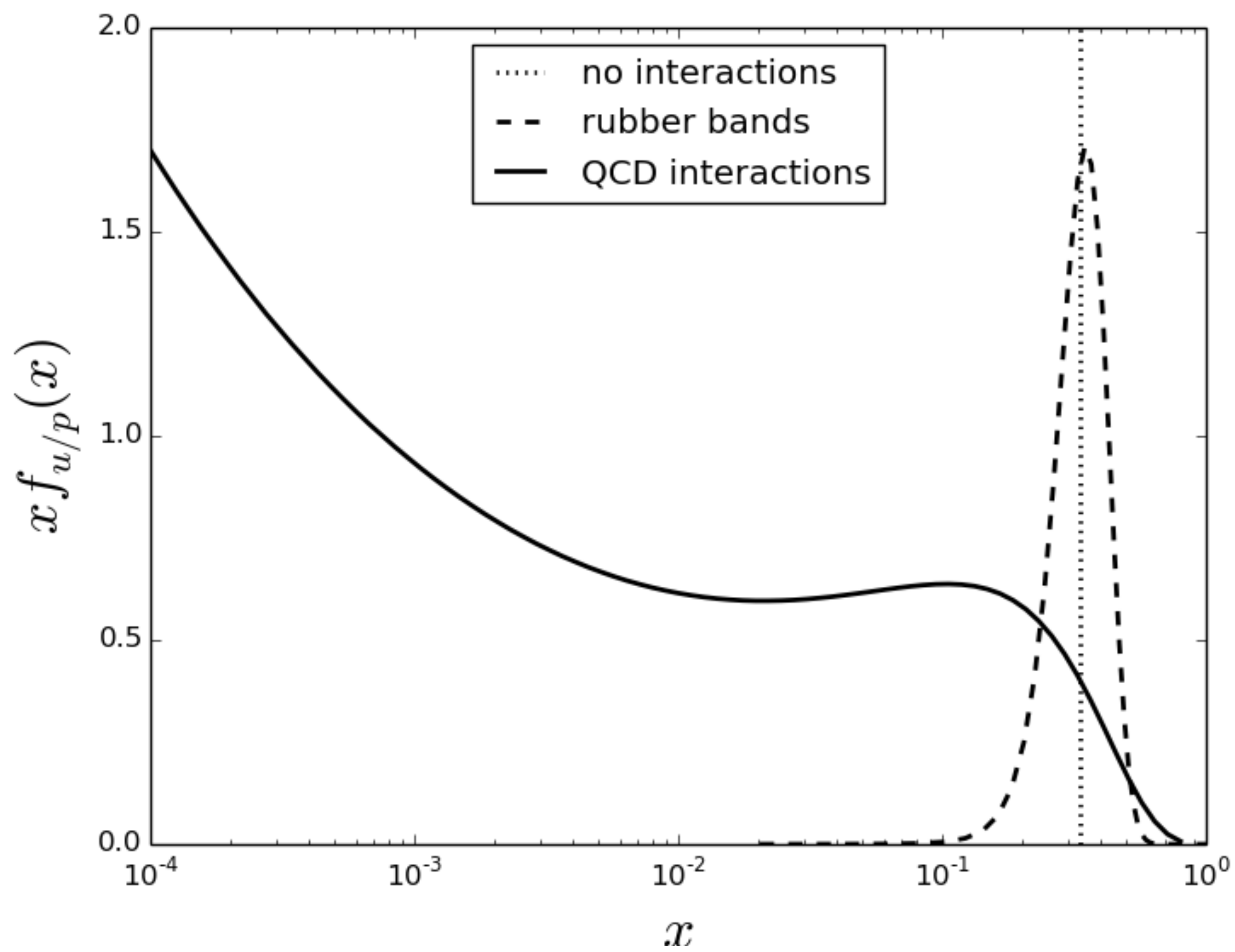
- A more sophisticated guess would be to imagine elastic interactions between the quarks, “rubber bands” holding them together
  - only effect would be to smear out the δ-function, smoothing the sharp peak at $x=1/3$.

QCD effects in pdfs

- In fact, the valence quarks inside the proton will emit gluons (that can further split into quark-antiquark pairs).
- These emissions will tend to be soft with respect to the original quark, meaning that the additional sea partons will be more likely to be found at small values of *x*.
- In fact, to a fair approximation:

$$f_{\mathrm{sea}/p}(x,\,\mu^2) \propto x^{-\lambda}$$

λ=1 (gluons, sea quarks)
λ=-1/2 (valence quarks)



- Effect of QCD interactions:
  - pdfs increase at small *x*
  - valence peak shifts to lower x ≈ 0.1 and broadens (due to emission)

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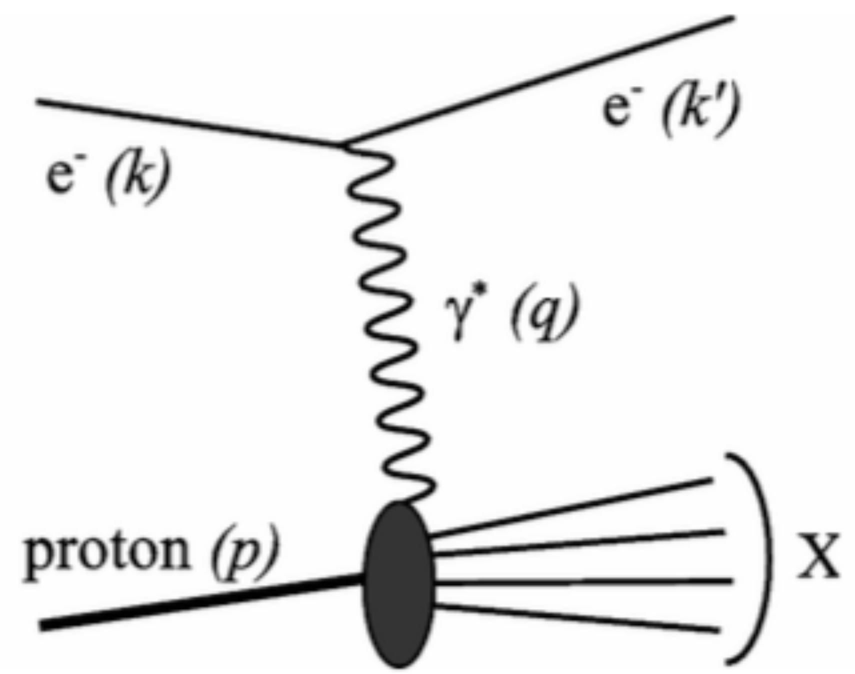
Probing pdfs

- Since they represent truly soft, non-perturbative, physics the pdfs cannot be calculated from first principles.
- However, the factorization procedure is based on the fact that they are universal: independent of the hard scattering and the rest of the collision.
  - therefore they can be extracted from experimental data.
- Deep inelastic scattering in electron-proton collisions, historically at HERA, is an ideal environment for this.
  - pdf enters only in part of the initial state.
  - the rest is well-known QED.



- This process is called "deep" due to the fact that the probing photon is of very high virtuality:

$$Q^2 = -q^2 \gg 1 \text{ GeV}$$

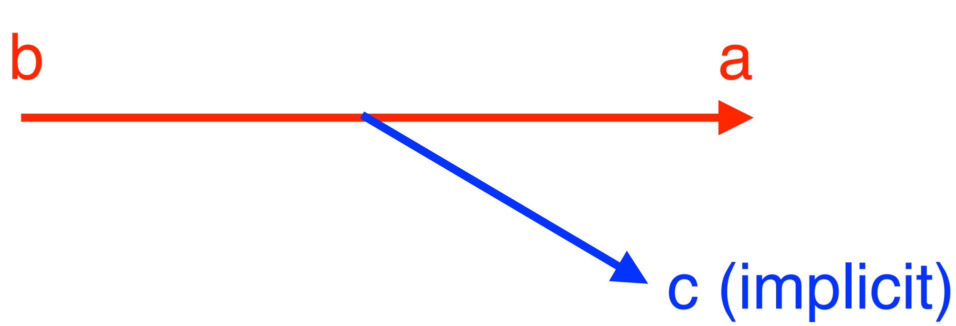
- This is the scale of the pdf that is probed.

Pdf evolution

- Although they are essentially non-perturbative objects, their evolution — the dependence on the probing scale — depends on the emission of quarks and gluons and is calculable in perturbative QCD.
- Just like the strong coupling, the pdfs obey (coupled) evolution equations. At first order these take the form:

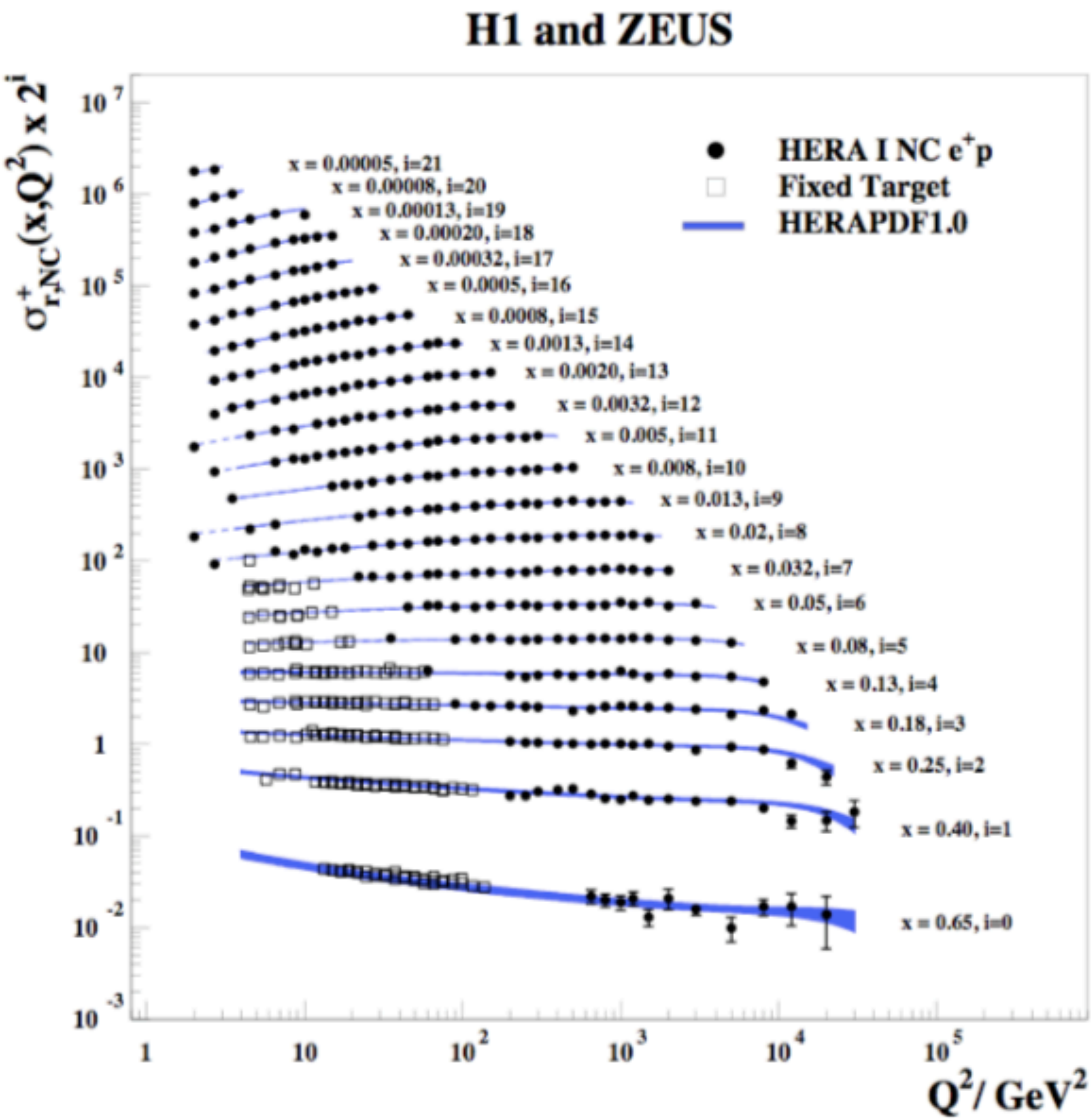
$$\frac{\partial}{\partial \log Q^2} \begin{pmatrix} f_{q/h}(x,\,Q^2) \\ f_{g/h}(x,\,Q^2) \end{pmatrix} = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{\mathrm{d}z}{z} \begin{pmatrix} \mathcal{P}_{qq}\left(\frac{x}{z}\right) & \mathcal{P}_{qg}\left(\frac{x}{z}\right) \\ \mathcal{P}_{gq}\left(\frac{x}{z}\right) & \mathcal{P}_{gg}\left(\frac{x}{z}\right) \end{pmatrix} \begin{pmatrix} f_{q/h}(z,\,Q^2) \\ f_{g/h}(z,\,Q^2) \end{pmatrix}$$

- Called the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation.
- The kernels of this evolution equation, the quantities P_{ab}, are called splitting functions (more on these later).
- They represent the parton splitting:



QCD-improved parton model

- Taking account of this evolution results in the QCD-improved parton model.
- It gives rise to so-called scaling violation, which is clearly visible in experimental data.
- See for example the combination of HERA data (from experiments H1 and ZEUS) taken over the period 1994-2000.



Pdf fitting: general strategy

- Since the Q^2 evolution of the pdfs is known, the traditional approach is to parametrize them at some reference scale, typically $Q_0 = 1\text{-}2$ GeV.
- Typically starting ansatz is:

$$F(x, Q_0) = x^{A_1}(1-x)^{A_2} P(x; A_3, A_4 \ldots).$$

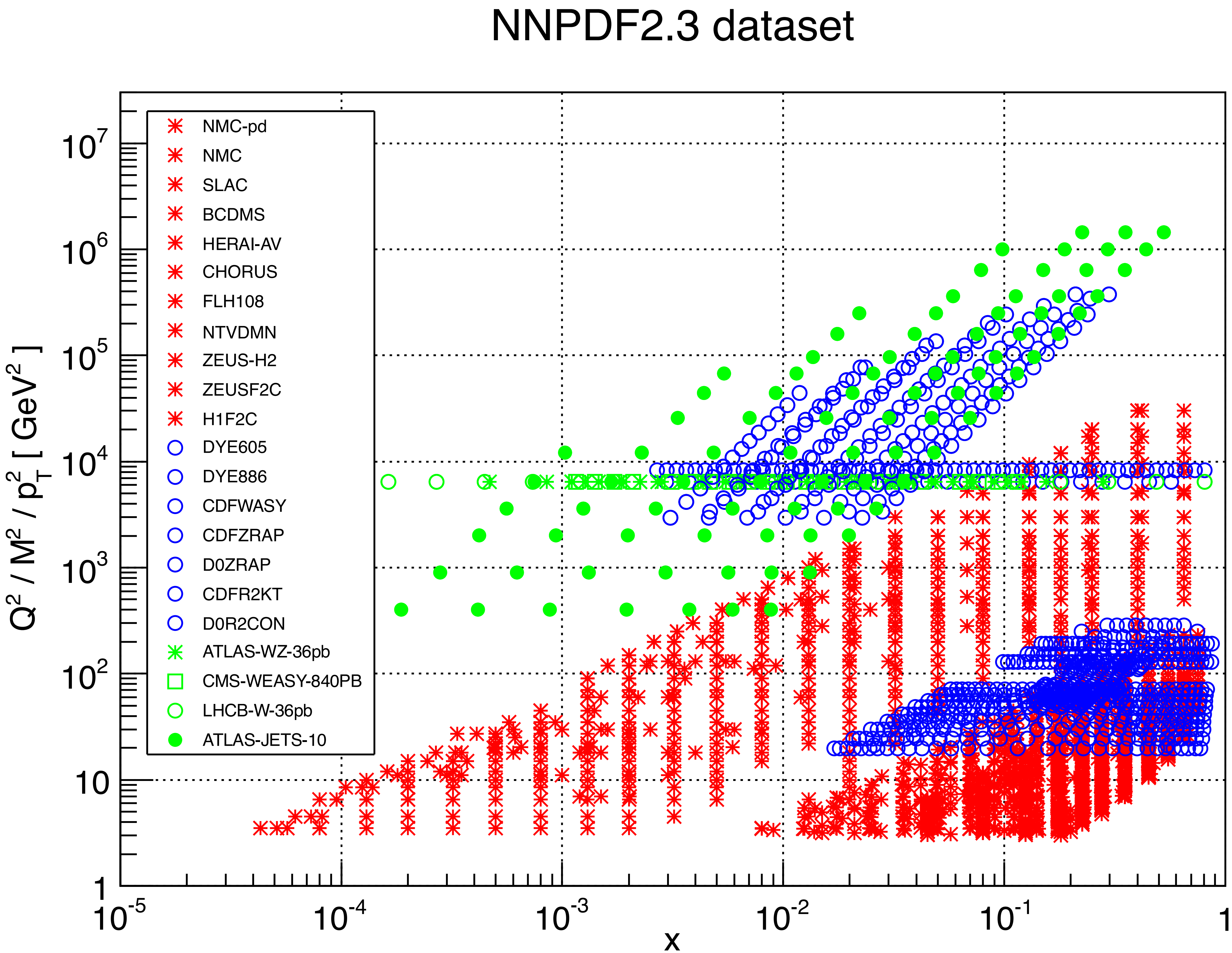
  with a smooth function P and free parameters A_1, A_2, …
- Perform a global fit to available data, using DGLAP equation to evolve the pdfs to the appropriate scale first.
- Lots of room for interpretation:
  - choice of input data sets (especially in cases of conflict)
  - order of perturbation theory (in theory predictions and DGLAP evolution)
  - input parametrization and other theoretical prejudice (e.g. always positive or not).
- Global fitting industry: continuous improvements to the fitting procedure and theoretical input. Main groups are CTEQ, MSTW/MMHT and NNPDF.
- NNPDF has a different approach to starting ansatz, instead using a sample of pdf replicas generated by neural network to try to avoid parametrization bias.

Typical data sets

LHC

fixed-larget, Tevatron

deep inelastic scattering



Pdf requirements

- Simplest case: production of a single particle with mass *M* and rapidity *y*.
- Kinematics:

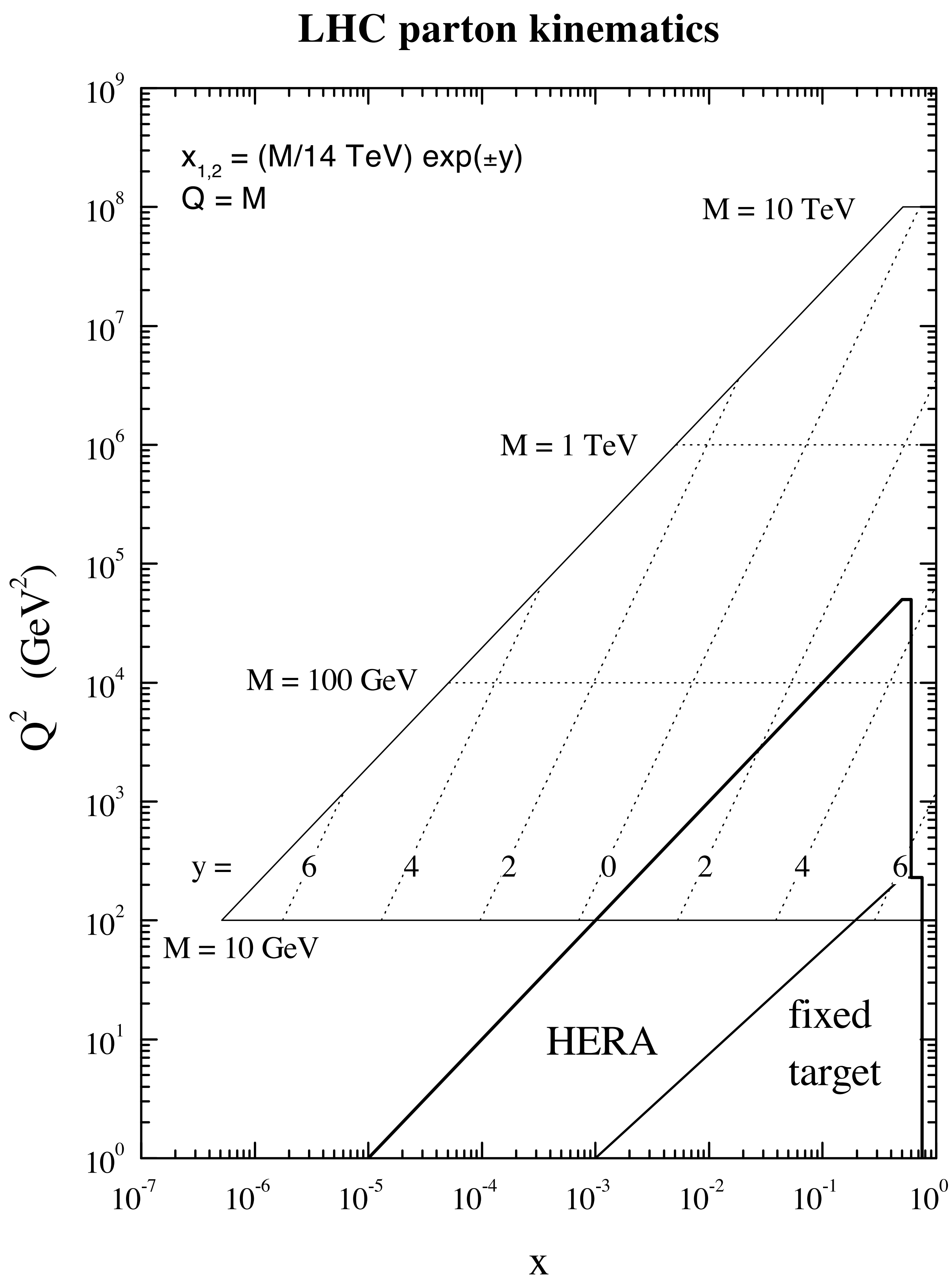
$$p_1 = x_1 \frac{\sqrt{s}}{2} (1, 0, 0, 1)$$

$$p_2 = x_2 \frac{\sqrt{s}}{2} (1, 0, 0, -1)$$

$$p_f = M (\cosh y, 0, 0, \sinh y)$$

$$\Longrightarrow \quad x_{1,2} = \frac{M}{\sqrt{s}} e^{\pm y}$$

- High-mass or high-rapidity particle production may be outside fit range and suffer from larger pdf uncertainties.



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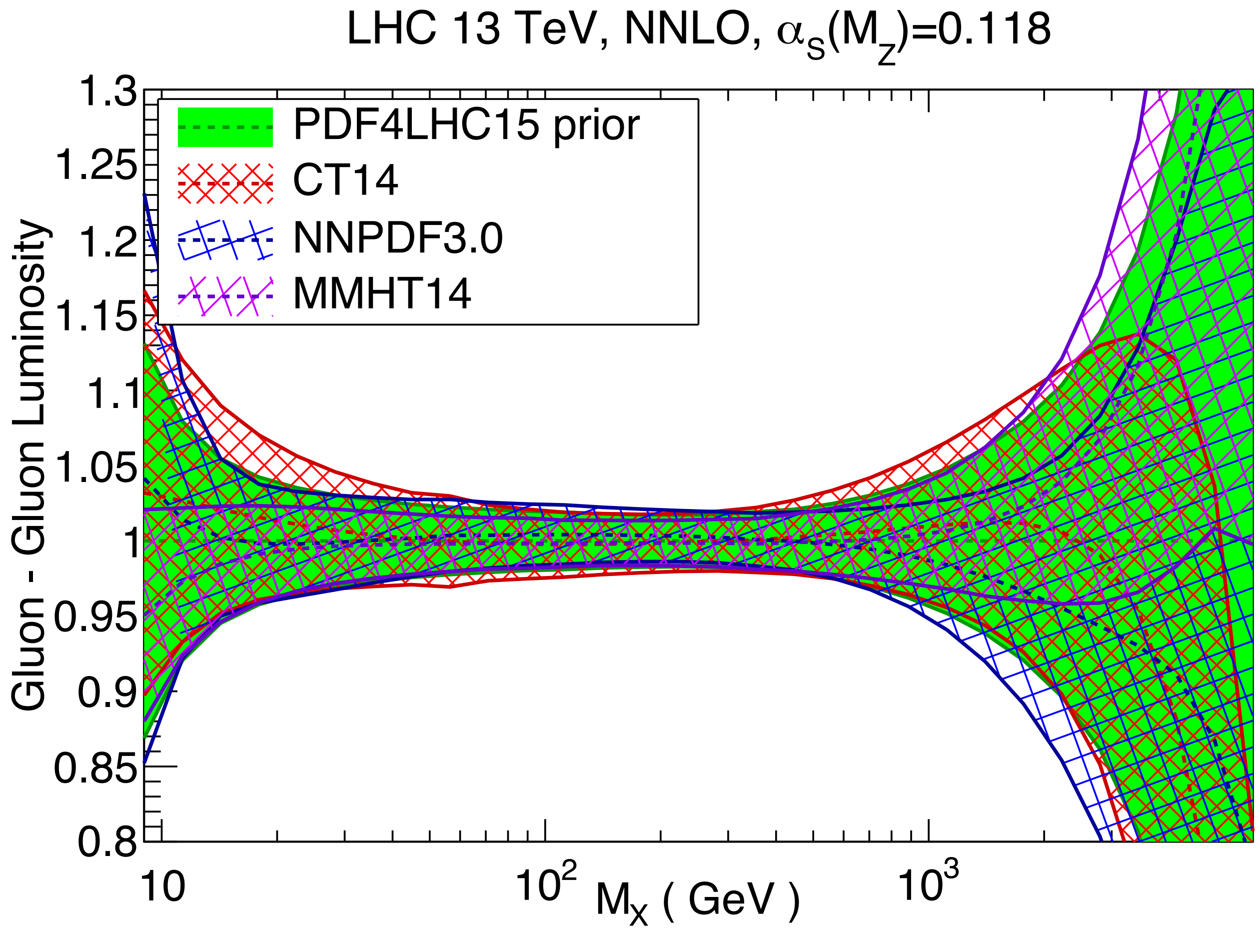
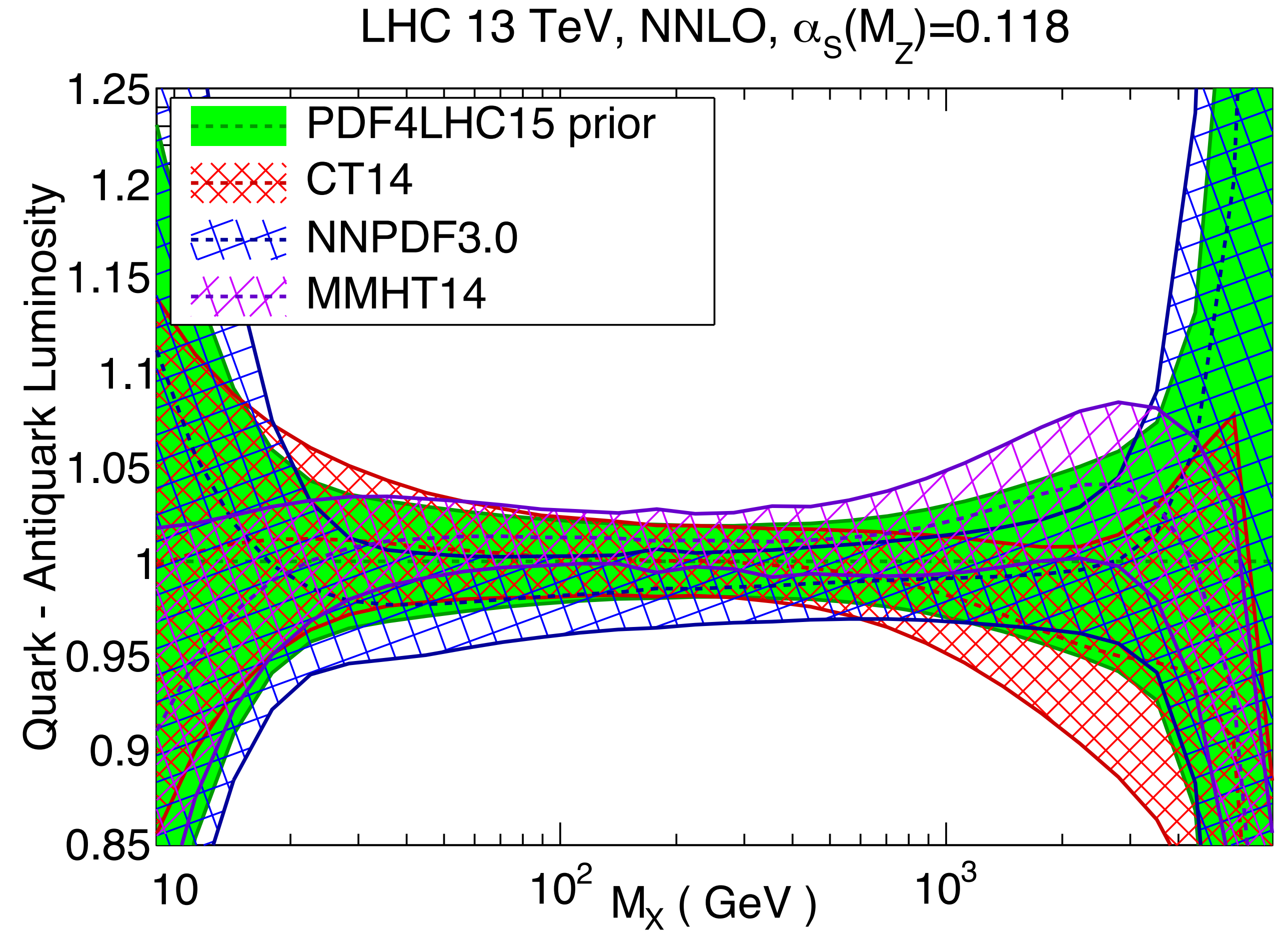
Uncertainties and consistency

- The associated pdf uncertainties typically cover the spread between different fitting groups, at least in the well-constrained region 50 - 500 GeV.
  - uncertainties on cross-sections at the level of 2–4% (important for modern precision!)
- Beyond that, differences begin to emerge and uncertainties are O(10%).
  - prescriptions for combining them to capture the spread exist, e.g. PDF4LHC.

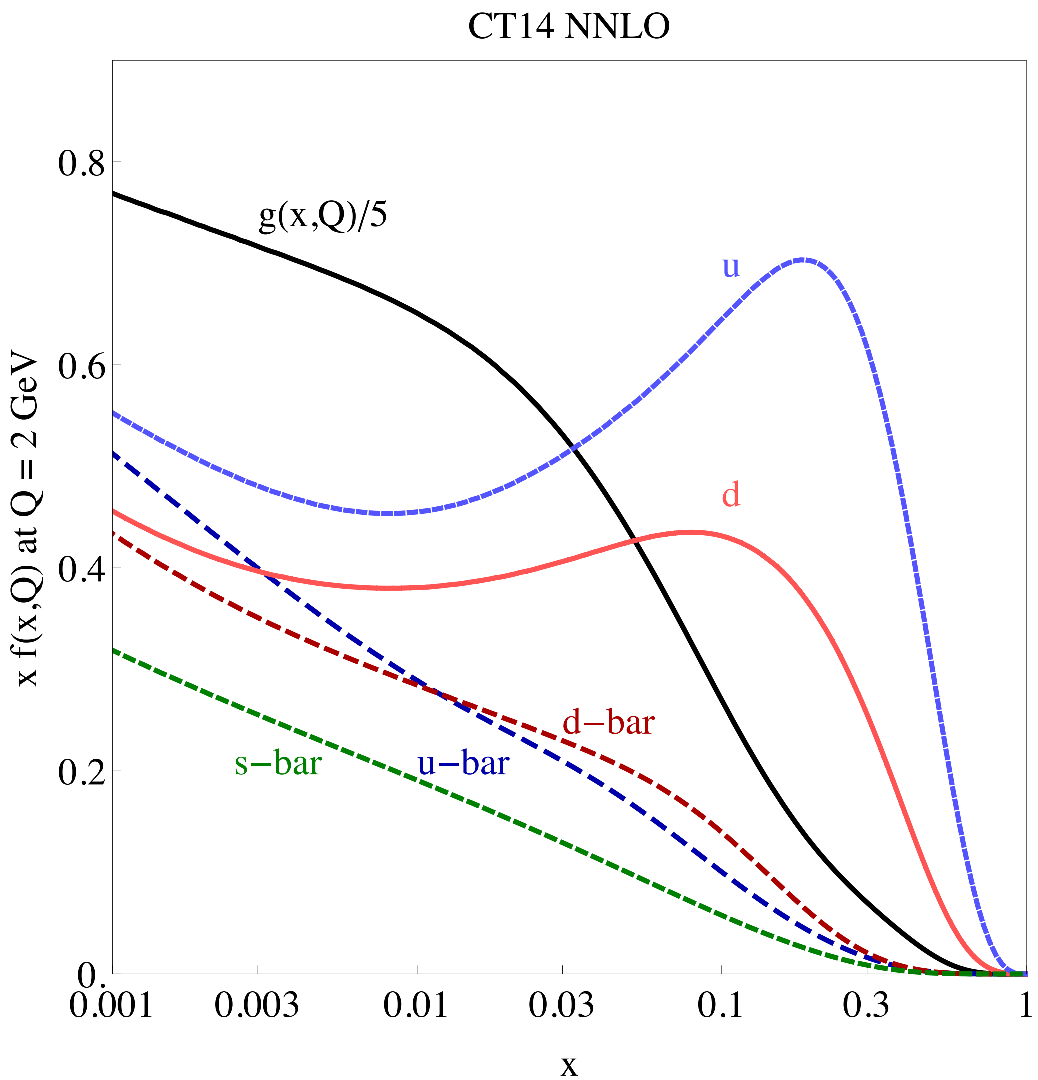
uncertainty for $gg \to X$

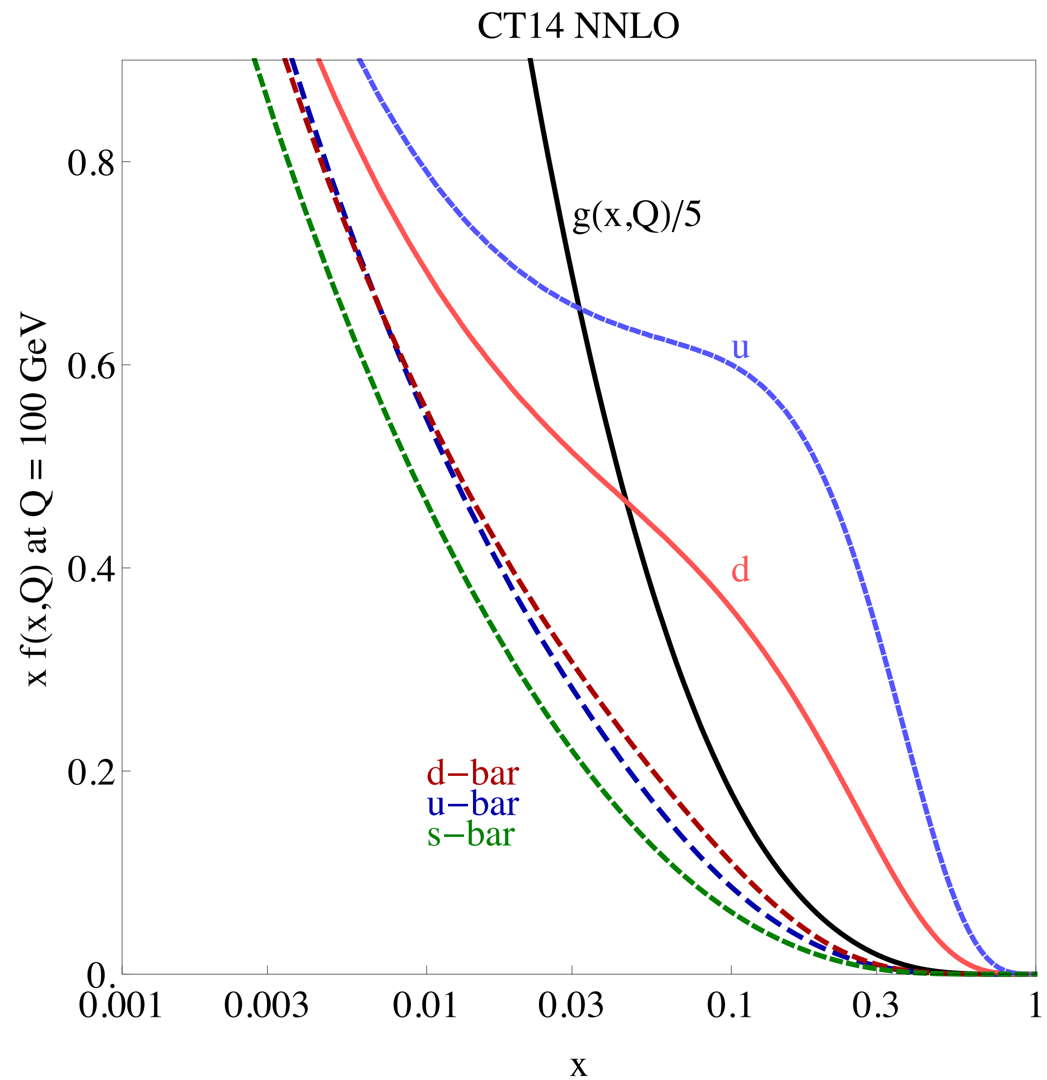


uncertainty for $q\bar{q} \to X$



Example pdfs





Q = 2 GeV

- Near starting scale for the evolution.
- u, d still peaked near x=1/3.

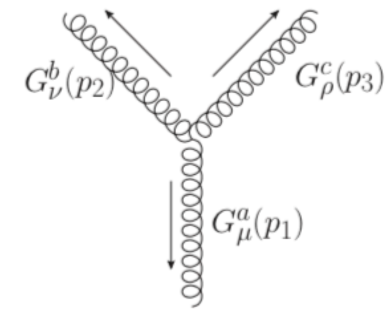
Q = 100 GeV

- Typical LHC kinematics.
- u,d flattened, less important.
- gluon dominant for x < 0.1.

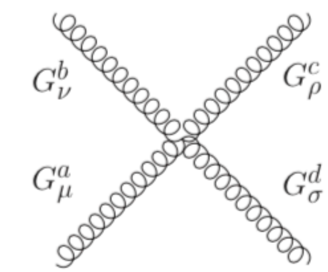
Summary so far

- Have illustrated how the QCD Lagrangian can be translated into Feynman rules, with an emphasis on the special role of color.

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} + \sum_{\text{flavors}} \bar{q}_i \left(i D_\mu \gamma^\mu - m \right)_{ij} q_j$$

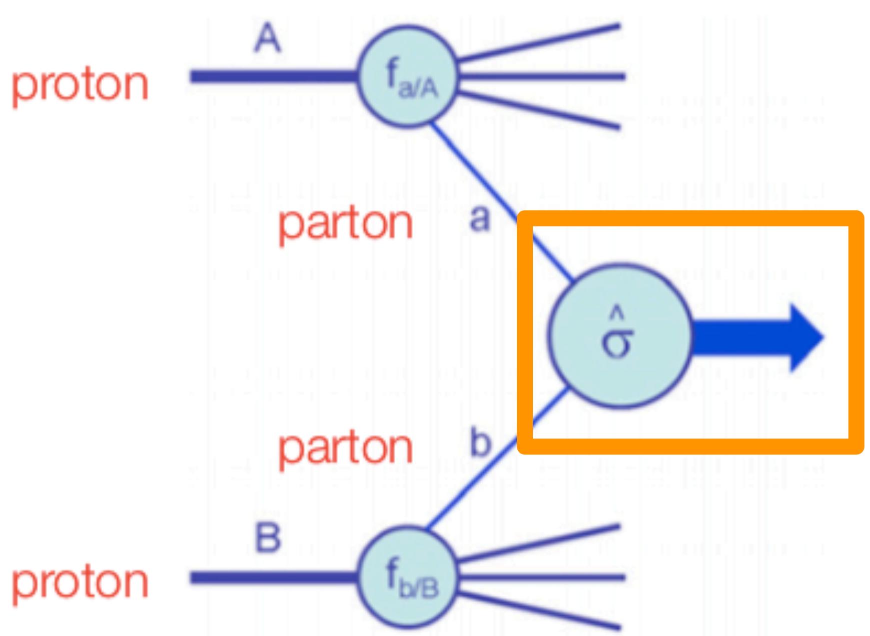


$$g_s f^{abc}[(p_1 - p_2)_\rho\, g_{\mu\nu} + (p_2 - p_3)_\mu\, g_{\nu\rho} + (p_3 - p_1)_\nu\, g_{\rho\mu}]$$



$$i g_s^2 [f^{eac} f^{ebd}(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho}) + f^{ead} f^{ebc}(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma}) + f^{eab} f^{ecd}(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})]$$

$$\sigma_{2\to n} = \sum_{a,b} \int_0^1 \mathrm{d}x_a \mathrm{d}x_b\, f_{a/h_1}(x_a, \mu_F)\, f_{b/h_2}(x_b, \mu_F)\, \hat{\sigma}_{ab\to n}(\mu_F, \mu_R)$$



- Have discussed the strong coupling and the idea of collinear factorization for hadron collisions and the introduction of pdfs.

- Will now spend some time on the calculation of the **hard scattering process**.

Hard scattering calculations

- First we have to break down the partonic cross-section we identified into a few constituent parts:

$$\hat{\sigma}_{ab\to n}(\mu_F,\mu_R) = \frac{1}{2\hat{s}}\int \mathrm{d}\Phi_n\,|\mathcal{M}_{ab\to n}|^2(\Phi_n;\mu_F,\mu_R)$$

- Incoming partonic flux: $\dfrac{1}{4\sqrt{(p_a\cdot p_b)^2 - p_a^2 p_b^2}} \xrightarrow{m_{a,b}\to 0} \dfrac{1}{2\hat{s}} = \dfrac{1}{2x_a x_b s}$

- Transition amplitude (or matrix element) squared: $|\mathcal{M}_{ab\to n}|^2(\Phi_n;\mu_F,\mu_R)$

- Integrated over the available n-parton phase-space element, $\mathrm{d}\Phi_n$.

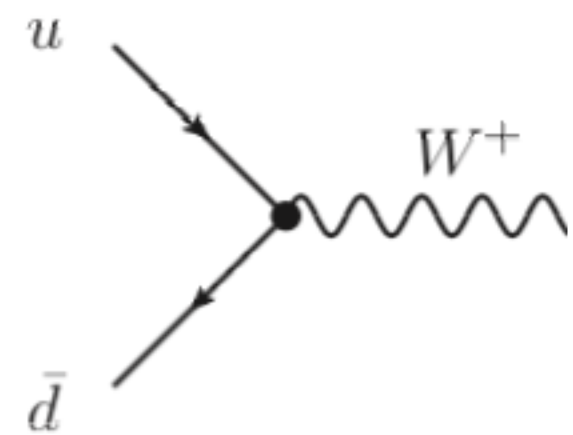
$$\mathrm{d}\Phi_n = \prod_{i=1}^{n}\left[\frac{\mathrm{d}p_i}{(2\pi)^4}(2\pi)\,\delta(p_i^2 - m_i^2)\Theta(p_i^{(0)})\right]\,(2\pi)^4\delta^4(p_a+p_b-\sum_{i=1}^{n}p_i)$$

Lorentz-invariant phase-space element for each final state particle

ensure overall four-momentum conservation

W-production

- Consider one of the simplest-possible hadron-collider processes, which is primarily mediated by up-anti-down annihilation.



- Application of the (mostly EW) Feynman rules gives the matrix element:

$$\mathcal{M}_{u\bar{d}\to W^+} = -\frac{iV_{ud}g_W\delta_{ij}}{\sqrt{2}}\bar{d}_i(p_2)\gamma^\mu\frac{1-\gamma_5}{2}u_j(p_1)\epsilon_\mu^{(W)}$$

CKM element (V_{ud}); weak coupling (g_W); trivial color factor (δ_{ij}); quark spinors; Dirac algebra (LH current); polarization vector

- Squaring and summing over spins and colors is an exercise in Dirac algebra:

$$\overline{\sum}|\mathcal{M}_{u\bar{d}\to W^+}|^2 = \frac{3}{9\cdot 4}\frac{|V_{ud}|^2 g_W^2}{2}\mathrm{Tr}\left[\not{p}_2\gamma^\mu\not{p}_1\gamma^\nu\frac{1-\gamma_5}{2}\right]\left[-g_{\mu\nu}+\frac{Q_\mu Q_\nu}{m_W^2}\right]$$
$$= \frac{|V_{ud}|^2 g_W^2}{12}Q^2 = \frac{|V_{ud}|^2 g_W^2}{12}m_W^2\,,$$

color sum (3); averaged over initial colors and spins ($9\cdot 4$); $Q = p_1+p_2$

Partonic cross-section

- Putting the ingredients together we have:

$$\sigma^{(\mathrm{LO})}_{h_1 h_2 \to W^+} = \int_0^1 \mathrm{d}x_u \mathrm{d}x_{\bar{d}} \sum_{u,\bar{d}} f_{u/h_1}(x_u,\, \mu_F)\, f_{\bar{d}/h_2}(x_{\bar{d}},\, \mu_F)\, \hat{\sigma}^{(\mathrm{LO})}_{u\bar{d} \to W^+}$$

where

$$\hat{\sigma}^{(\mathrm{LO})}_{u\bar{d} \to W^+} = \frac{1}{2\hat{s}} \int \frac{\mathrm{d}^4 p_W}{(2\pi)^4} (2\pi)^4 \delta^4(p_u + p_{\bar{d}} - p_W)(2\pi)\delta(p_W^2 - m_W^2)|\mathcal{M}|^2_{u\bar{d}\to W^+}$$
$$= \frac{\pi\delta(\hat{s} - m_W^2)}{\hat{s}} |\mathcal{M}|^2_{u\bar{d}\to W^+} = \frac{\pi\delta(\hat{s} - m_W^2)}{\hat{s}} \frac{g_W^2 |V_{ud}|^2 m_W^2}{12}$$

- Recalling our earlier kinematics we also have

$$\hat{s} = x_u x_{\bar{d}}\, s \qquad\qquad y_W = \frac{1}{2} \log \frac{x_u}{x_{\bar{d}}}$$

so that we can perform the convenient change of variable: $\mathrm{d}x_u \mathrm{d}x_{\bar{d}} = \frac{\mathrm{d}\hat{s}}{s} \mathrm{d}y_W$

Final result

$$\sigma^{(\mathrm{LO})}_{h_1 h_2 \to W^+} = \int_0^1 \mathrm{d}x_u \mathrm{d}x_{\bar{d}} \sum_{u,\bar{d}} f_{u/h_1}(x_u, \mu_F)\, f_{\bar{d}/h_2}(x_{\bar{d}}, \mu_F)\, \hat{\sigma}^{(\mathrm{LO})}_{u\bar{d} \to W^+}$$

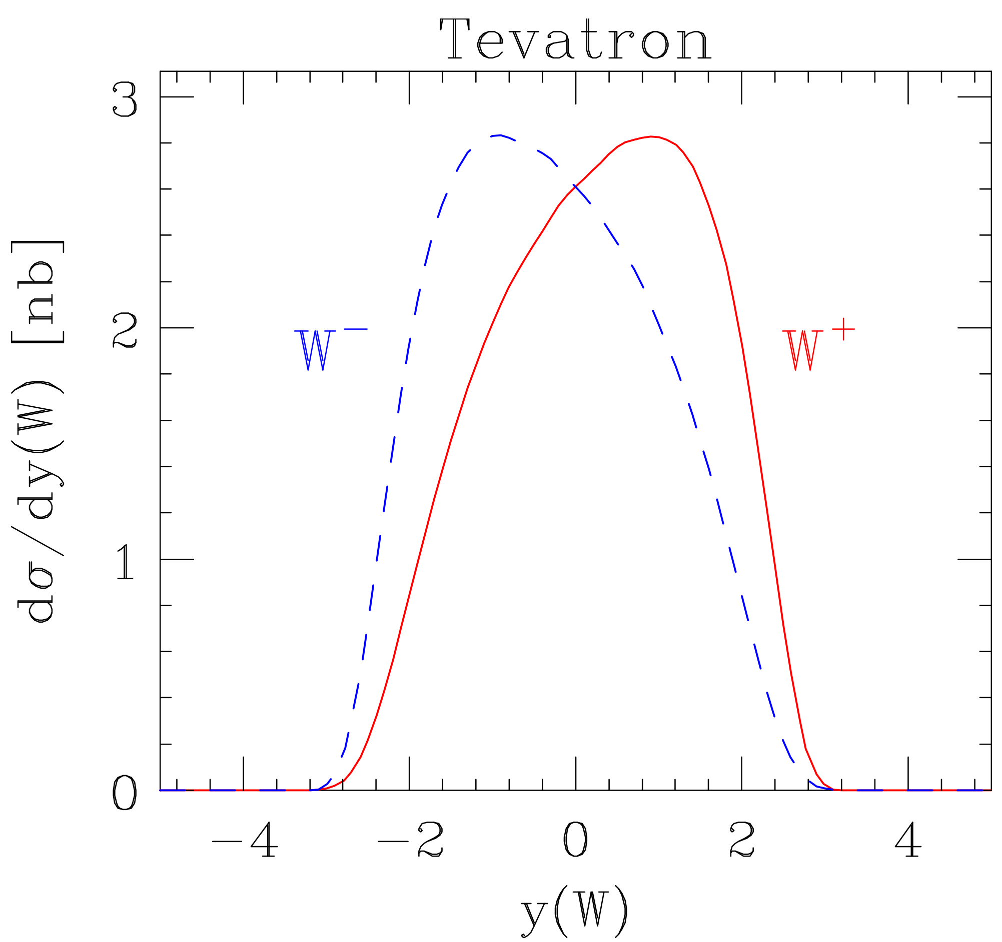
$$= \frac{\pi g_W^2 \, |V_{ud}|^2}{12} \int \frac{m_W^2 \mathrm{d}\hat{s}}{\hat{s}^2}\, \delta(\hat{s} - m_W^2)$$

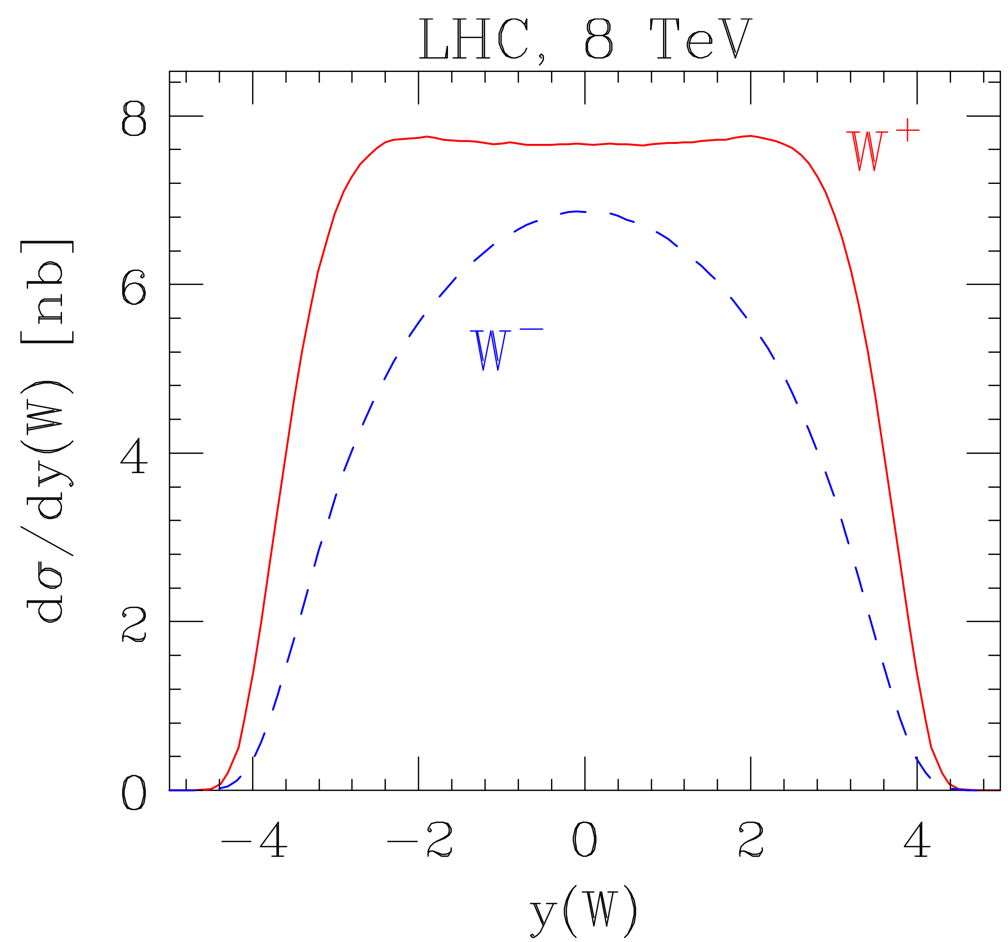
$$\times \int_{-y_{\max}}^{y_{\max}} \mathrm{d}y_W \sum_{u,\bar{d}} x_u f_{u/h_1}(x_u, \mu_F)\, x_{\bar{d}} f_{\bar{d}/h_2}(x_{\bar{d}}, \mu_F)\Big|_{x_u x_{\bar{d}} s = m_W^2}$$

$$= \frac{\pi g_W^2 \, |V_{ud}|^2}{12 s} \int_{-y_{\max}}^{y_{\max}} \mathrm{d}y_W \sum_{u,\bar{d}} f_{u/h_1}(x_u, \mu_F)\, f_{\bar{d}/h_2}(x_{\bar{d}}, \mu_F)\,,$$

- The maximum rapidity is constrained by x<1 to be: $y_{\max} = \frac{1}{2} \log \frac{s}{m_W^2}$
- This is the lowest order (tree-level) result for the inclusive cross-section.
  - the result for W^- is obtained by interchanging u and anti-d quarks.
- In this form we immediately see that the rapidity distribution of the W-boson is entirely defined by the (quark) pdfs.

W rapidity distribution: Tevatron vs. LHC





- Tevatron: valence quarks in protons drive production of W^+ to positive rapidity and anti-protons favor W^- at negative rapidity.
  - asymmetry is used to constrain high-*x* valence quark pdfs (although indirectly, through diluted lepton asymmetry)
- LHC: no preferred direction and sea quarks play an important role; impact of valence quarks still evident in wider plateau for W^+.

W rapidity at the LHC and beyond

- As energy of collisions increases, so does accessible range of W rapidities.
- The value of *x* required to produce a W boson decreases, leading to more important role for sea quarks.
- Eventually sea quarks dominate and, at central rapidities, W^+ and W^- cross sections become similar.