Weak coupling limit of $2+1, SU(2)$ lattice gauge theory and mass gap

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Lattice 2018

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Introduction

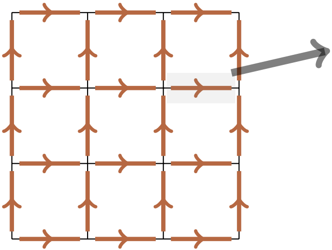
- Attempts to describe Yang mills theory in terms of Gauge invariant Wilson loops.
  - Non-local.
  - Over-complete.
- We will describe gauge theory in 'dual' electric loop representation.
  - local
  - complete.

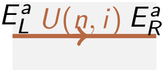
The plan of the talk.

1. A quick look at Hamiltonian LGT .
2. Point split lattice - PSlattice.
3. Local gauge invariant states.
4. Path integral in phase space.
5. Weak coupling limit and mass gap.

Hamiltonian SU(2) Gauge theory on a lattice

(Kogut and Susskind, 1976)





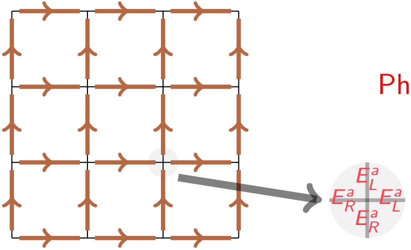
- $U \sim e^{iA}$ – $SU(2)$ parallel transport operator
- $E_L \equiv$ lattice analogue of E
- $E_R \equiv -E_L$ parallel transported by U
- $E_R^2 = E_L^2$ [link constraint]

- $E_L/E_R \in$ SU(2) algebra.

Hamiltonian SU(2) Gauge theory on a lattice continued..

- Hamiltonian is:

$$H = \frac{\tilde{g}^2}{2}\sum_{links} E^a E^a + \frac{1}{2g^2}\sum_{plaq}[2 - TrU_p]$$



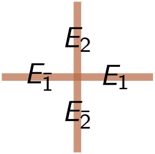
Physical states are gauge invariant.

Gauss Law Constraints!

$$\sum_i \left[E^a_L(i) + E^a_R(i)\right]|\psi_{phys}\rangle = 0$$

- Gauss law operator generates gauge transformations at each site.
- Gauss law says: at each site, incoming electric flux = outgoing electric flux.

Gauge invariant, local Hilbert space

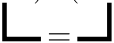


$E_i^a \in su(2)$ algebra

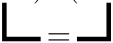
Gauss law:

$$\vec{E}_1 + \vec{E}_{\bar{1}} + \vec{E}_2 + \vec{E}_{\bar{2}} = 0$$

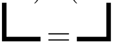
$$\left(\vec{E}_1 + \vec{E}_2\right) + \left(\vec{E}_{\bar{1}} + \vec{E}_{\bar{2}}\right) = 0$$

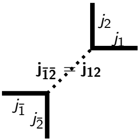


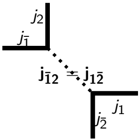
$$\left(\vec{E}_1 + \vec{E}_{\bar{2}}\right) + \left(\vec{E}_{\bar{1}} + \vec{E}_2\right) = 0$$

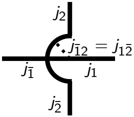


$$\left(\vec{E}_1 + \vec{E}_{\bar{1}}\right) + \left(\vec{E}_2 + \vec{E}_{\bar{2}}\right) = 0$$



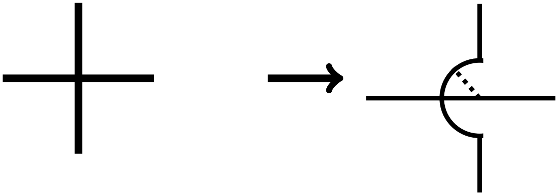






(Ramesh Anishetty and H. S. Sharatchandra,PRL,65, 813 (1990))

Splitting of point

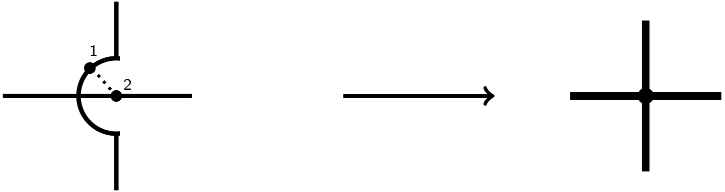


- Split the site into two sites and introduce a new link.
- Introduce Link operator and link constraint at the new link.
- All sites have 3 links and Gauss law constraint at each site.
- Dynamics is much more transparent on the split lattice.

(Ramesh Anishetty and T P Sreeraj, PRD, 97, 074511 (2018))

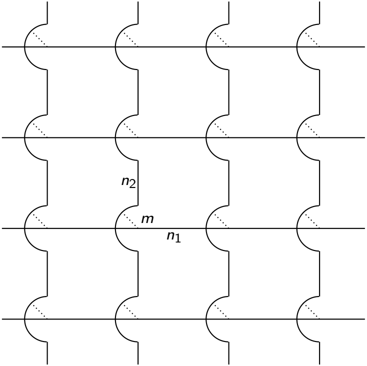
PS-lattice=original lattice

- PS lattice reduces to the original lattice by a gauge fixing.

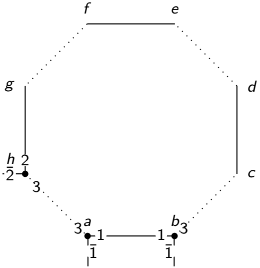


PS-lattice

- Lattice after splitting each site:



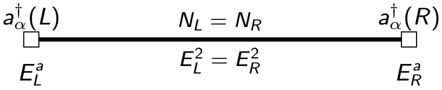
- plaquette → octagon



- 3 possible point splitting schemes at each site → large number of unitarily equivalent Hilbert spaces.

Schwinger Bosons.

- $E_L, U, E_R \rightarrow a^\dagger_\alpha(L), a^\dagger_\alpha(R)$; $a^\dagger_\alpha(L/R)-$ Harmonic oscillator doublets!



$$E_L^a \equiv a^\dagger(L)\frac{\sigma^a}{2}a(L), \qquad E_R^a \equiv a^\dagger(R)\frac{\sigma^a}{2}a(R).$$

$$E^2 = \frac{N}{2}\left(\frac{N}{2}+1\right)$$

$$U = \underbrace{\frac{1}{\sqrt{\hat{N}+1}}\begin{pmatrix} a_2^\dagger(L) & a_1(L) \\ -a_1^\dagger(L) & a_2(L)\end{pmatrix}}_{U_L}\underbrace{\begin{pmatrix} a_1^\dagger(R) & a_2^\dagger(R) \\ a_2(R) & -a_1(R)\end{pmatrix}\frac{1}{\sqrt{\hat{N}+1}}}_{U_R} \quad (prepotential\ \ rep)$$

[Manu Mathur, J.phys A(2005), Phys. Lett. B (2007), Nucl.Phys.B(2007)
Ramesh Anishetty, Manu Mathur, Indrakshi. R, JMP(2009),J.Phys(2009),JMP(2010)]

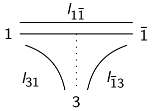
- Under gauge transformations:

$$U \rightarrow \Lambda_L U \Lambda_R^\dagger$$

$$a(L) \rightarrow \Lambda_L a(L) \quad , \quad a(R) \rightarrow \Lambda_R a(R)$$

Gauge invariant basis with Schwinger Bosons

- At a 3-vertex:



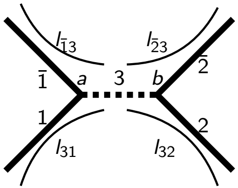
- Normalized gauge invariant states at a 3-vertex:

$$|l_{1\bar{1}}, l_{\bar{1}3}, l_{31}\rangle = \frac{\left(a^\dagger[1]\epsilon a^\dagger[\bar{1}]\right)^{l_{1\bar{1}}} \left(a^\dagger[\bar{1}]\epsilon a^\dagger[3]\right)^{l_{\bar{1}3}} \left(a^\dagger[3]\epsilon a^\dagger[1]\right)^{l_{31}}}{\sqrt{(l_{1\bar{1}} + l_{31} + l_{\bar{1}3} + 1)!(l_{1\bar{1}})!(l_{31})!(l_{23})!}} |0\rangle \equiv |n_1, n_{\bar{1}}, n_3 = m\rangle$$

- $n_1, n_{\bar{1}}, n_3$ gives the number of harmonic oscillators on the link $1, \bar{1}, 3$.

$$n_1 = l_{12} + l_{31} \qquad n_2 = l_{23} + l_{12} \qquad n_3 = m = l_{31} + l_{23}$$

(Ramesh Anishetty and T P Sreeraj, PRD, 97, 074511 (2018))



- Equivalent descriptions based on:
  1. l_{ij} satisfying the link condition :

$$l_{31}[a] + l_{\bar{1}3}[a] = n_3 (\equiv m) = l_{32}[b] + l_{\bar{2}3}[b]$$

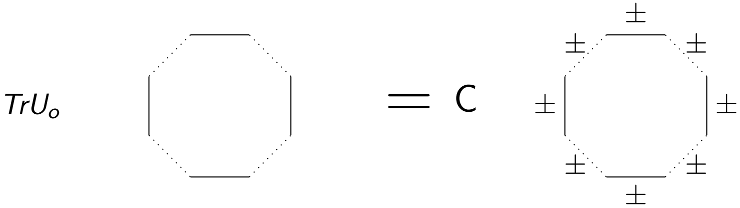
l_{ij} into a link $=$ l_{ij} going out $\Longrightarrow$ Closed Electric flux loops.

  2. n_i, m -local quantum numbers satisfying triangle inequalities at each site:

$$|n_i - n_{\bar{i}}| \leq m \leq n_i + n_{\bar{i}}$$

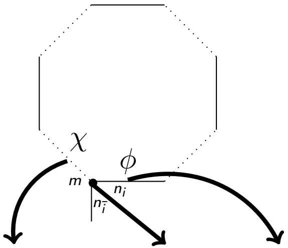
Action of Hamiltonian on the number basis.

- $E_i^2 = \frac{\hat{N}_i}{2}\left(\frac{\hat{N}_i}{2}+1\right)$ diagonal.
- $TrU_p = TrU_o$ changes n_i, m at each link along a plaquette by ± 1.



Phases

- We define phase operators satisfying : $[\hat{N}_i, e^{i\hat{\phi}}] = e^{i\hat{\phi}}$ $\qquad [\hat{M}, e^{i\hat{\chi}}] = e^{i\hat{\chi}}$



$$TrU_o|n_i, n_{\bar{i}}, m\rangle = Tr\prod_{oct} \underbrace{\begin{pmatrix} e^{i\hat{\chi}} & 0 \\ 0 & e^{-i\hat{\chi}} \end{pmatrix} \begin{pmatrix} D & F \\ F & D \end{pmatrix} \begin{pmatrix} e^{i\hat{\phi}} & 0 \\ 0 & e^{-i\hat{\phi}} \end{pmatrix}}_{\hat{P}=\hat{L}_\chi \hat{V} \hat{L}_\phi} |n_i, n_{\bar{i}}, m\rangle$$

$$D = \sqrt{\frac{(n_i + n_{\bar{i}} + m + 3)(n_i - n_{\bar{i}} + m + 1)}{4(m+1)(n_i+1)}} \qquad F = \sqrt{\frac{(n_{\bar{i}} - n_i + m + 1)(n_{\bar{i}} + n_i - m + 1)}{4(m+1)(n_i+1)}}$$

Path integral in phase space

• Path integral is constructed in phase space by usual time slicing and sandwiching eigenbasis of the number and phase basis.

• Path integral in phase space is :

$$Z = \int D\phi_i D\chi \sum_{n_1,n_2,m}{}' e^{-\int dt \left[\sum_s \left[i(n_1\dot{\phi}_1 + n_2\dot{\phi}_2 + m\dot{\chi}) + \frac{\tilde{g}^2}{2}\left(n_1^2(s) + n_2^2(s)\right)\right] + \frac{1}{2g^2}\sum_{oct}\left[2 - Tr\left(\prod_{oct} P\right)\right]\right]}$$

n_1, n_2, m should satisfy triangle inequality.

Weak coupling analysis

- When $g \to 0$, $\langle n_1 \rangle = \langle n_2 \rangle = N$, $\langle m \rangle = 2N$, N large ,ϕ_i, χ small gives

$$P = \begin{pmatrix} e^{i\hat{\chi}} & \\ & e^{-i\hat{\chi}} \end{pmatrix} \begin{pmatrix} D & F \\ F & D \end{pmatrix} \begin{pmatrix} e^{i\hat{\phi}} & \\ & e^{-i\hat{\phi}} \end{pmatrix} \quad \to \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1}$$

$$D = \sqrt{\frac{(n_i + n_{\bar{i}} + m + 3)(n_i - n_{\bar{i}} + m + 1)}{4(m+1)(n_i+1)}} \sim 1$$

$$F = \sqrt{\frac{(n_{\bar{i}} - n_i + m + 1)(n_{\bar{i}} + n_i - m + 1)}{4(m+1)(n_i+1)}} \sim \frac{1}{2\sqrt{N}}$$

attains the minimum of the magnetic term.

- Splitting fields into mean field and fluctuations.

$$\begin{aligned} n_i &= N + \tilde{n}_i \qquad & m &= 2N + \tilde{m} \\ D &\sim o(1) \qquad & F &\sim o(1/2\sqrt{N}) \end{aligned} \tag{2}$$

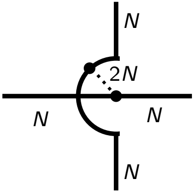
- Redefine $\phi_i, \chi \to g\phi_i,\ g\chi$.

Weak coupling Vacuum

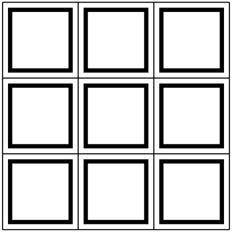
- $\langle n_1 \rangle = \langle n_2 \rangle = N, \langle m \rangle = 2N$

$\implies$ all electric flux into a site in x direction goes to y direction and vice versa

$\implies$ small electric loops.



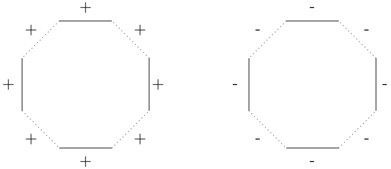
- Vacuum dominated by small (spatially) electric flux loops containing huge fluxes.



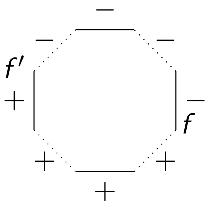
(in the unsplit lattice)

Fluctuations

- Dominant fluctuations:



- sub dominant fluctuations of order $\frac{1}{N}$:



$+ \cdots$

Each flip gives a factor of $\frac{1}{2\sqrt{N}}$.

• We now make an expansion in $\frac{1}{N}$ and g. After a few field redefinitions gives :

$$\left[2 - Tr\Big(\prod P\Big)\right] \approx \left[\frac{1}{4N^2}\tilde{m}^2 + V(\phi_1, \phi_2, \chi)\right] \qquad (3)$$

$$\begin{aligned} V(\phi_1,\phi_2,\chi) = \frac{g^2}{2}\Bigg\{ & \Big[(\Delta_1(\phi_2 - \frac{1}{2}\Delta_2\chi) - \Delta_2(\phi_1 + \frac{1}{2}\Delta_1\chi))\Big]^2 \\ & + \frac{1}{N}\Big[16\Big[(\phi_1 + \frac{1}{2}\Delta_1\chi)^2 + (\phi_2 - \frac{1}{2}\Delta_2\chi)^2 + \chi^2\Big] \\ & - \Big[\Delta_1(\phi_2 - \frac{1}{2}\Delta_2\chi) - \Delta_2(\phi_1 + \frac{1}{2}\Delta_1\chi) + \Delta_1\Delta_2\chi\Big]^2 - (\Delta_1\Delta_2\chi)^2\Big]\Bigg\} \end{aligned}$$

$$\begin{aligned} = \frac{g^2}{2}\Bigg\{ & \Big(\Delta_1\phi'_2 - \Delta_2\phi'_1\Big)^2 + \frac{1}{N}\Big[16\Big(\phi'^2_1 + \phi'^2_2 + \chi^2\Big) - \Big(\Delta_1\phi'_2 - \Delta_2\phi'_1 + \Delta_1\Delta_2\chi\Big)^2 \\ & - (\Delta_1\Delta_2\chi)^2\Big] \end{aligned} \qquad (4)$$

• Performing the Gaussian summation over $\tilde{n}_1, \tilde{n}_2, \tilde{m}$, and making the transformation: $\phi_i' = \frac{1}{\sqrt{-\Delta^2}}(\Delta_i \eta + \epsilon_{ij}\delta_j \psi)$
Path integral becomes:

$$Z = \int D\psi D\eta D\chi \; e^{-\int dt \sum_{sites} \left[\frac{2g^2}{\tilde{g}^2}\left(\dot{\eta}^2+\dot{\psi}^2\right)+2g^4N^2\dot{\chi}^2+V'(\psi,\eta,\chi)\right]+o(a^4)}$$

$$V'(\psi,\eta,\chi) = \left\{\frac{1}{4}(\Delta\psi)^2 + \frac{1}{4N}\left[16(\eta^2+\psi^2+\chi^2) - (\Delta\psi)^2\right]\right\}$$

• Casting ψ in canonical form by $\psi \to \sqrt{2}\psi$ gives:

$$\left(\frac{g}{\tilde{g}}\right)^2 = \frac{a^2}{8}$$

$$\frac{16}{N} = M^2 a^2 \qquad\qquad N = \frac{16}{M^2 g^4}$$

Dispersion relations.

- The euclidean inverse propagators in the energy-momentum space to the leading order are

$$\psi : p_0^2 + M^2 + \vec{p}^2 + O(a^2)$$
$$\eta : p_0^2 + M^2 + O(a^4)$$
$$\chi : M^2 + O(a^4); p_0 = 0.$$

- ψ is a relativistic particle with mass M
- η may propagate due to higher order corrections.
- χ do not fluctuate.

On going work

1. Calculation of string tension.
2. Extending the same methods to higher dimensions.
3. Inclusion of fermions.
4. Extension to SU(3)

Thanks

Thank You for your Attention.