Simulation of Scalar Field Theories with Complex Actions

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Outline

- PT Symmetry and a new algorithm
- Models: Bose gas, positivity violations and pattern formation
- The special problem of $i\phi^3$

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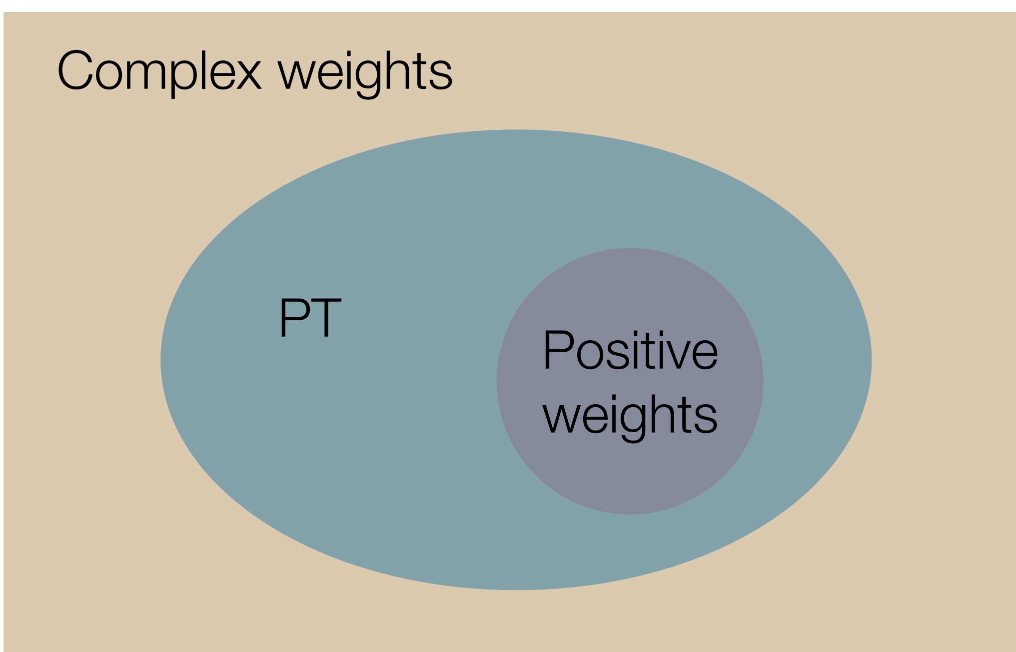
- Carl Bender and Zohar Nussinov: advice and discussions

The Sign Problem

Within the general class of problems with complex weights, there is a distinguished class: those with a **PT symmetry**:

$$\mathscr{L}(\chi) = \mathscr{L}(-\chi)^{*}$$



Bender & Boettcher, PRL 1998
Meisinger & mco 1208.5077

- Motivated by Lee-Yang theory and the iϕ^3 field theory
- Many favorite sign problem models are PT-symmetric, *e.g.*, the charged Bose gas and QCD at finite density. In those cases, the symmetry is CK where C is charge conjugation and K is complex conjugation.
- Unbroken PT symmetry implies that transfer matrix eigenvalues are either real or form a complex conjugate pair. Z is real but not necessarily positive.

An algorithm for PT-symmetric models

- Any PT symmetric theory can be transformed into a form where all weights are real.
- Key steps are rewriting the kinetic and potential terms as Fourier transforms. This can be understood as a partial duality transform.
- Local, easy to implement, works in any dimension
- ***For models satisfying the positive weight condition, the sign problem is completely solved.***

$$S(\chi) = \sum_x \left[\tfrac{1}{2}(\partial_\mu \chi(x))^2 + V(\chi(x)) - ih(x)\chi(x)\right]$$

PT Symmetry: $V(\chi)^* = V(-\chi)$

$$\exp\left[\frac{1}{2}\left(\partial\chi_x\right)^2\right] = \int d\pi_\mu(x) \exp\left[\frac{1}{2}\pi_\mu(x)^2 + i\pi_\mu(x)\partial_\mu \chi_x\right]$$

$$w(\chi) \equiv \exp[-V(\chi)]$$

$$w(\chi)^* = w(-\chi) \implies \tilde{w}(\tilde{\chi}) \equiv F[w] \in R$$

$$\tilde{w} > 0 \implies \tilde{V} \equiv -\log[\tilde{w}] \in R$$

$$Z = \int \prod_x d\pi_\mu(x) \exp\left\{-\sum_x \left[\tfrac{1}{2}\pi_\mu^2(x) + \tilde{V}(\partial \cdot \pi(x) - h(x))\right]\right\}$$

Two Components and the Bose Gas

The Bose gas at finite density has the same anti-linear symmetry as QCD: CK where C is charge conjugation and K is complex conjugation. The key is treatment of the temporal derivative term:

$$\Psi = \frac{1}{\sqrt{2}}\left(\phi + i\chi\right)$$

PT Symmetry: $\mathscr{L}\left(\phi,\chi\right)^{*} = \mathscr{L}\left(\phi,-\chi\right)$

Key: Fourier transform time derivative and mass term together

$$K_0 = \left(e^{\mu}\Psi'^{*} - \Psi^{*}\right)\left(e^{-\mu}\Psi' - \Psi\right) + \frac{1}{2}m^2\left(\Psi'^{*}\Psi' + \Psi^{*}\Psi\right)$$

After Fourier transformation of χ:

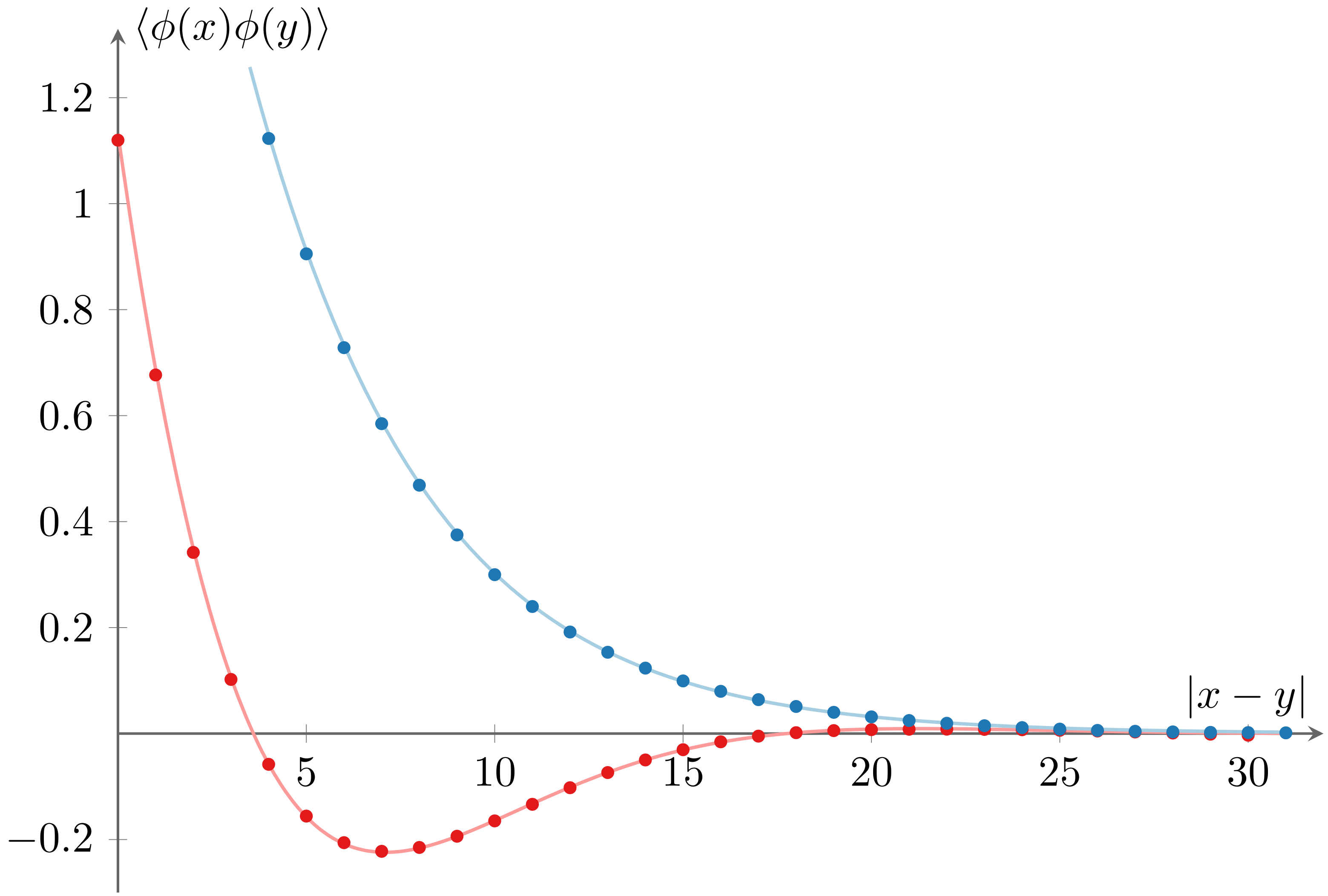
$$\tilde{S} = \frac{1}{2}\cosh\mu\,(\partial_4\phi)^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{1}{2}\pi_\mu^2 + \tilde{V}(\phi,\pi_\mu)$$

$$\tilde{V}(\phi,\pi_\mu) = \frac{1}{2m^2}\left(\cosh\mu\,(\partial_4\pi_4) + \nabla\cdot\vec{\pi} - \sinh\mu\left[\phi(x+\hat{e}_4) - \phi(x-\hat{e}_4)\right]\right)^2$$

Exactly solvable quadratic model (ICQ)

$$V(\phi,\chi) = m_\phi^2\phi^2/2 + m_\chi^2\chi^2/2 - ig\phi\chi$$

$$\tilde{V}(\phi,\partial\cdot\pi) = m_\phi^2\phi^2/2 + (\partial\cdot\pi - g\phi)^2/2m_\chi^2$$

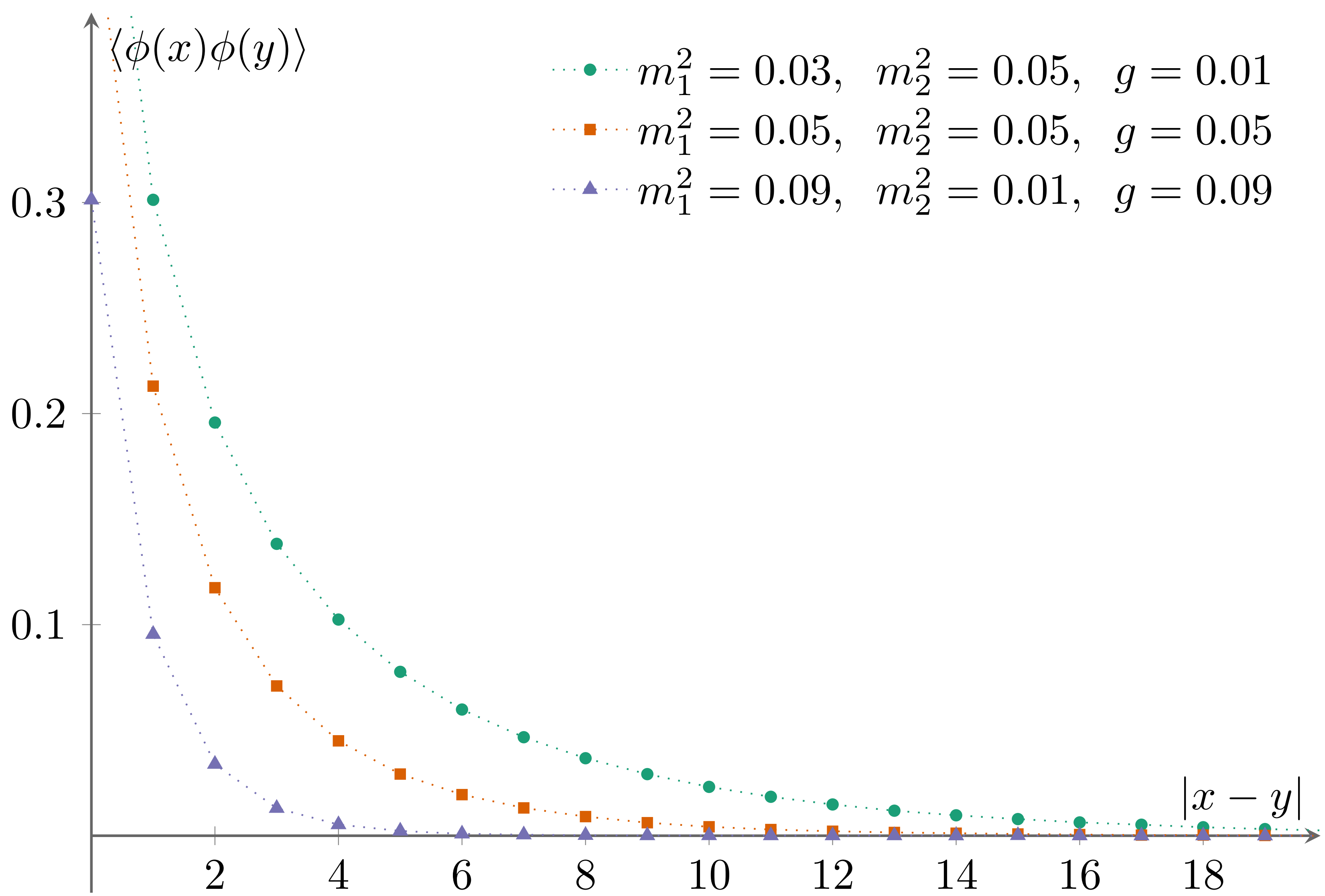


- Exactly solvable in any dimension
- PT symmetry implies masses are either both real OR form a conjugate pair
- Spectral positivity always broken
- Data points: 1d simulations, error bars negligible
- Lines: exact continuum solution

Imaginary Yukawa coupling (ICY)

$$V(\phi,\chi) = m_\phi^2\phi^2/2 + m_\chi^2\chi^2/2 - ig\chi\phi^2$$

$$\tilde{V}(\phi,\pi_\mu) = m_\phi^2\phi^2/2 + (\partial\cdot\pi - g\phi^2)^2/2m_\chi^2$$



- No sign of any complex mass pairs in d=1,2 or 3.
- This model goes smoothly into a phi fourth model in a scaled limit where g and m chi go to infinity.

Double Well with imaginary coupling (ICDW)

local & complex form:

$$S = \sum_x \left[\frac{1}{2}\left(\partial\phi_x\right)^2 + \frac{1}{2}\left(\partial\chi_x\right)^2 + \lambda(\phi^2 - v^2)^2 + m_\chi^2\chi^2/2 - ig\chi\phi\right]$$

local & real form:

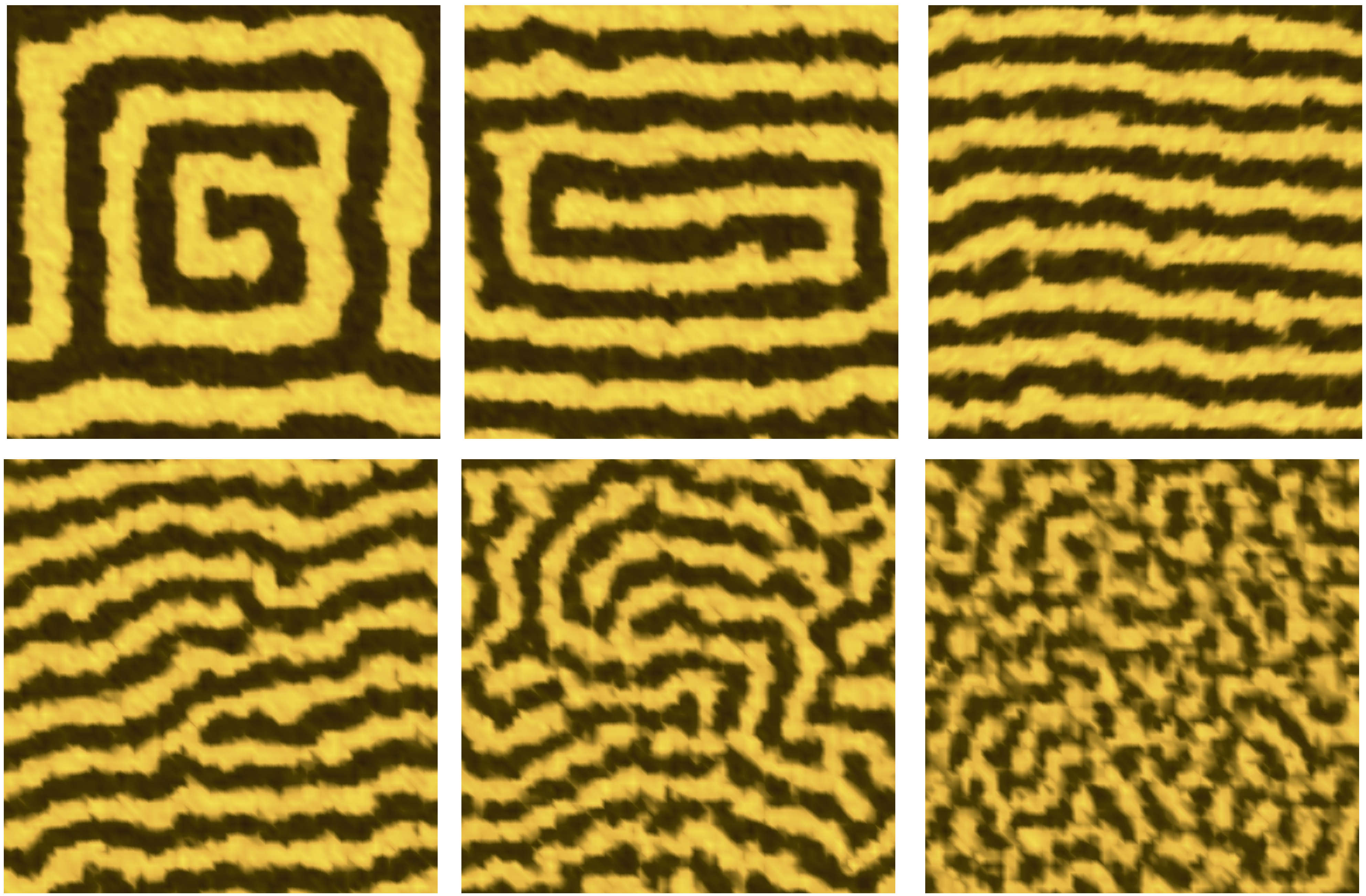
$$\tilde{S} = \sum_x \left[\frac{1}{2}\left(\partial\phi_x\right)^2 + \frac{1}{2}\pi_{x\mu}^2 + \frac{1}{2m^2}\left(\partial\cdot\pi - g\phi\right)^2 + \lambda(\phi^2 - v^2)^2\right]$$

nonlocal real form:

$$S_{eff} = \sum_x \left[\tfrac{1}{2}(\partial_\mu\phi(x))^2 + U(\phi)\right] + \frac{g^2}{2}\sum_{x,y}\phi(x)\Delta(x-y)\phi(x)$$

Configuration snapshots of ICDW model

$g = 0.9$ $g = 1.0$ $g = 1.1$



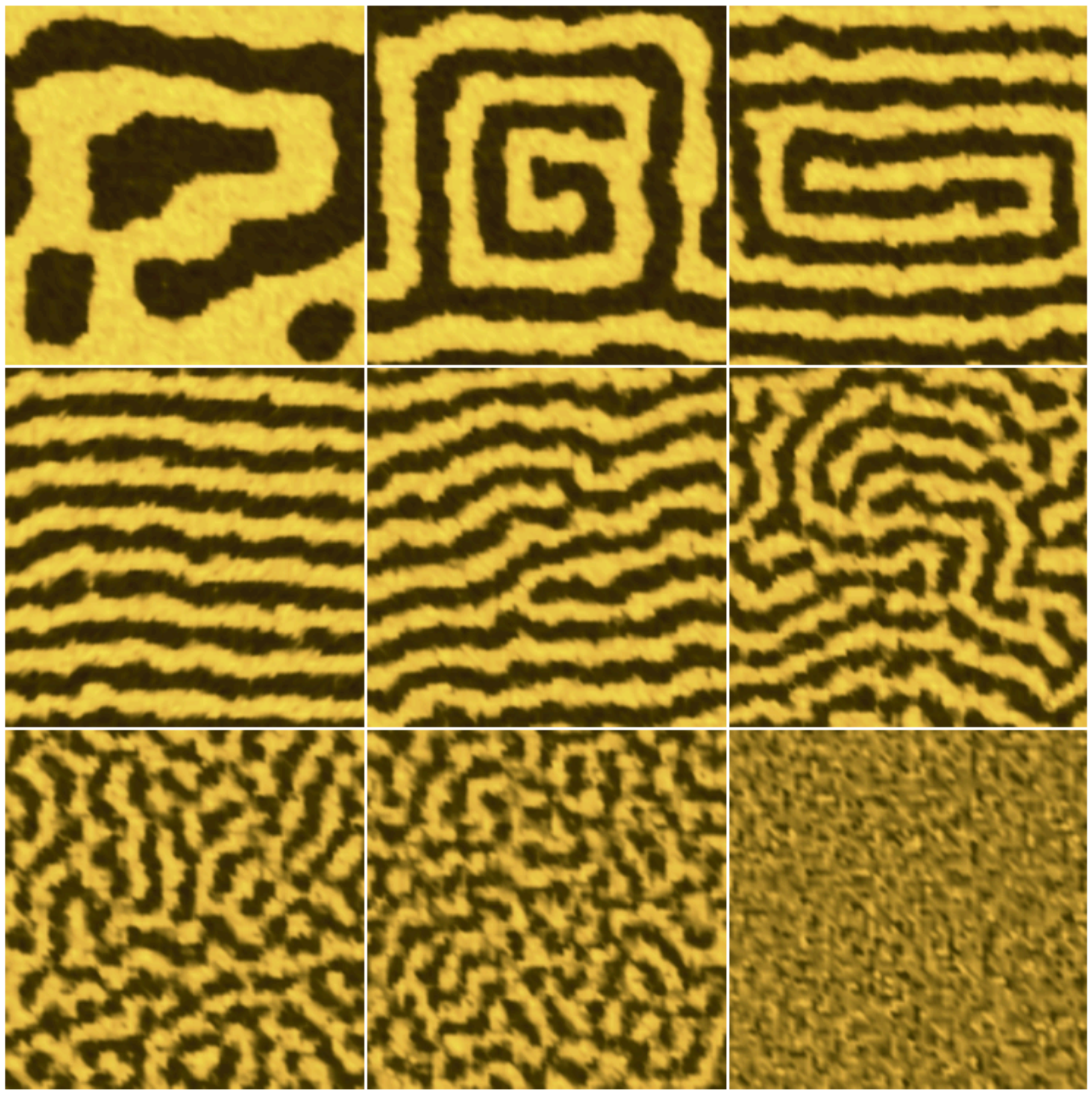
$g = 1.2$ $g = 1.3$ $g = 1.5$

$m_\chi^2 = 0.5, \lambda = 0.1 \,\&\, v = 3$

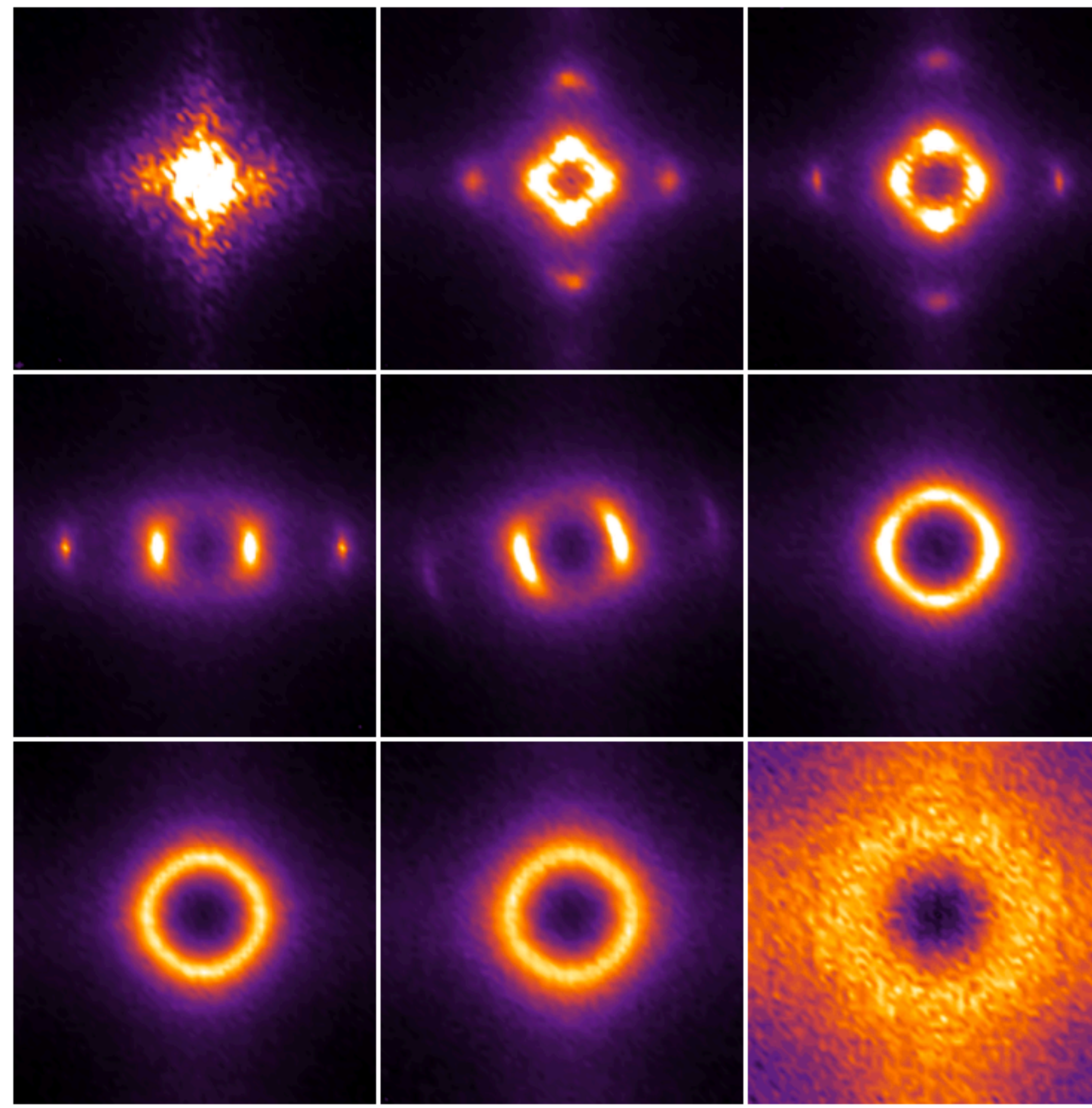
- Many different striping behaviors
- Rotational symmetry breaking effects obvious at smaller g
- Larger g, behavior is similar to that seen in spinodal decomposition
- These patterns change very slowly in the course of simulations, and this is physical: **slow modes**

ICDW & the Ring of Fire

Configurations

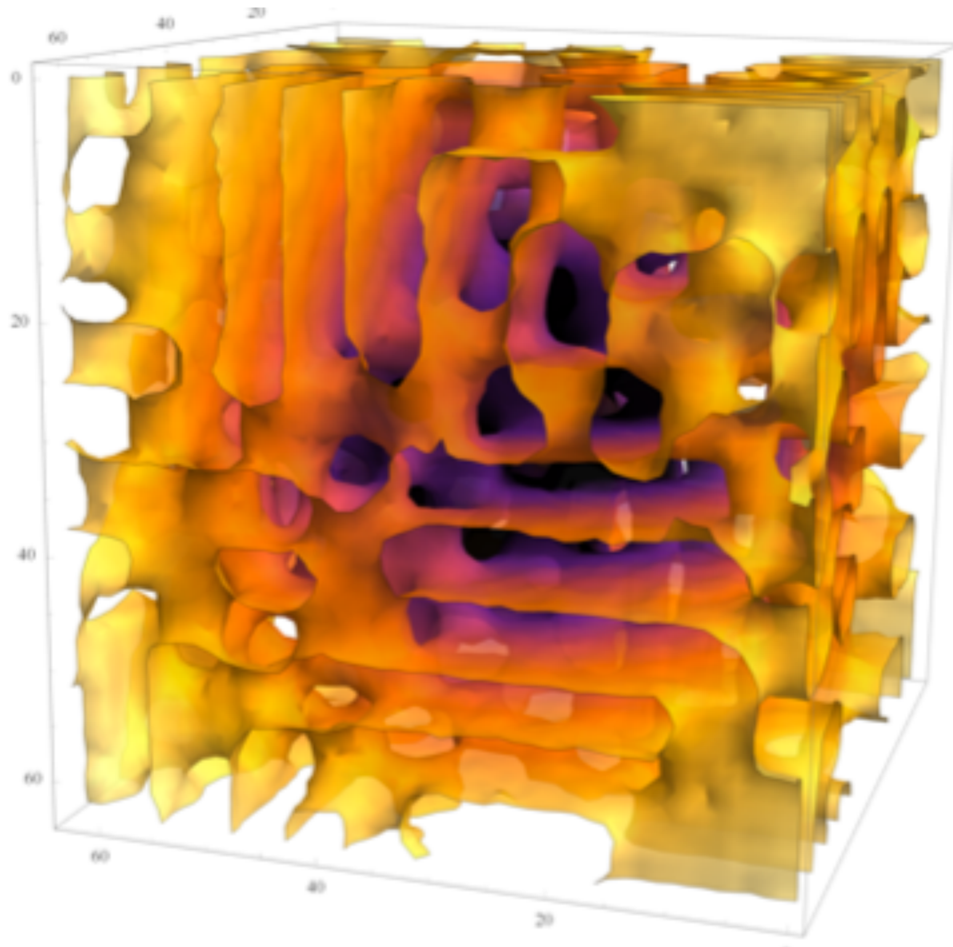


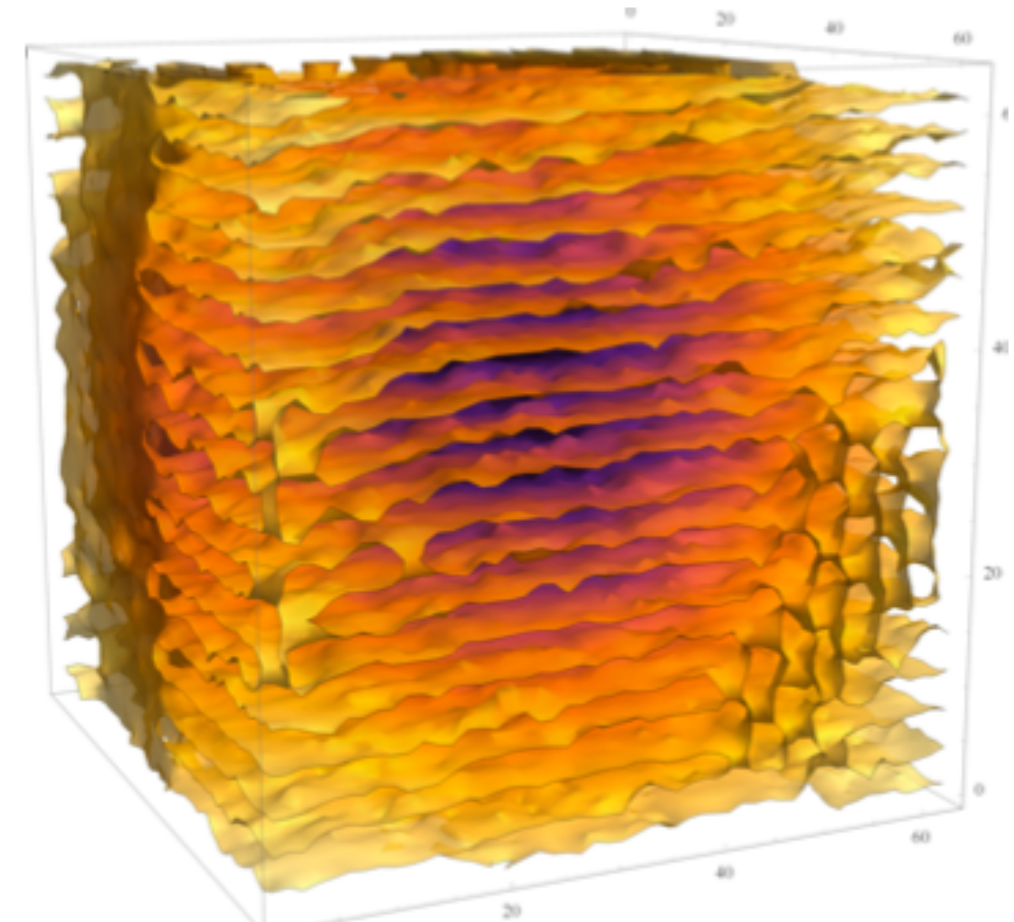
Averaged Fourier transform of configurations



Pattern formation due to a tachyonic instability!

3d Pattern formation





g=1.0 g=1.4

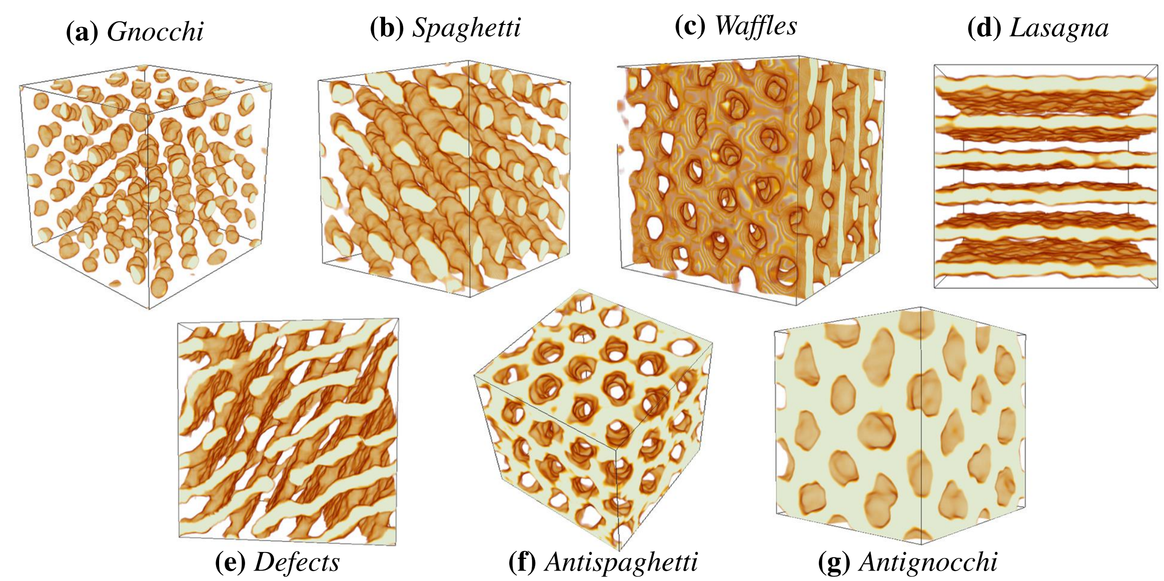
Configuration snapshots of ϕ in the three-dimensional ICDW model on a 64^3 lattice two values of g. The other parameters are $m_\chi^2 = 0.5$, $\lambda = 0.1$ and $v = 3$. The surfaces represent the domain walls between $\phi > 0$ and $\phi < 0$ regions. The color has no meaning and is meant only to guide the eye.

Connections

Nuclear pasta

- Predicted in inner crust of neutron stars
- Due to competition between nuclear and Coulomb forces.

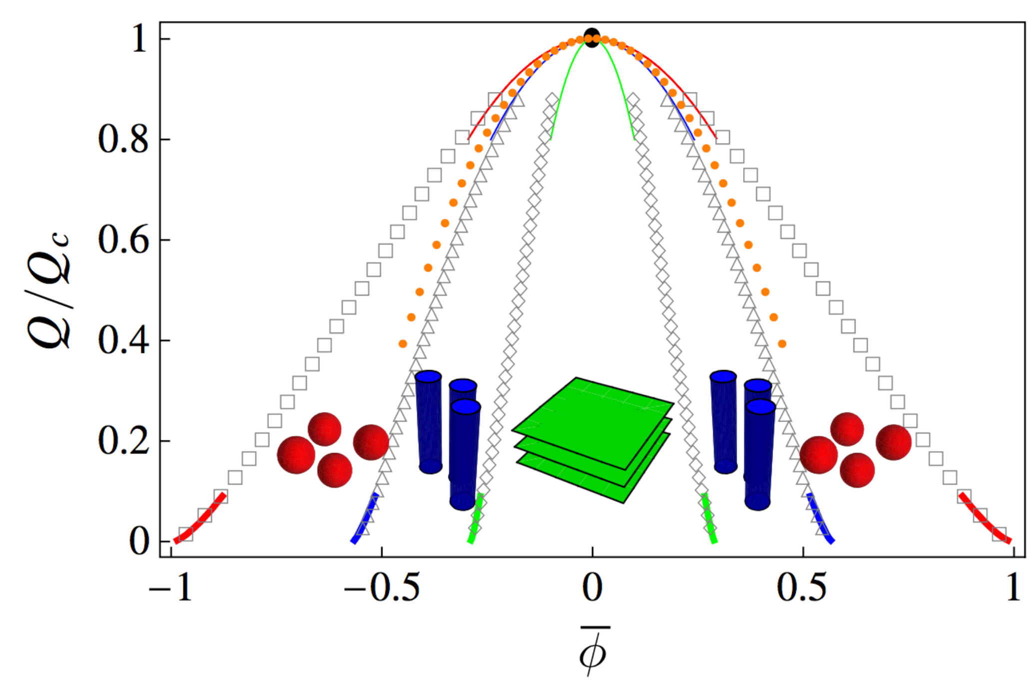
Ravenhall *et al.*, 1983; Hashimoto *et al.* 1984; Horowitz *et al.* 2004



Caplan and Horowitz 1606.03646

Other complex systems

Seul and Andelman, Science 1995



Ortix et al., PRL .100.246402 (2008)

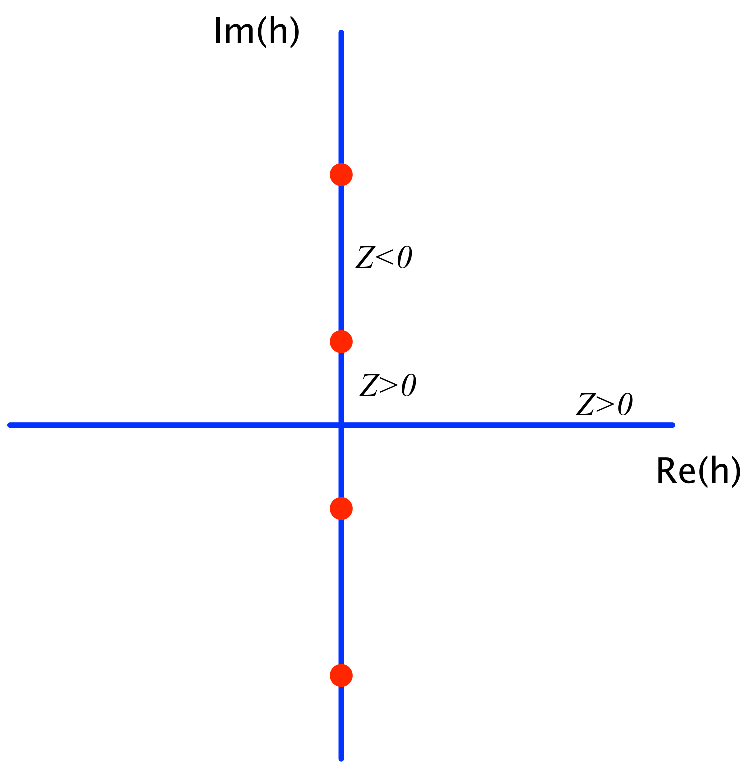
Common Feature: slow modes

The case of iϕ^3 transitions

- Models in the iϕ^3 universality class are problematic. The original case of the Ising model in an imaginary field is instructive: the partition function must vary in sign as Lee-Yang zeros are crossed.

- This behavior has a natural explanation using PT symmetry: zeros arise when the largest eigenvalues of the transfer matrix form a conjugate pair.

- See also: de Forcrand & Rindlisbacher, 1711.00042



$$Z = \sum_p \left(\lambda_p^N + \lambda_p^{*N} \right) + \sum_r \lambda_r^N$$

$$Z \approx \lambda_0^N + \lambda_0^{*N} = \left| \lambda_0 \right|^N 2\cos(N\phi_0)$$

Spin models: Z(3)

Generic spin model with complex spins

$$Z = Tr_z \exp\left[\bar{z}_x G^{-1}_{xy} z_y + h_{Rx} z_{Rx} + i h_{Ix} z_{Ix}\right] \qquad \tilde{G} \simeq J/\left(q^2 + m^2\right)$$

Real representation

$$\tilde{S} = \frac{1}{2J}\left(\partial\phi_x\right)^2 + \frac{J}{2}\pi^2_{x\mu} + \frac{m^2}{2J}\phi_x^2 + \frac{J}{2m^2}(\partial\cdot\pi_x)^2 + \sum_x \tilde{U}\left(\phi_x, \partial\cdot\pi_x\right)$$

$$\exp\left[-\tilde{U}\left(\phi_x, \tilde{\chi}_x\right)\right] = Tr_{z_x}\exp\left[\frac{J}{2m^2} z_{Ix}^2 + \left(h_{Rx} + \phi_x\right) z_{Rx} + i\left(h_{Ix} - \frac{J}{m^2}\tilde{\chi}\right) z_{Ix}\right]$$

Z(3): $$\exp\left[-\tilde{U}\left(\phi_x, \tilde{\chi}_x\right)\right] = 1 + \exp\left[\frac{3J}{4m^2} - \frac{3}{2}\left(h_{Rx} + \phi_x\right)\right] 2\cos\left[\frac{\sqrt{3}}{2}\left(h_{Ix} - \frac{J}{m^2}\tilde{\chi}\right)\right]$$

- This factor becomes negative when ϕ_x becomes sufficiently negative. This regions can be eliminated by a mild deformation of the model.
- Models that have $i\phi^3$ transitions somewhere in their parameter space will typically have negative contributions to Z which dominate when Z<0.
- For Z(3), the $i\phi^3$ transition is obtained when h_R becomes very negative.

Conclusions

- There is a simple, local algorithm that allows for simulation of a large class of scalar models with complex actions. This provides benchmark results for all proposed algorithms.The presence of an i phi cubed transition embedded within a particular model's parameter space is an obstacle to a successful resolution of a sign problem.

- Simulation indicates that pattern formation is an expected feature associated with models with sign problems and a phase transition. It can be driven by a tachyonic instability of the global translation-invariant phase. This is closely related to spinodal decomposition.

- Simulation of models with features in common with QCD at finite density indicate pattern formation around critical region.

- Pattern formation gives a new perspective on computational complexity but represents an opportunity to connect lattice field theory with other areas of physics.