Review: Lattice Muon $g-2$ HLbL Calculation

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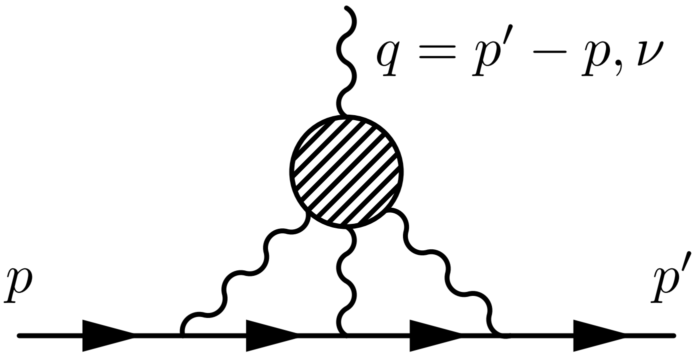
Outline

- Introduction
- $\mathrm{QED_L}$ (RBC-UKQCD)
  - Method
  - Results at $m_\pi = 139\,\mathrm{MeV}$, $a = 0.113\,\mathrm{fm}$, $L = 5.4\,\mathrm{fm}$
  - Continuum limit
  - Infinite volume limit
- Infinite volume QED
  - Mainz approach
  - RBC-UKQCD approach
  - Check between Mainz and RBC
  - Combine with pion pole contribution from $\pi^0 \to \gamma\gamma$ form factor
- Summary and outlook

	$a_\mu \times 10^{10}$	
QED incl. 5-loops	11658471.8853 ± 0.0036	Aoyama, et al, 2012
Weak incl. 2-loops	15.36 ± 0.10	Gnendiger et al, 2013
HVP LO	692.5 ± 2.7	RBC/UKQCD and FJ17 combined
HVP NLO	-9.93 ± 0.07	FJ17
HVP NNLO	1.22 ± 0.01	FJ17
Hadronic Light by Light	10.3 ± 2.9	FJ17
Standard Model	11659181.3 ± 4.0	
Experiment (0.54 ppm)	11659208.9 ± 6.3	E821, The $g-2$ Collab. 2006
Difference ($\mathrm{Exp} - \mathrm{SM}$)	27.6 ± 7.4	

Table 1. Standard model theory and experiment comparison

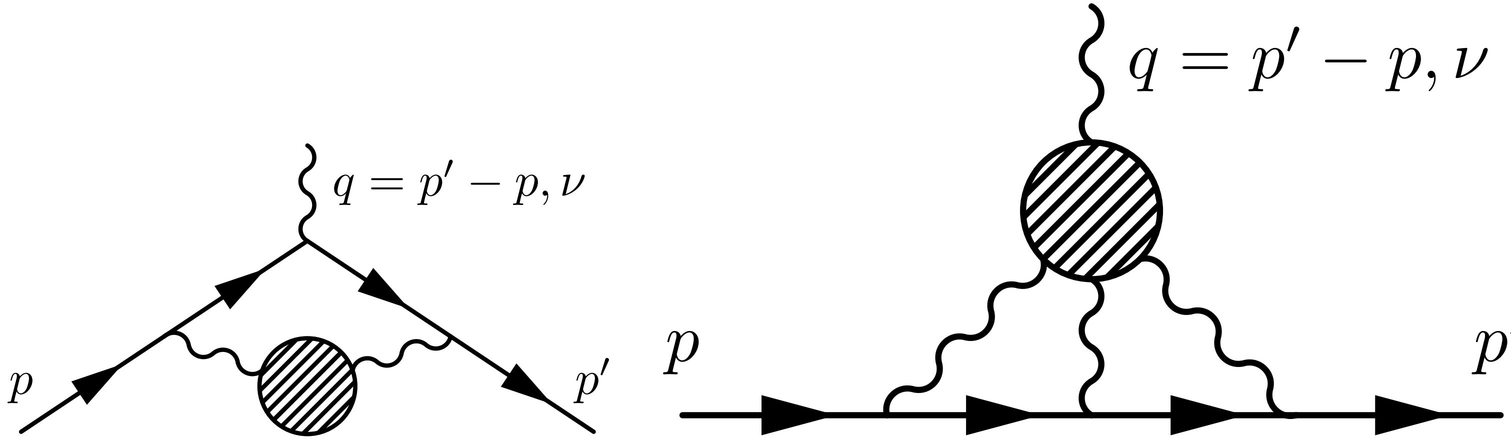
- HLbL is becoming the leading source of uncertainty in the theoretical prediction!
- To fully profit from future $g-2$ experimental results at Fermilab (E989) and J-PARC (E34) with four-fold improvement $\Rightarrow \delta a_\mu^{\mathrm{theory}} = 1.6 \times 10^{-10}$.



Various contributions to $a_\mu^{\mathrm{HLbL}} \times 10^{10}$

	PdRV09 (Glasgow consensus)	JN09	FJ17
π^0, η, η'	11.4 ± 1.3	9.9 ± 1.6	9.5 ± 1.2
π, K loops	-1.9 ± 1.9	-1.9 ± 1.3	-2.0 ± 0.5
axial-vector	1.5 ± 1.0	2.2 ± 0.5	0.8 ± 0.3
scalar	-0.7 ± 0.7	-0.7 ± 0.2	-0.6 ± 0.1
quark loops	0.2 (charm)	2.1 ± 0.3	2.2 ± 0.4
tensor	-	-	0.1 ± 0.0
NLO	-	-	0.3 ± 0.2
Total	10.5 ± 4.9 10.5 ± 2.6 (quadrature)	11.6 ± 3.9	10.3 ± 2.9

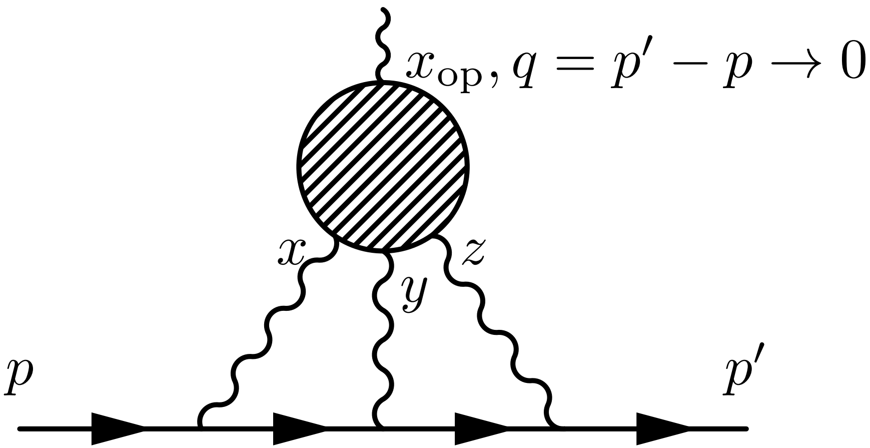
Hadronic contribution: HLbL



(L) HVP (R) HLbL

HLbL: hadronic light-by-light scattering.

- Combinations of models for different processes: 10.3 ± 2.9 FJ17, arXiv:1705.00263.
- Dispersive approach.
- **Lattice approach**.



$a_\mu^{\rm HLbL} = F_2(0) =$ some factors

$\otimes$

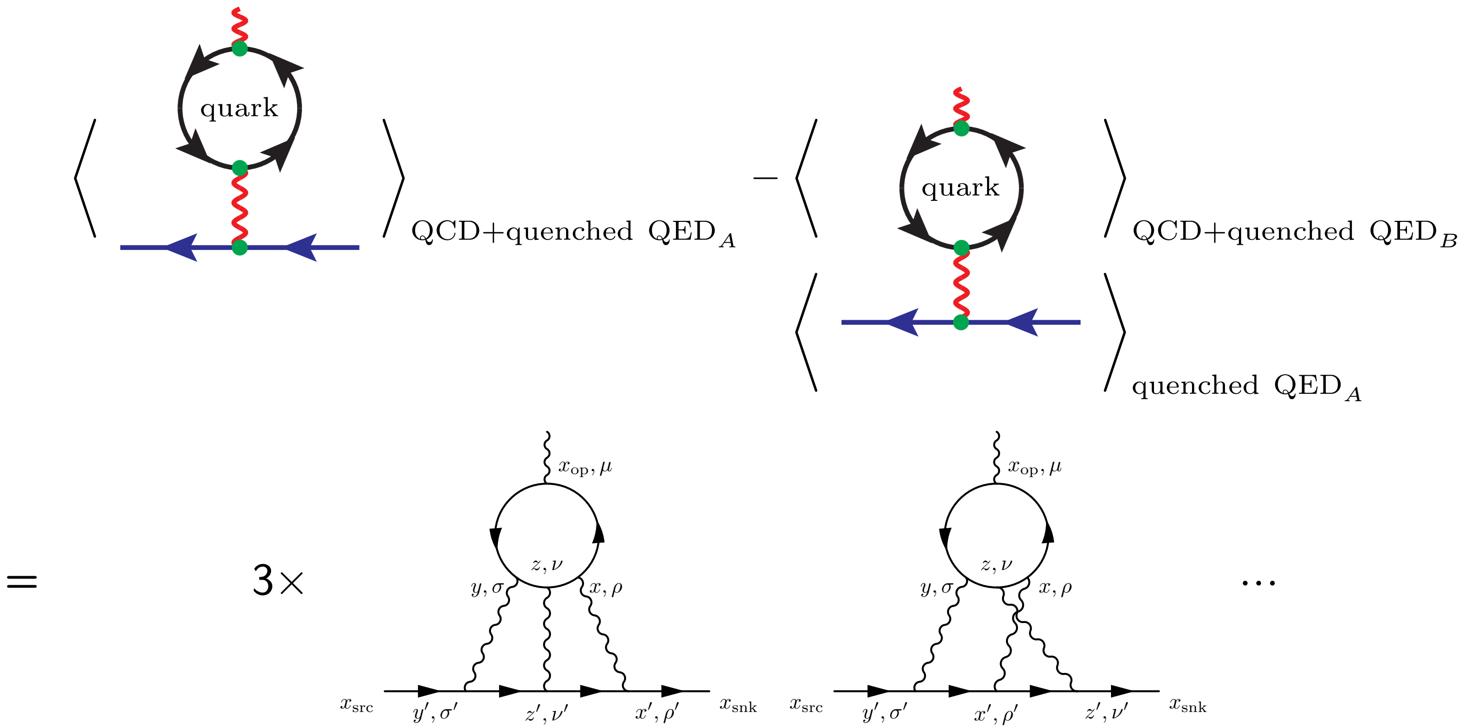
Hadronic 4-point function ($\mathcal{H}$)

$\otimes$

Muon line function with photons ($\mathcal{M}$)

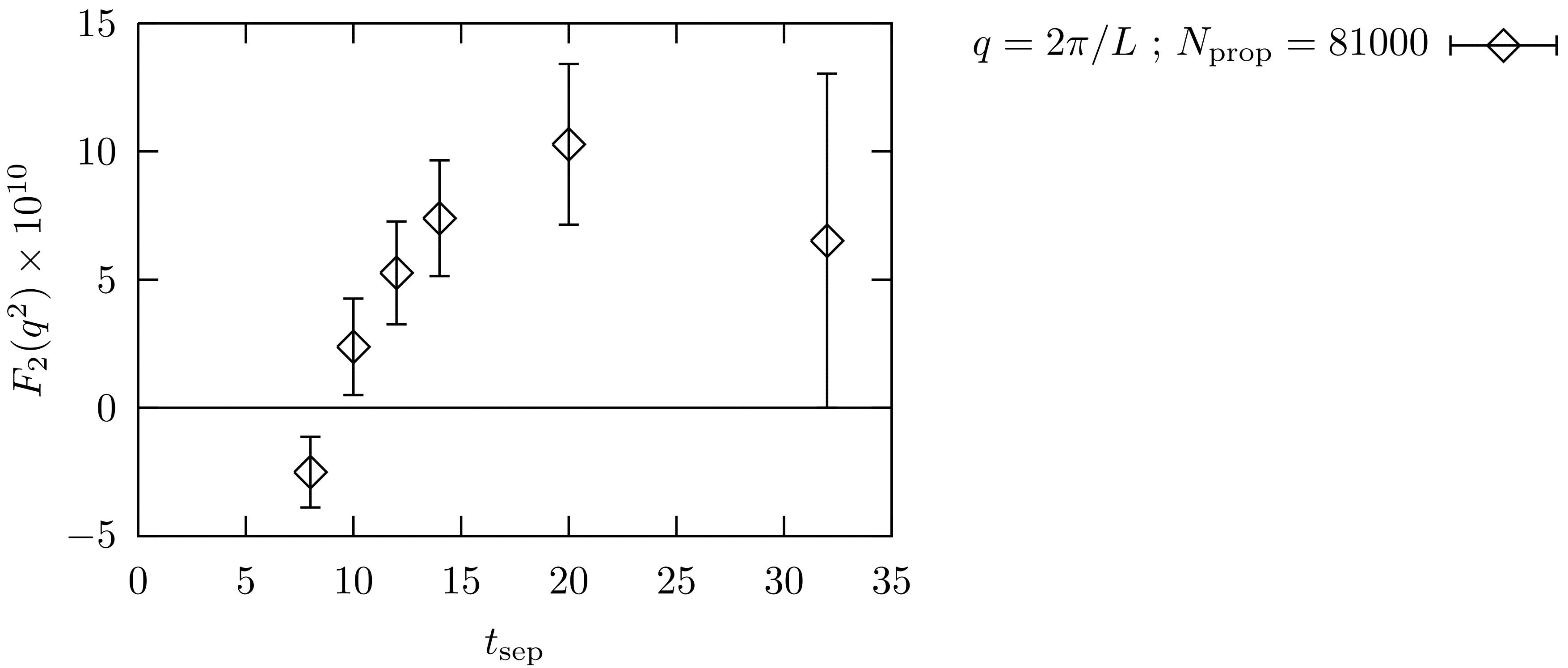
Group :	**RBC-UKQCD**		Mainz
	↓	↘	↓
$\mathcal{M}$:	**$\mathbf{QED_L}$**		Infinite volume

- Introduced in LATTICE 2005.
- PoS LAT2005 (2006) 353. hep-lat/0509016.
- T. Blum, S. Chowdhury, M. Hayakawa, T. Izubuchi.



- Evalutate the quark and muon propagators in the background quenched QED fields, generating all kinds of diagrams.

- Ten years after the method is proposed.
- Phys.Rev.Lett. 114 (2015) 1, 012001. arXiv:1407.2923.
- T. Blum, S. Chowdhury, M. Hayakawa, T. Izubuchi.

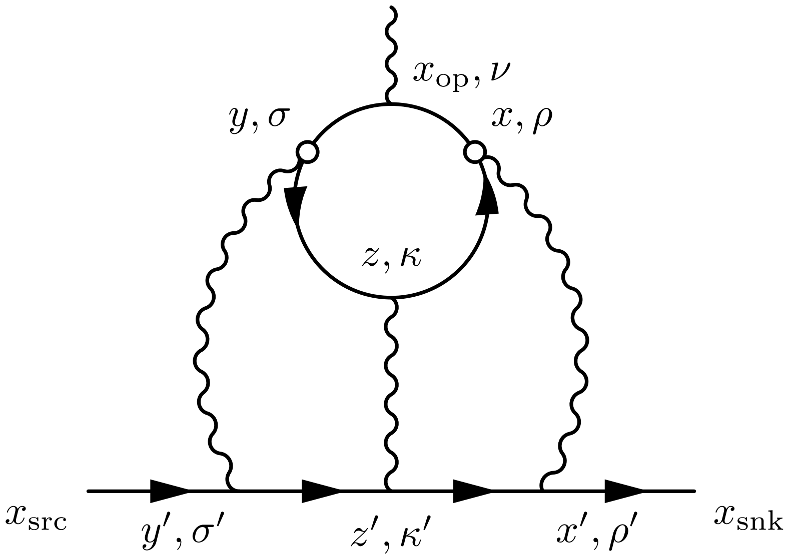


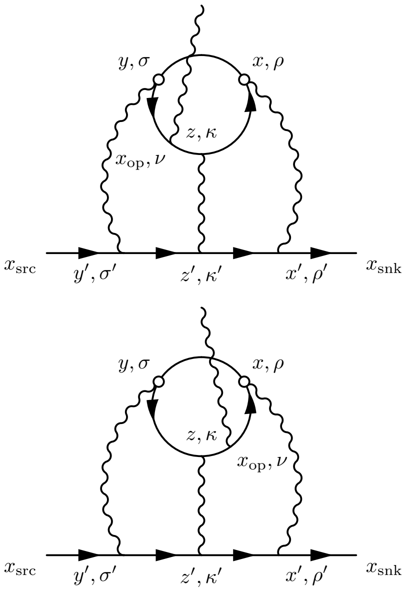
- RBC/UKQCD $24^3 \times 64$ DWF, with $a^{-1} = 1.785\,\text{GeV}$, $m_\pi = 342\,\text{MeV}$. $m_\mu = 178.5\,\text{MeV}$.
- Only **connected diagrams** is calculated.

Improving the method (RBC-UKQCD - QED_L)

- More people joined the HLbL project.
- Phys.Rev. D93 (2016) no.1, 014503.
- T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Lehner.

1. Explicitly calculated all the photon propagators.
2. Use **two point sources** to evaluate the hadronic 4-point function. All photon propagator become exact. **Importance sampling** is used to perform the 4-D summation.
3. Introduce the "**moment method**" to directly calculate the $q \to 0$ limit.





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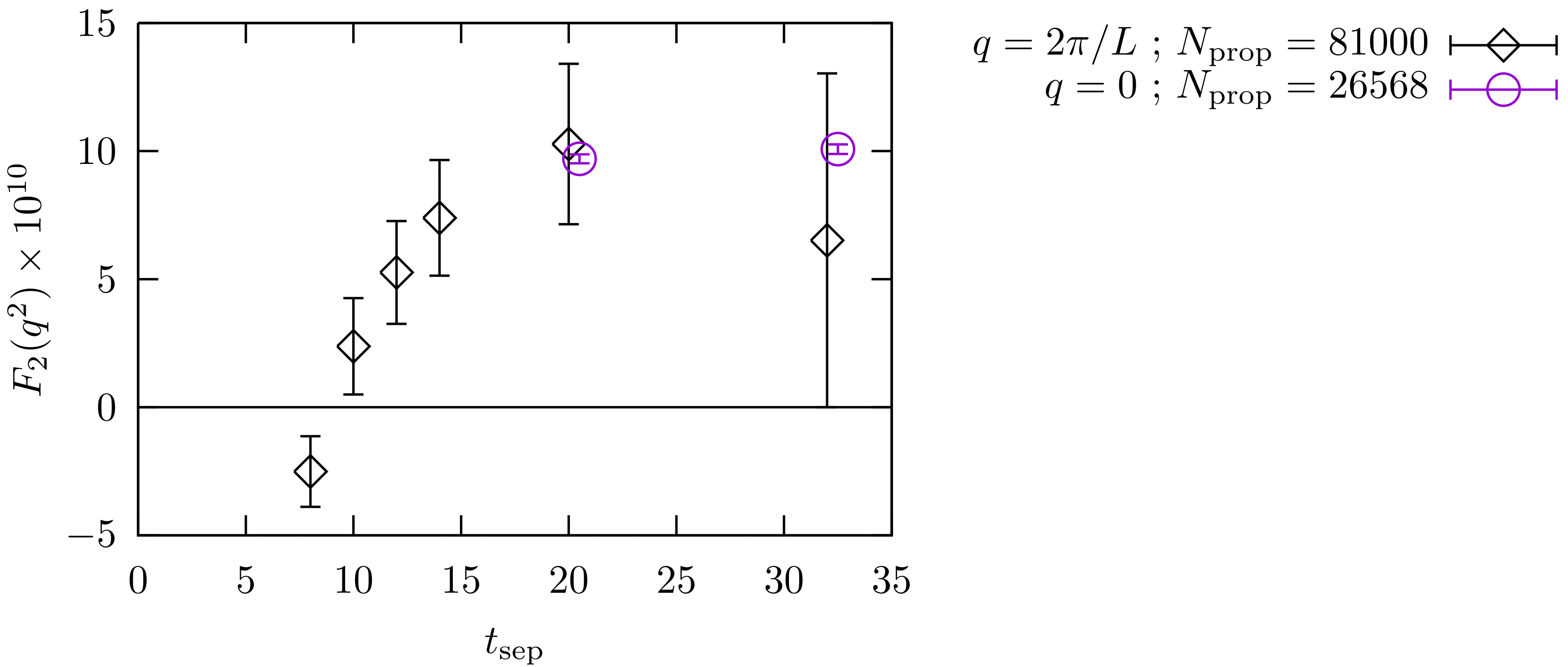
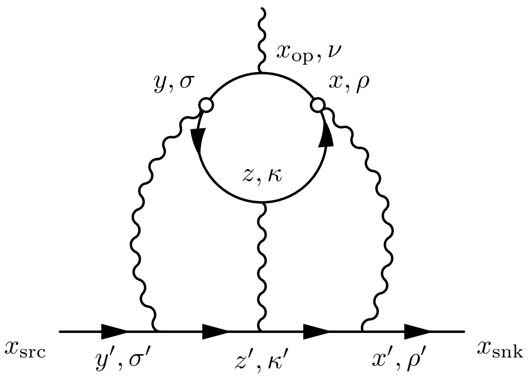
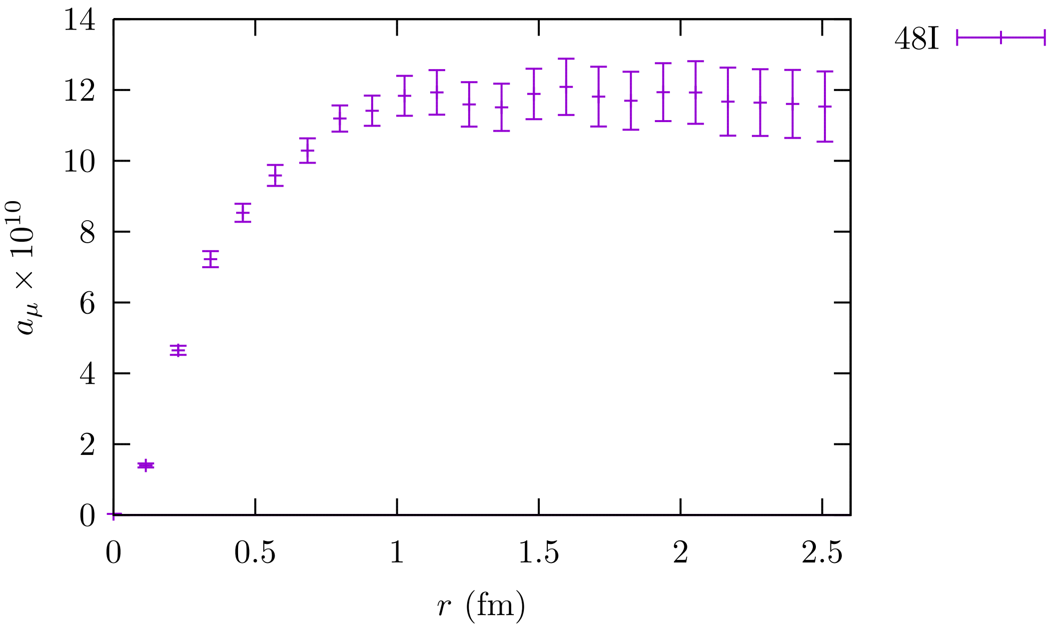


- RBC/UKQCD $24^3 \times 64$ DWF, with $a^{-1} = 1.785\,\mathrm{GeV}$, $m_\pi = 342\,\mathrm{MeV}$. $m_\mu = 178.5\,\mathrm{MeV}$.
- Only **connected diagrams** is calculated.
- Statistical error significantly reduced with less cost. Error do not increase with increasing t_{sep}. In the future, we will always compute with $t_{\mathrm{sep}} = T/2$.

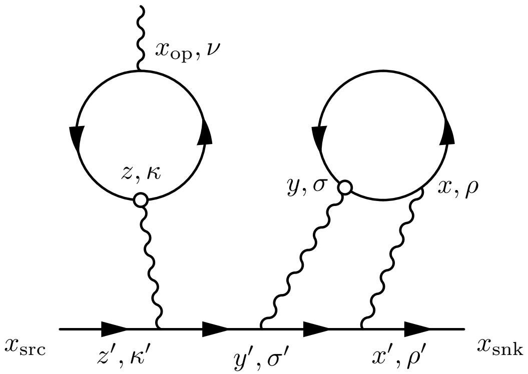
- Phys.Rev.Lett. 118 (2017) no.2, 022005
- T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, C. Lehner.
- Connected diagrams: $a_\mu^{\mathrm{cHLbL}} = (11.60 \pm 0.96) \times 10^{-10}$.
- Additional info available: the separation between the two point sources $r = |x - y|$.



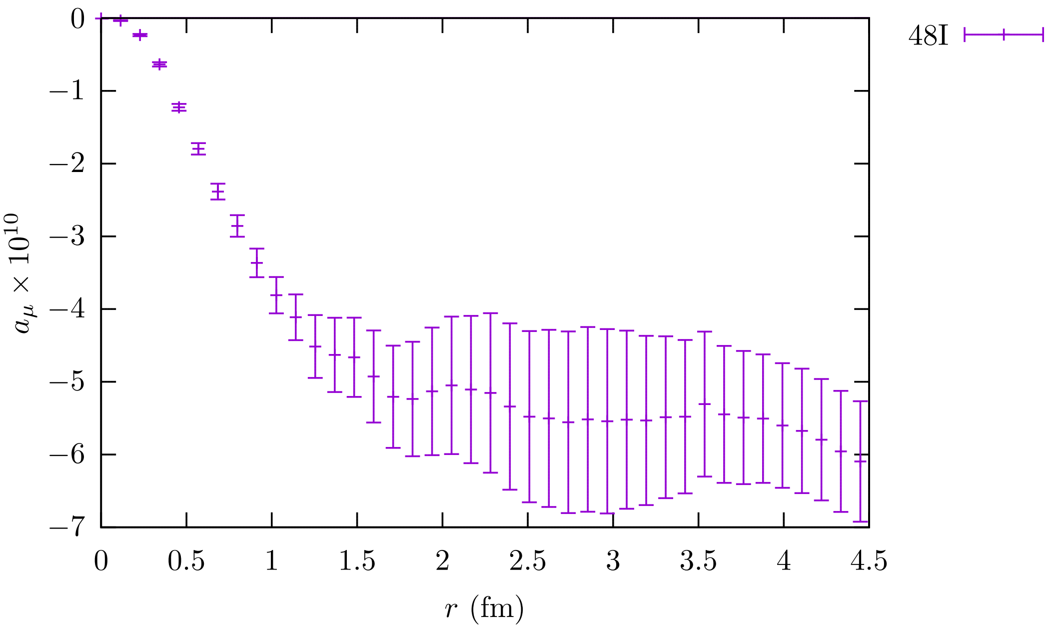


- RBC/UKQCD $48^3 \times 96$ DWF, with $a^{-1} = 1.73\,\mathrm{GeV}$, $m_\pi = 139\,\mathrm{MeV}$, $m_\mu = 106\,\mathrm{MeV}$.

- Phys.Rev.Lett. 118 (2017) no.2, 022005
- T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, C. Lehner.
- Disconnected diagrams: $a_\mu^{\mathrm{dHLbL}} = (-6.25 \pm 0.80) \times 10^{-10}$.
- Why negative? See: JHEP 1609 (2016) 113, PoS LATTICE2016 (2016) 181.



$M = 1024$ propagators/config
M^2 trick to reduce noise.



- $48^3 \times 96$ lattice, with $a^{-1} = 1.73\,\mathrm{GeV}$, $L = 5.47\,\mathrm{fm}$, $m_\pi = 139\,\mathrm{MeV}$, $m_\mu = 106\,\mathrm{MeV}$.
- Total: $a_\mu^{\mathrm{HLbL}} = (5.35 \pm 1.35) \times 10^{-10}$, at fixed $a = 0.114\,\mathrm{fm}$ and fixed $L = 5.47\,\mathrm{fm}$.

Summarize 48I (RBC-UKQCD - QED_L)

- Phys.Rev.Lett. 118 (2017) no.2, 022005
- T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, C. Lehner.
- Connected diagrams: $a_\mu^{\text{cHLbL}} = (11.60 \pm 0.96) \times 10^{-10}$.
- Disconnected diagrams: $a_\mu^{\text{dHLbL}} = (-6.25 \pm 0.80) \times 10^{-10}$.
- Total: $a_\mu^{\text{HLbL}} = (5.35 \pm 1.35) \times 10^{-10}$.
- 48I at physical pion mass, but fixed $a = 0.114\,\text{fm}$, $L = 5.5\,\text{fm}$.

Next:

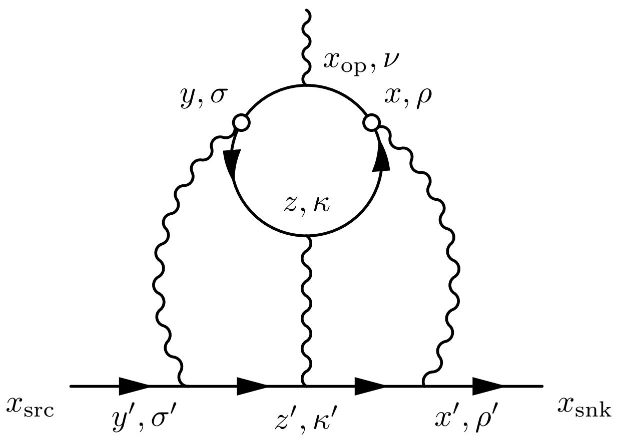
- Continuum limit.
- Infinite volume limit

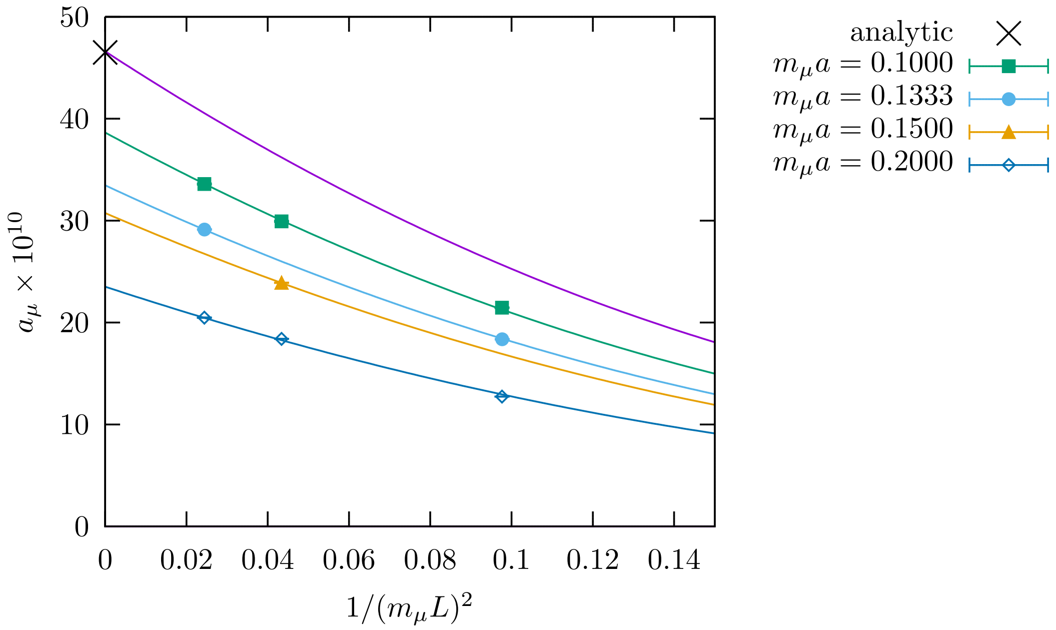
Study these two effects in a simpler setting: muon leptonic LbL.

(replace the quark loop by a muon loop)

- Phys.Rev. D93 (2016) no.1, 014503.
- T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Lehner.
- Three Ls: $6.0\,\mathrm{fm}$, $9.0\,\mathrm{fm}$, $12.0\,\mathrm{fm}$, each with three lattice spacings.

$$F_2(a,L) = F_2\left(1 - \frac{c_1}{(m_\mu L)^2} + \frac{c_1'}{(m_\mu L)^4}\right)(1 - c_2 a^2 + c_2' a^4) \;\rightarrow\; F_2 = 46.6(2)\times 10^{-10} \quad (1)$$

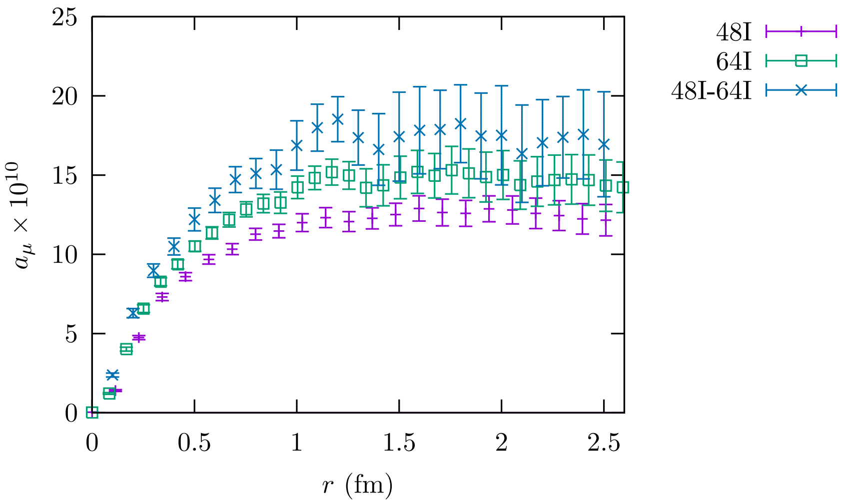




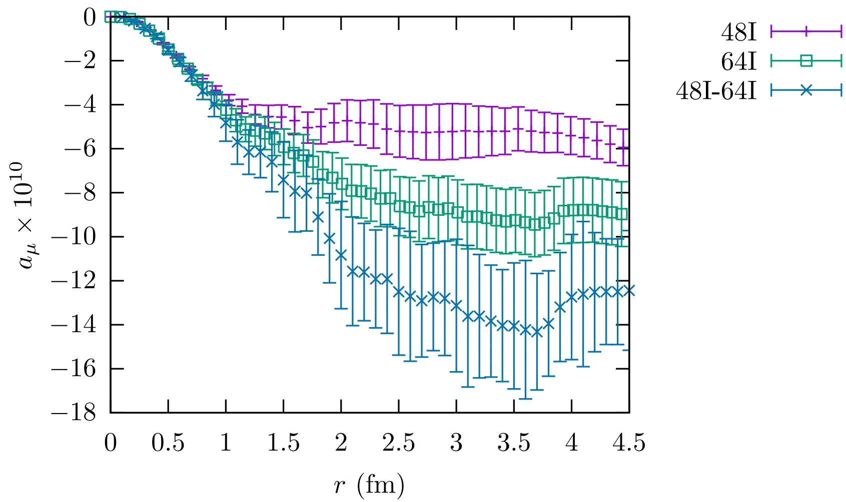
- Analytic results: $0.371 \times (\alpha/\pi)^3 = 46.5 \times 10^{-10}$.

Continuum limit (RBC-UKQCD - $\mathrm{QED_L}$ - preliminary)

- $48^3 \times 96$ lattice, $a^{-1} = 1.73\,\mathrm{GeV}$, $L = 5.47\,\mathrm{fm}$, $m_\pi = 135\,\mathrm{MeV}$ (corrected from $139\,\mathrm{MeV}$).
- $64^3 \times 128$ lattice, $a^{-1} = 2.36\,\mathrm{GeV}$, $L = 5.34\,\mathrm{fm}$, $m_\pi = 135\,\mathrm{MeV}$.



Connected diagrams



Disconnected diagrams

- Performing the continuum extrapolation assuming a^2 scaling, plotted with label "48I-64I".

$$a_\mu^{\mathrm{cHLbL}} = (16.9 \pm 3.8) \times 10^{-10} \tag{2}$$
$$a_\mu^{\mathrm{dHLbL}} = (-12.3 \pm 3.4) \times 10^{-10} \tag{3}$$
$$a_\mu^{\mathrm{HLbL}} = (4.6 \pm 5.0) \times 10^{-10} \tag{4}$$

- Very large statistical error (48I: $(5.35 \pm 1.35) \times 10^{-10}$). Can we do better than this?

Hybrid continuum limit (RBC-UKQCD - QED_L - preliminary)

- Split the a_μ into two parts:

$$a_\mu = a_\mu^{\text{short}} + a_\mu^{\text{long}} \tag{5}$$

- $a_\mu^{\text{short}} = a_\mu(r \leqslant 1\,\text{fm})$: most of the contribution, small statistical error.
- $a_\mu^{\text{long}} = a_\mu(r > 1\,\text{fm})$: small contribution, large statistical error.

Treat a_μ^{short} and a_μ^{long} separately:

- a_μ^{short}: just like before, continuum extrapolation assuming a^2 scaling.
- a_μ^{long}: average to minimize the statistical error, estimate the $\mathcal{O}(a^2)$ error.

$$a_\mu^{\text{cHLbL}} = (16.9 \pm 3.8) \times 10^{-10} \Rightarrow (17.3 \pm 1.7_{\text{stat}} \pm 0.2_{\text{sys}}) \times 10^{-10} \tag{6}$$
$$a_\mu^{\text{dHLbL}} = (-12.3 \pm 3.4) \times 10^{-10} \Rightarrow (-8.9 \pm 1.1_{\text{stat}} \pm 1.0_{\text{sys}}) \times 10^{-10} \tag{7}$$
$$a_\mu^{\text{HLbL}} = (4.6 \pm 5.0) \times 10^{-10} \Rightarrow (8.4 \pm 2.0_{\text{stat}} \pm 1.2_{\text{sys}}) \times 10^{-10} \tag{8}$$

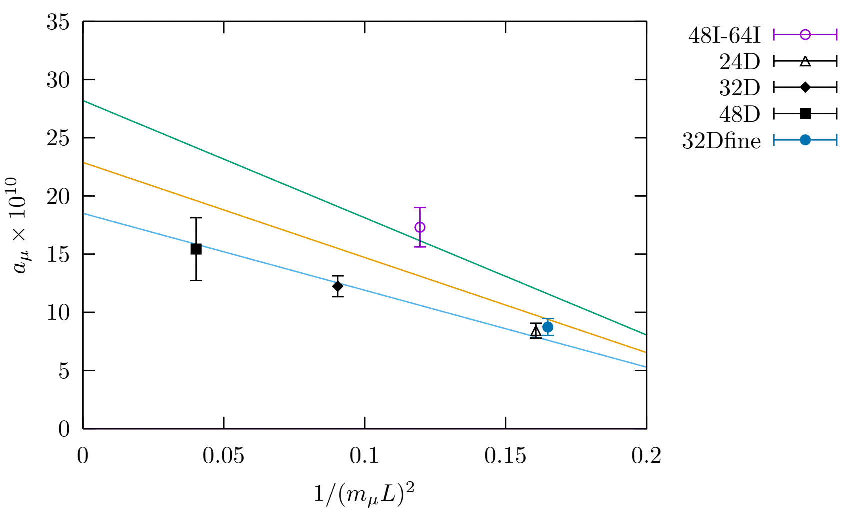
- The above systematic error is the estimated $\mathcal{O}(a^2)$ error.
- In addition, we estimate there could be additional 10% $\mathcal{O}(a^4)$ error:

$\Rightarrow a_\mu^{\text{HLbL}} = (8.4 \pm 2.0_{\text{stat}} \pm 1.2_{\text{sys},a^2} \pm 0.8_{\text{sys},a^4})$.

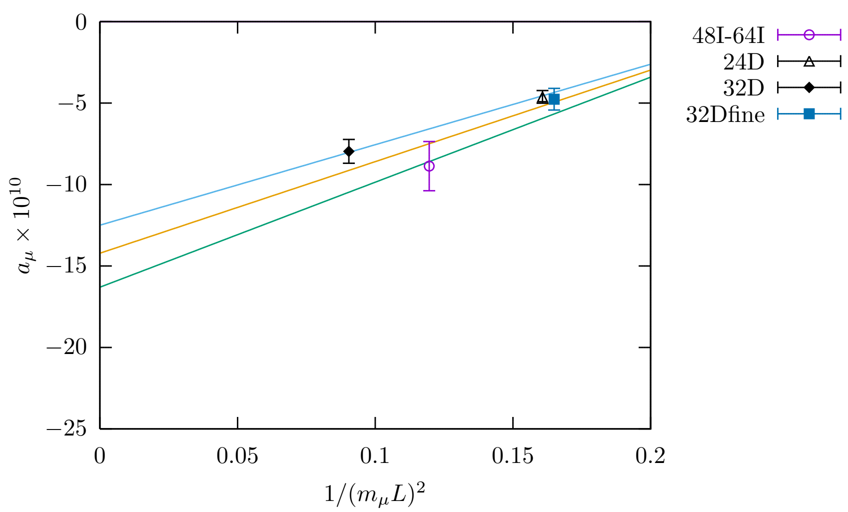
Infinite volume limit (RBC-UKQCD - QED_L - preliminary)

- MDWF+Iwasaki: continuum limit ($5.4\,\text{fm}$)
- MDWF+DSDR: $a^{-1} = 1.015\,\text{GeV}$: $24^3 \times 64$ ($4.8\,\text{fm}$), $32^3 \times 64$ ($6.4\,\text{fm}$), $48^3 \times 64$ ($9.6\,\text{fm}$).
- MDWF+DSDR: $a^{-1} = 1.371\,\text{GeV}$: $32^3 \times 64$ ($4.6\,\text{fm}$).

$$F_2(a, L) \;=\; F_2\left(1 - \frac{c_1}{(m_\mu L)^2}\right)(1 - c_2\, a^2) \tag{9}$$



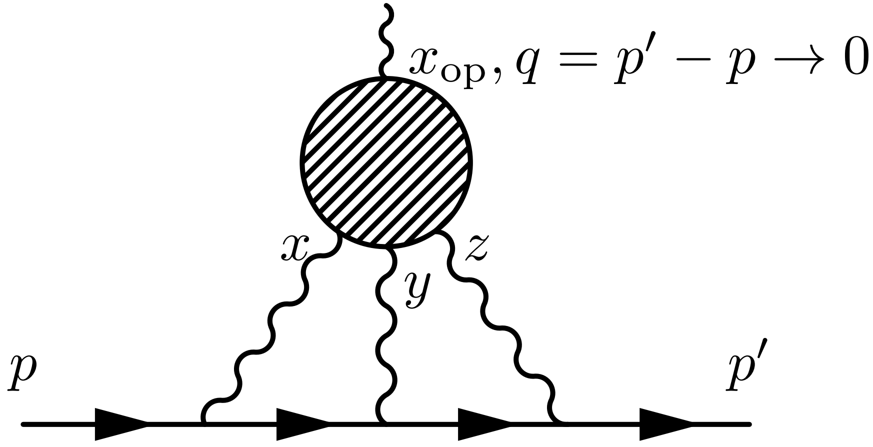
Connected diagrams



Disconnected diagrams

$$a_\mu^{\text{cHLbL}} = (28.2 \pm 4.0) \times 10^{-10} \qquad a_\mu^{\text{dHLbL}} = (-16.3 \pm 3.4) \times 10^{-10} \tag{10}$$

$$a_\mu^{\text{HLbL}} \;=\; (11.9 \pm 5.3) \times 10^{-10} \tag{11}$$



$a_\mu^{\mathrm{HLbL}} = F_2(0) =$ some factors

$\otimes$

Hadronic 4-point function ($\mathcal{H}$)

$\otimes$

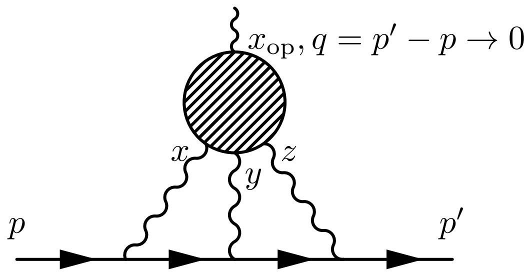
Muon line function with photons ($\mathcal{M}$)

Group	:	**RBC-UKQCD**	**Mainz**
		↓ ↘	↓
$\mathcal{M}$	:	$\mathrm{QED_L}$	**Infinite volume**

- So far only connected diagrams are calculated.

Methodology (Mainz - Infinite volume)

- Developed **independently** from **RBC**.
- Asmussen, Gérardin, Green, Meyer, Nyffeler (2015 - 2017).
- Ideas by Harvey Meyer after Muon $g-2$ workshop at Mainz in April 2014.



$a_\mu^{\rm HLbL} = F_2(0) =$ some factors

$\otimes \mathcal{H}$ (lattice regularization, using **two point sources** y and z)

New: sequential propagators not needed if certain permutations of point sources included.

$\otimes \mathcal{M}$ (**semi-analytically** in **continuum** and **infinite volume**)

pre-computed on a grid, saved to disk, once and for all

No power law effects $1/L^2$ in the volume

Lorentz covariance formulation:

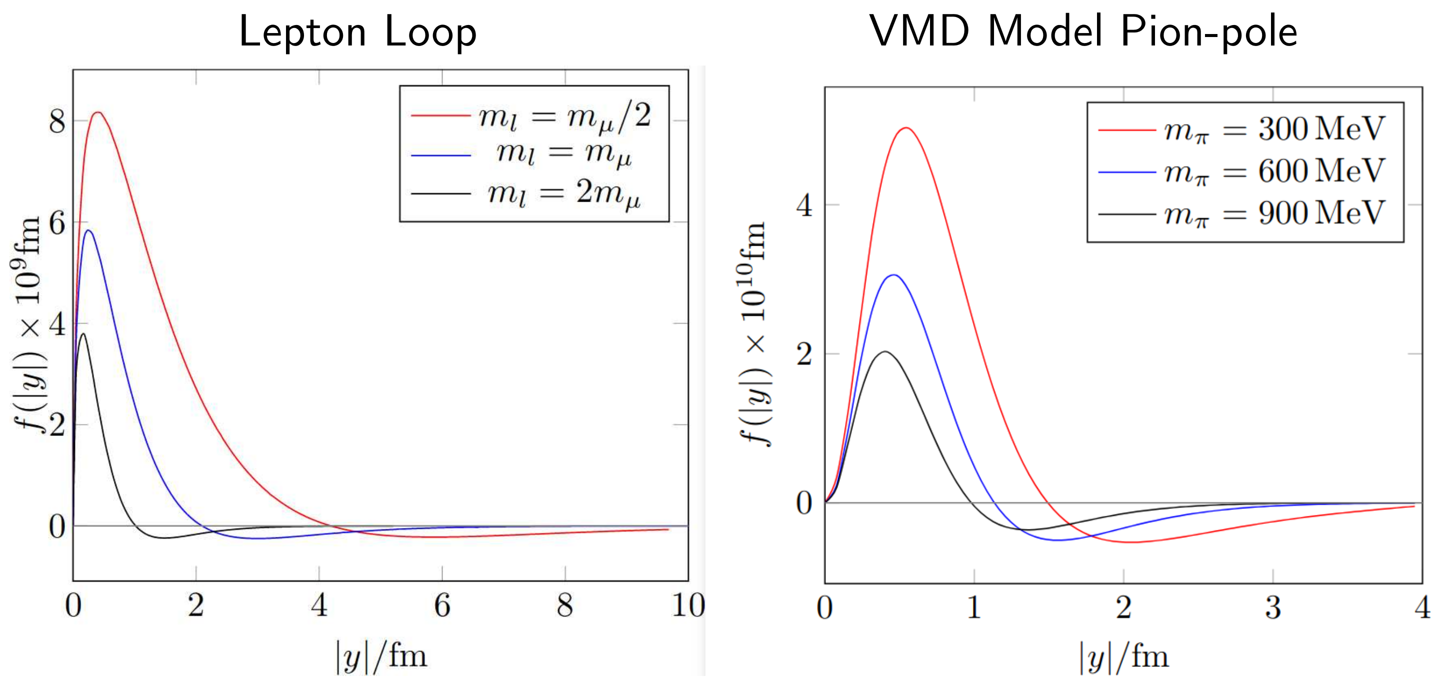
- The number of variables $\mathcal{M}$ depends on reduced from 5 to 3.
- The 4-D summation of the $y-z$ reduced to 1-D integral of $|y|$ (take $z=0$).
- Choose $y=(i,i,i,i)$ to reduce finite volume and discretization effects.

Mainz's notation is slightly different from RBC. In this talk, we try to use consistant notation when possible (mostly RBC convention).

Pion-pole and lepton loop (Mainz - Infinite volume)

EPJ Web Conf. 179 (2018) 01017.

Asmussen, Gérardin, Green, Meyer, Nyffeler.



- After performing the integration for $|y|$, the results agree with analytic results obtained in momentum space. $\Rightarrow$ QED kernel $\mathcal{M}$ successful.
- QED kernel will be publicly available.

Subtracted QED kernel (RBC-UKQCD - infinite volume)

- Phys.Rev. D96 (2017) no.3, 034515
- T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, C. Lehner.
- Motivated by Mainz's success, RBC also start the infinite volume QED kernel project.
- Major difference: muon only propagate in the time direction of the lattice.
- Find a way to reduce discretization effects and finite volume effects:

$$\mathcal{M}^{(\mathrm{sub})}(x,y,z) \;=\; \mathcal{M}(x,y,z) - \mathcal{M}(y,y,z) - \mathcal{M}(x,y,y) \tag{12}$$

Notice that **conserved current is a total derivative**: $J_\mu(x) = \partial_\nu^{(x)}(x_\mu J_\nu)$. Therefore:

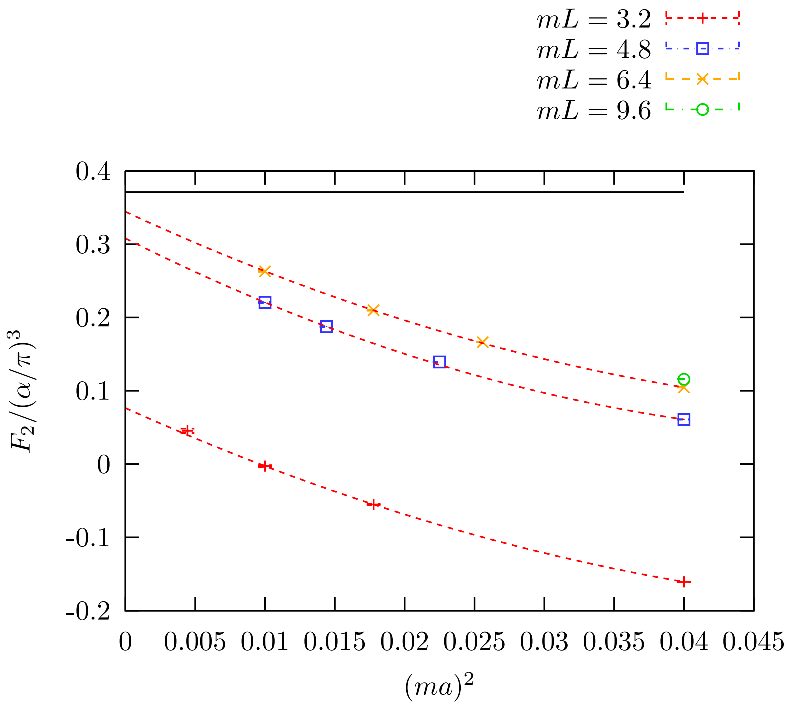
$$\int d^4x \, \mathcal{H}(x,y,z,x_{\mathrm{op}}) \;=\; 0 \qquad \text{(in infinite volume and in continuum)} \tag{13}$$

- The **subtraction terms vanishes in infinite volume and continuum limit**.
- At non-zero lattice spacing or finite volume:

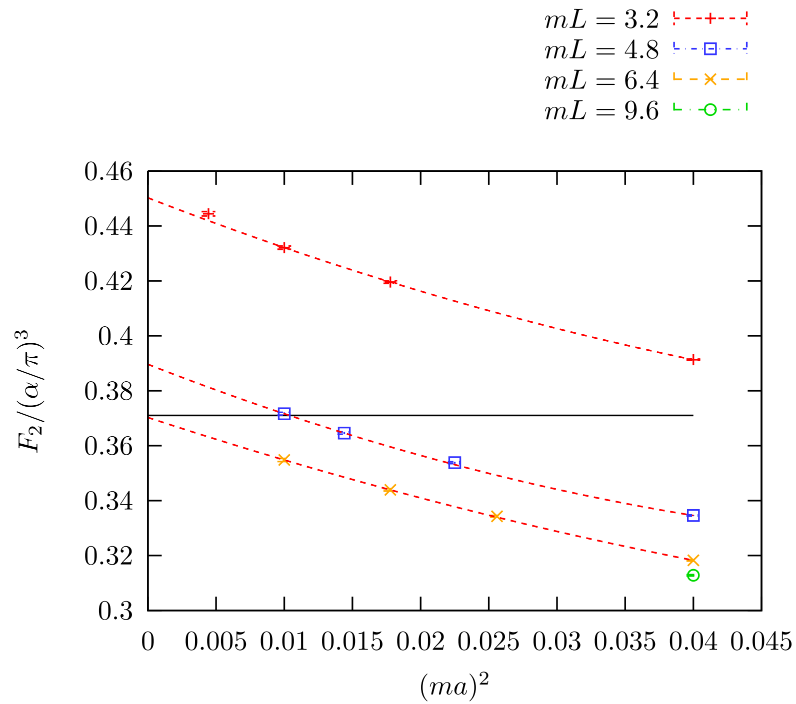
  proper choice of subtraction term can reduce the two systematics.
- The **integrand will be completely different** with the subtracted QED kernel.

Subtracted QED kernel (RBC-UKQCD - infinite volume)

- Phys.Rev. D96 (2017) no.3, 034515
- T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, C. Lehner.
- **Continuum extrapolation** for **muon leptonic light-by-light**.



Original QED kernel

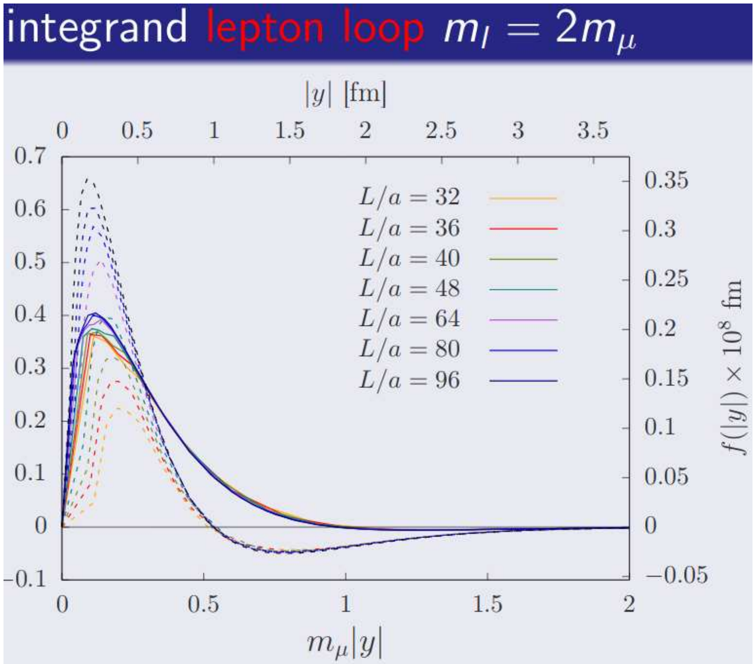


Subtracted QED kernel

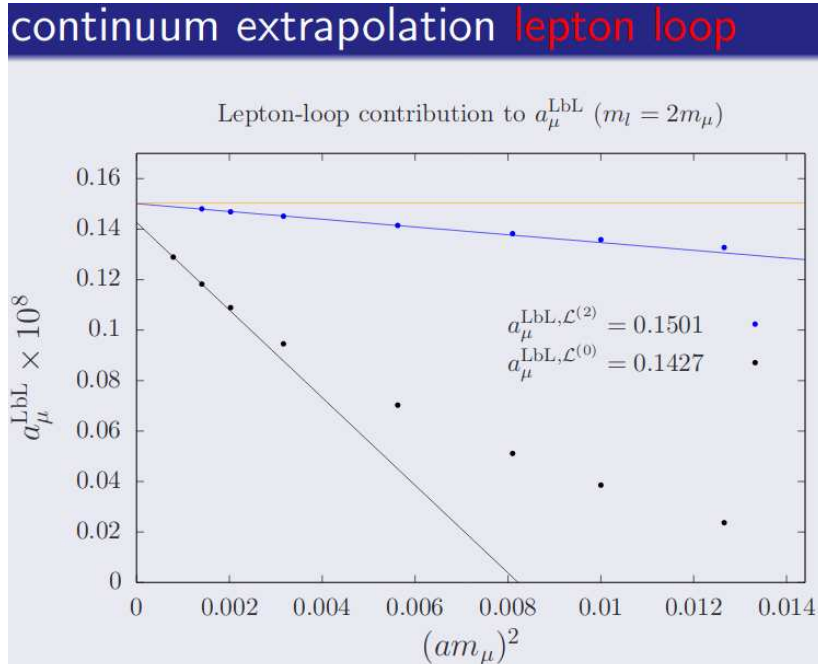
- Note the vertical scales of the two plots are different.
- **Discretization erorr** and **finite volume error** are greatly **reduced**.

- Talk by Nils Asmussen at Mainz $g-2$ workshop in June 2018.
- Subtraction on QED kernel also proved to be successful.
- The subtraction scheme is **not the same** as RBC.

$$\mathcal{M}^{(2)}(x,y,z) \;=\; \mathcal{M}(x,y,z) - \mathcal{M}(z,y,z) - \mathcal{M}(x,z,z)$$



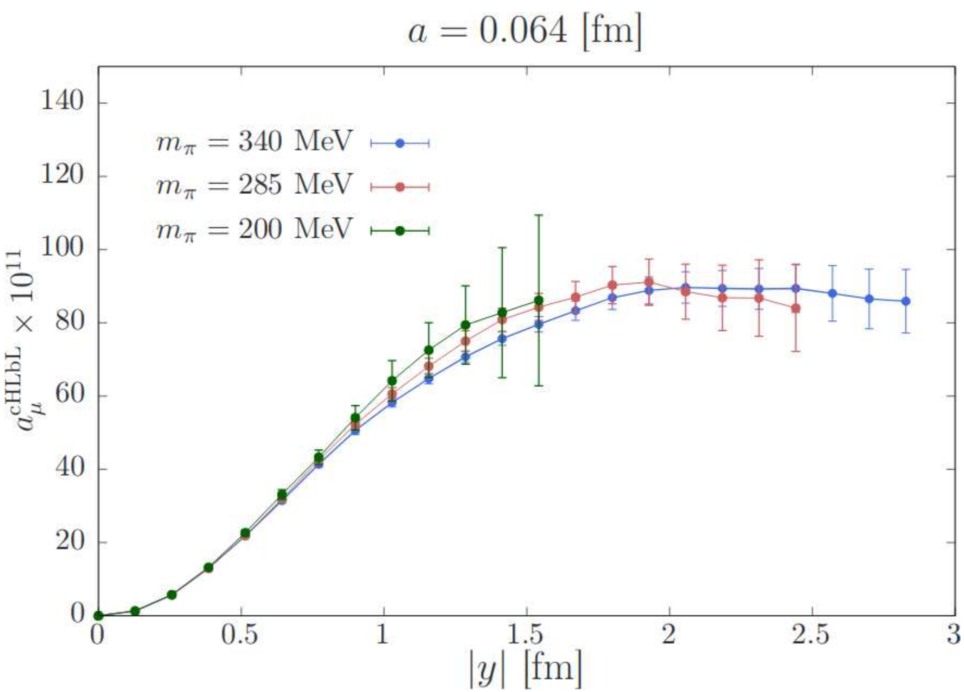
dashed line: standard kernel
solid line: subtracted kernel



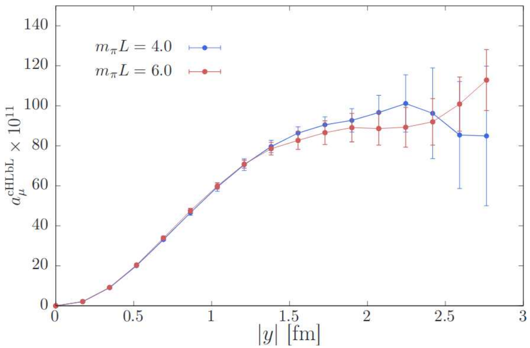
black: standard kernel
blue: subtracted kernel

Constant volume $m_\mu L = 7.2$.

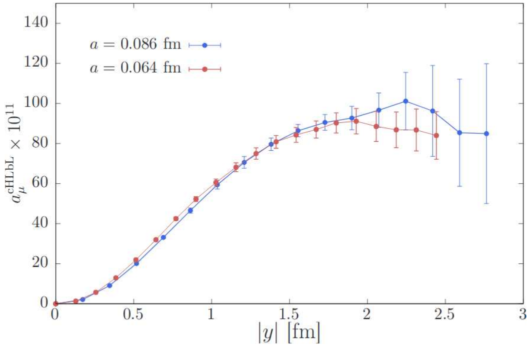
- Talk by Nils Asmussen at Mainz $g-2$ workshop in June 2018.
- Three different m_π: $200\,\mathrm{MeV}$, $285\,\mathrm{MeV}$, $340\,\mathrm{MeV}$. ($a = 0.064\,\mathrm{fm}$)
- Two different L: $2.75\,\mathrm{fm}$, $4.13\,\mathrm{fm}$. ($m_\pi = 285\,\mathrm{MeV}$, $a = 0.086\,\mathrm{fm}$).
- Two different a: $0.086\,\mathrm{fm}$, $0.064\,\mathrm{fm}$. ($m_\pi = 285\,\mathrm{MeV}$, $L = 2.7\,\mathrm{fm}{\sim}3.1\,\mathrm{fm}$).



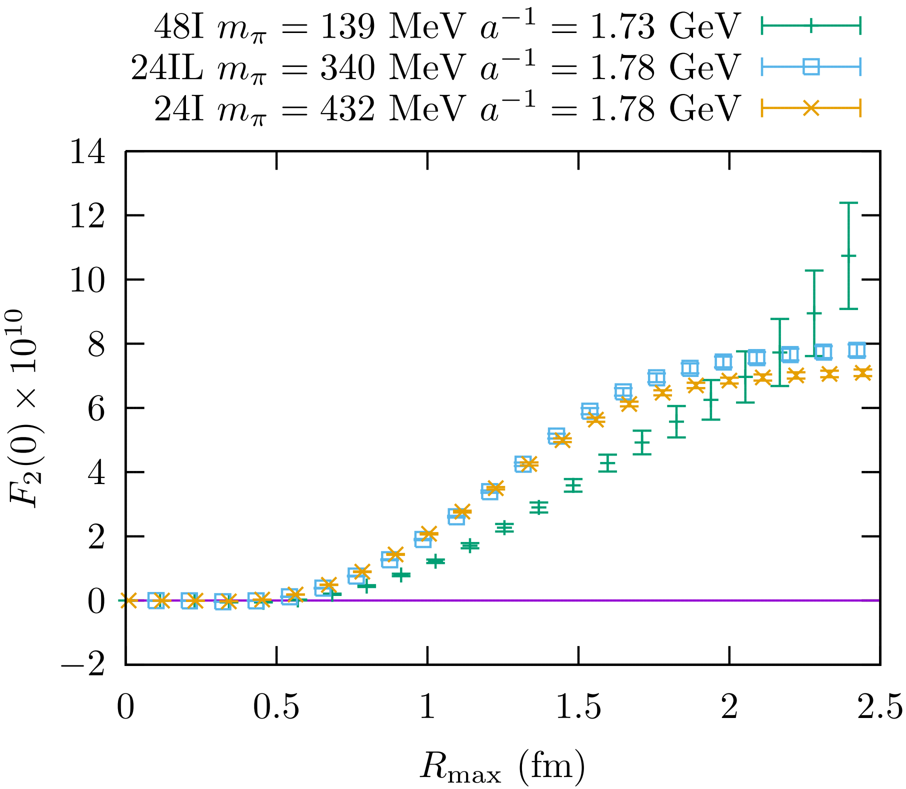
Upward trend for decreasing pion mass.



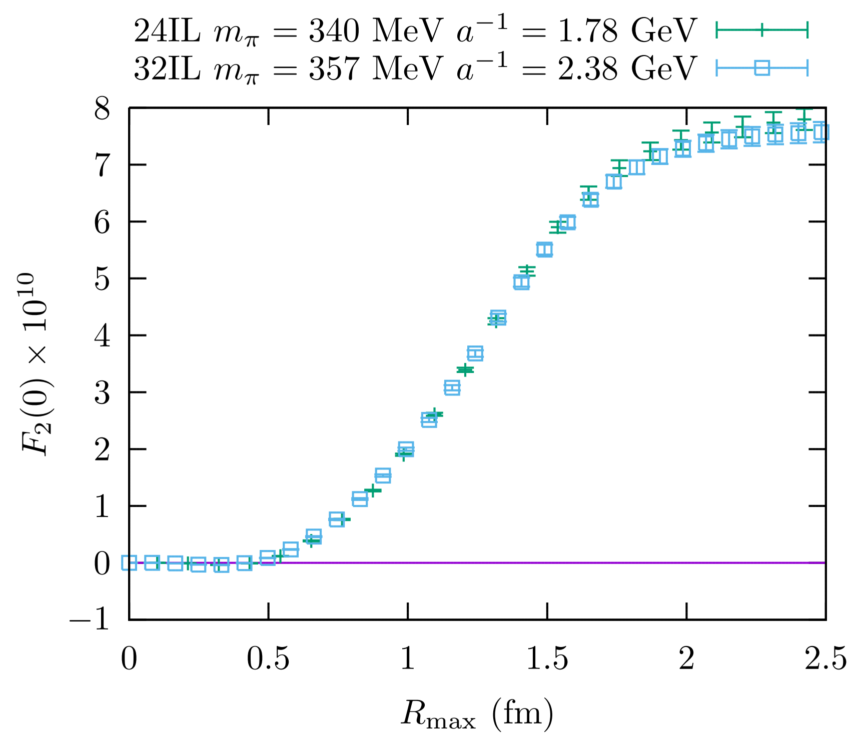
Small finite volume effects



Small discretization effects



Pion mass dependence



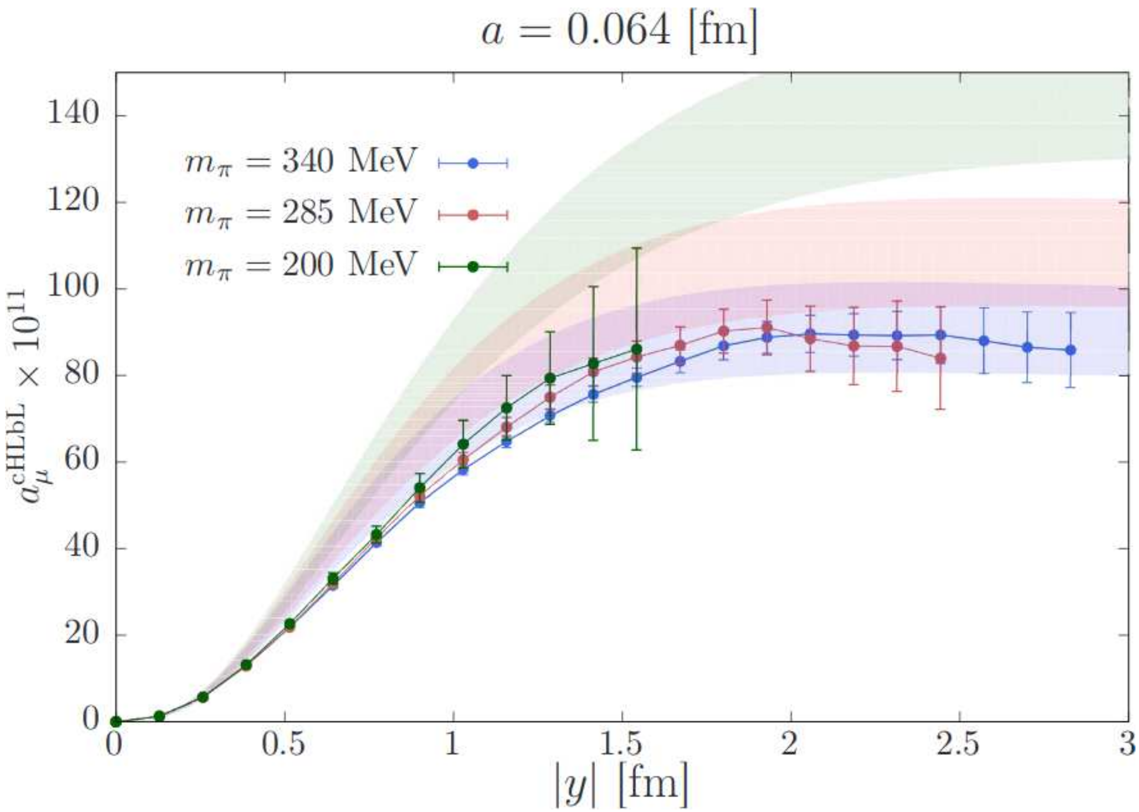
Small discretization effects

- $R_{\max} = \max\{|x-y|, |y-z|, |z-x|\}$.
- Physical pion mass significantly different from $m_\pi = 340\,\mathrm{MeV}$ or $432\,\mathrm{MeV}$.
- Plan to generate a larger $32^3 \times 64$, $m_\pi = 340\,\mathrm{MeV}$, $L = 3.5\,\mathrm{fm}$, $a^{-1} = 1.78\,\mathrm{GeV}$ ensemble to study the finite volume errors.

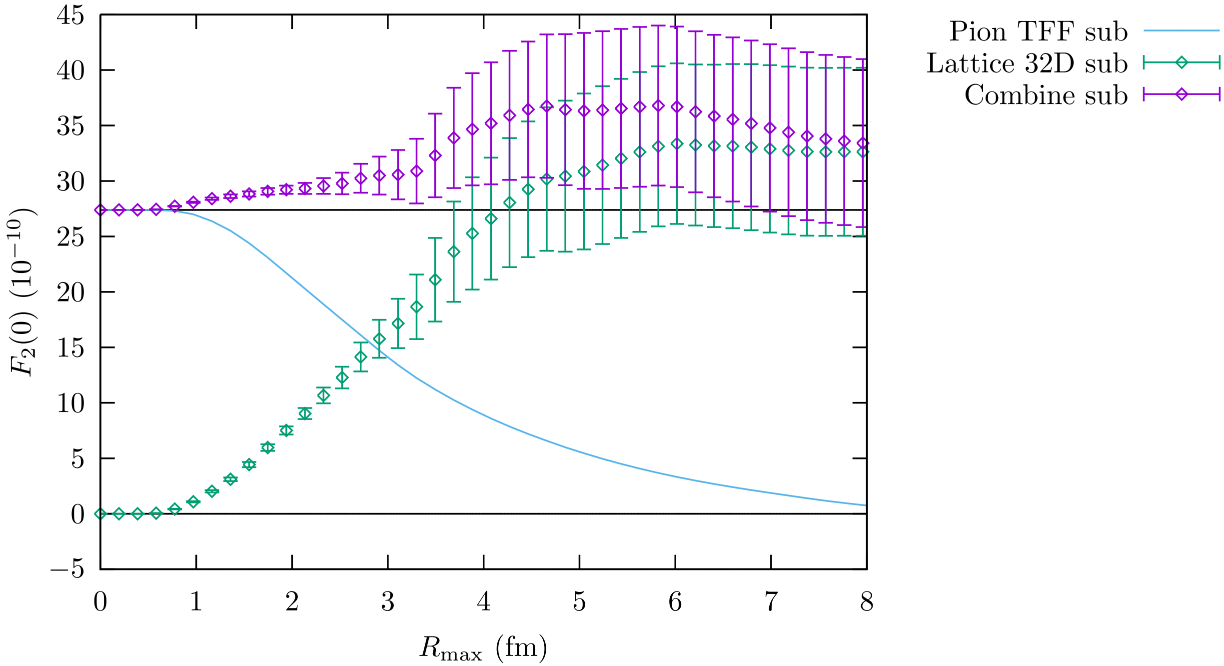
- **Target** $m_\pi = 340\,\mathrm{MeV}$, infinite volume and continuum limit.
- **Mainz**
  - $a_\mu^{\mathrm{cHLbL}} = 8.2(9) \times 10^{-10}$ at $L = 3.1\,\mathrm{fm}$, $a = 0.064\,\mathrm{fm}$.
  - Finite volume effects and discretization effects are found to be small at $m_\pi = 285\,\mathrm{MeV}$.
- **RBC-UKQCD**
  - $a_\mu^{\mathrm{cHLbL}} = 7.86(19) \times 10^{-10}$ at $L = 2.7\,\mathrm{fm}$, $a^{-1} = 1.78\,\mathrm{GeV}$.
  - Continuum extrapolation:
    - → $a_\mu^{\mathrm{cHLbL}} = 7.58(19) \times 10^{-10}$ at $L = 2.7\,\mathrm{fm}$, $a^{-1} = 2.38\,\mathrm{GeV}$, $m_\pi = 357\,\mathrm{MeV}$.
    - → $a_\mu^{\mathrm{cHLbL}} = 7.18(11) \times 10^{-10}$ at $L = 2.7\,\mathrm{fm}$, $a^{-1} = 1.78\,\mathrm{GeV}$, $m_\pi = 432\,\mathrm{MeV}$.
    - → $a_\mu^{\mathrm{cHLbL}}(a, m_\pi) = a_\mu^{\mathrm{cHLbL}}(0,0) + c_1\, a^2 + c_2\, m_\pi^2$
  - $a_\mu^{\mathrm{cHLbL}} = 7.49(46) \times 10^{-10}$ at $L = 2.7\,\mathrm{fm}$, continuum.
- Note that there are many differences between the Mainz treatment and RBC-UKQCD treatment: fermion action, how to treat boundary, choice for time direction, integration strategy, QED kernel subtraction scheme, QED kernel evaluation method, etc.

Compare with pion pole (Mainz - infinite volume)

- Talk by Nils Asmussen at Mainz $g-2$ workshop in June 2018.



- The bands are the results using the VMD model.
- The connected contribution from the pion pole is given by the factor between $34/9$ [two light quark flavors, JHEP 1609 (2016) 113] and 3 [three light quark flavors, 1712.00421].

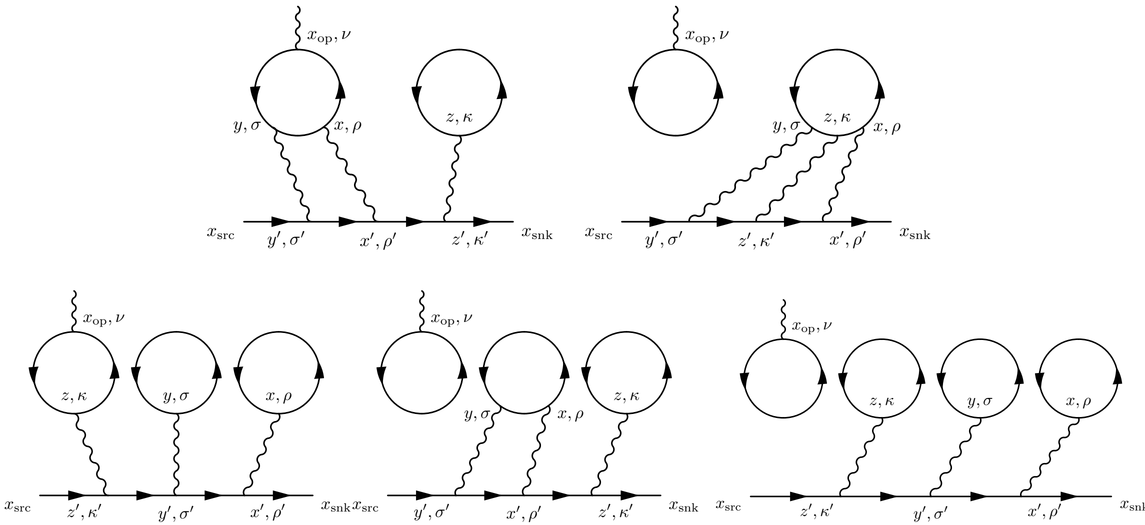


Combine **Lattice** with **Pion pole** LMD model.

- Physical pion mass. $R_{\max} = \max\{|x-y|, |y-z|, |z-x|\}$.
- Short distance: lattice calculation with 32D (6.4 fm, 1.015 GeV) (partial sum upto $R_{\max}$).
- Long distance: LMD (Lowest Meson Dominance) pion TFF model [Talk by Luchang Jin and Taku Izubuchi at Mainz $g-2$ workshop in June 2018] multiplied by $34/9$ [JHEP 1609 (2016) 113, PoS LATTICE2016 (2016) 181]. (partial sum from $R_{\max}$).
- At $R_{\max} = 2.0\,\mathrm{fm}$, the combined result is $29.2(0.4)_{\mathrm{stat}} \times 10^{-10}$.
- For 48I (5.4 fm, 1.73 GeV), also at $R_{\max} = 2.0\,\mathrm{fm}$, we obtain $27.9(0.8)_{\mathrm{stat}} \times 10^{-10}$.

Missing disconnected diagrams

- Missing disconnected diagrams which are $\mathrm{SU}(3)$ suppressed.
- First diagram in progress (RBC-UKQCD).



Summary

- Lots of progress on conceptual and technical side achieved for HLbL on the lattice in the last 3-4 years by **Mainz** and **RBC-UKQCD** group.
- **RBC-UKQCD**: Many results already obtained at physical pion mass with $\mathrm{QED_L}$, fully connected and leading disconnected diagrams.

  Progress is also made using the infinite volume QED kernel method, only connected diagrams are calculated so far.
- **Mainz**: Pioneered the semi-analytical approach for QED kernel in continuum and infinite volume to have full control over non-QCD part of HLbL. Results for QCD connected diagram contribution at several pion masses (lattice spacings, and volumes) are obtained.
- Cross-check at $m_\pi = 340\,\mathrm{MeV}$ between Mainz and RBC-UKQCD works (preliminary)!
- Pion-pole contribution by direct lattice calculation of $\pi^0 \to \gamma\gamma$ form factors helps control the QCD finite volume effects and reduce the statistical noise at long-distance region.

  RBC-UKQCD: Cheng Tu's talk, $\pi^0 \to \gamma\gamma$ in coordiante space. (Tuesday)

  Mainz: Gérardin et al, lattice calculation of $\pi^0 \to \gamma\gamma$. Phys.Rev. D94 (2016), 074507.

  Mainz: Gérardin et al, HLbL scattering amplitudes lattice v.s. sum rules, arXiv1712.00421

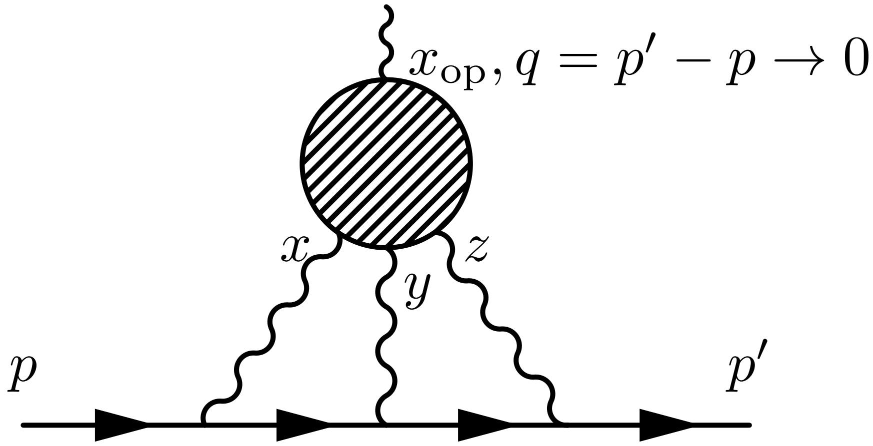
Outlook

- Timeframes: **End of 2018 (Whitepaper)** and final result by Fermilab $g-2$ experiment in **2-3 years**.
  - **RBC-UKQCD**: first number for a_μ^{HLbL} with physical pion mass, extrapolated to continuum and infinite volume, with some control over systematics by end of 2018 (Whitepaper).
  - **Mainz**: try to cross-check these numbers, at least for fully connected contribution and extrapolated to physical pion mass, with completely different approach and different lattice action.
  - More consolidated number: 10% uncertainty with controlled systematics by both collaborations seems within reach by the time the Fermilab experiment publishes its final result.

There was a claim that muon $g-2$ is not a very use quantity because one can never control HLbL uncertainty. Now, we know we will be able to nail down HLbL.

Thank You!

- This subject is started by T. Blum, S. Chowdhury, M. Hayakawa, T. Izubuchi more than 10 years ago. **hep-lat/0509016**, **Phys.Rev.Lett. 114 (2015) no.1, 012001**.
- A series of improvements in methodology is made. **RBC-UKQCD** computed the connected diagram of HLbL with 171 MeV pion mass. **Phys.Rev. D93 (2016) no.1, 014503**.
- **Mainz** group independently come up with a similar method to compute HLbL. **PoS LATTICE2015 (2016) 109**.
- With the improved methods, **RBC-UKQCD** calculated HLbL using the physical pion mass, $48^3 \times 96$, ensemble, and for the first time, calculated the leading disconnected diagrams contribution. **Phys.Rev.Lett. 118 (2017) no.2, 022005**.
- **Mainz** group announces the significant progress on the method to reduce the finite volume effects by treating the QED part of the HLbL diagram semi-analyticly in infinite volume. Part of the results are given in **PoS LATTICE2016 (2016) 164**.
- Encouraged by Mainz's success, **RBC-UKQCD** use a different approach to the QED part of the HLbL in infinite volume, and exploit a way to furthur reduce the lattice discretization error and finite volume error. **Phys.Rev. D96 (2017), 034515**.
- In the meanwhile, there are some very related work by **Mainz**: $\pi^0 \to \gamma\gamma$ transition form factors **Phys.Rev. D94 (2016), 074507**, comparing the hadronic light-by-light scattering amplitudes with model **Phys.Rev.Lett. 115 (2015) no.22, 222003**, **arXiv:1712.00421**.



$a_\mu^{\rm HLbL} = F_2(0) =$ some factors

$\otimes$

Hadronic 4-point function ($\mathcal{H}$)

$\otimes$

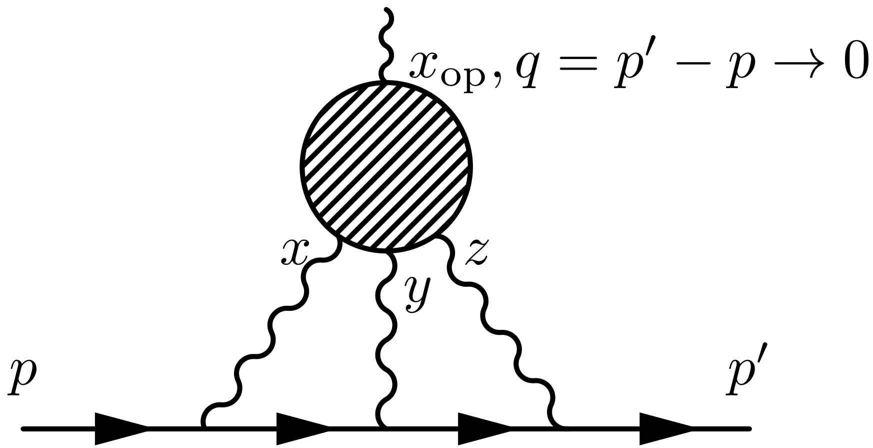
Muon line function with photons ($\mathcal{M}$)

- Muon line function with photons ($\mathcal{M}$):
  - The muon and photon propagator can be evaluated in:
    - $\to$ Finite volume using lattice with QED_L (RBC).

      $1/L^2$ finite volume systematic error, smaller statistical error.
    - $\to$ Infinite volume in the continuum semi-analytically (RBC and Mainz).

      Finite volume error only from the hadronic part $\mathcal{H}$, exponentially suppressed.
  - Need to perform proper spin projection to obtain magnetic moment:
    - $\to$ Muon propagate in the time direction (RBC).
    - $\to$ Average the muon propagation direction (Mainz).

Detail comparison - muon line part



$a_\mu^{\rm HLbL} = F_2(0) =$ some factors

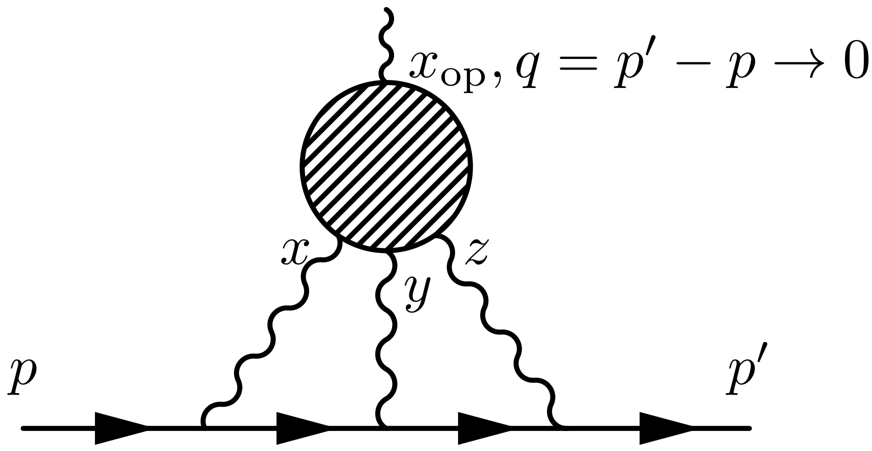
$\otimes$

Hadronic 4-point function ($\mathcal{H}$)

$\otimes$

Muon line function with photons ($\mathcal{M}$)

- Muon line function with photons ($\mathcal{M}$) subraction scheme:
  - RBC: $\mathcal{M}^{(\rm sub)}(x,y,z) = \mathcal{M}(x,y,z) - \mathcal{M}(y,y,z) - \mathcal{M}(x,y,y)$
  - Mainz
    1. $\mathcal{M}^{(1)}(x,y,z) = \mathcal{M}(x,y,z) - \frac{1}{2}\mathcal{M}(x,x,z) - \frac{1}{2}\mathcal{M}(y,y,z)$
    2. $\mathcal{M}^{(2)}(x,y,z) = \mathcal{M}(x,y,z) - \mathcal{M}(z,y,z) - \mathcal{M}(x,z,z)$
    3. $\mathcal{M}^{(3)}(x,y,z) = \mathcal{M}(x,y,z) - \mathcal{M}(z,y,z) - \mathcal{M}(x,x,z) + \mathcal{M}(z,x,z)$
- The $\mathcal{M}(x,y,z)$ should be understand as an tensor of photon polarizations.
- $\mathcal{M}(z,z,z) = 0$.



$a_\mu^{\rm HLbL} = F_2(0) =$ some factors

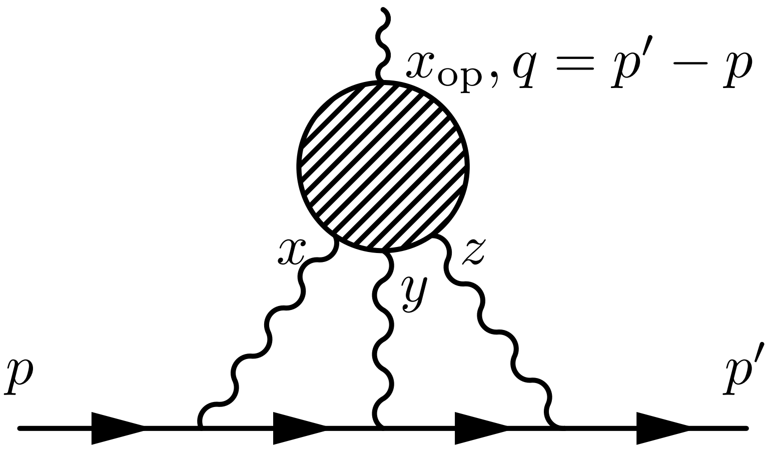
$\otimes$

Hadronic 4-point function ($\mathcal{H}$)

$\otimes$

Muon line function with photons ($\mathcal{M}$)

- Hadronic 4-point function ($\mathcal{H}$):
  - Evaluated the four point function using two point sources:
    - → The two closest points among the three points x, y, z (RBC).
    - → Old: y and z, New: all three possible combinations (Mainz).
  - Perform integration over the relative coordinate r between the two point sources.
    - → Complete summation upto $r \leqslant 5$, plus stochastic summation with importance sampling for $r > 5$ (RBC).
    - → $\int d^4 r \Rightarrow 2\pi^2 \int d|r| \, |r|^3$, always pick r to be $(1,1,1,1)$ direction (Mainz).



$a_\mu = F_2(0) =$ some factors

$\otimes$

Hadronic 4-point function ($\mathcal{H}$)

$\otimes$

Muon line function with photons ($\mathcal{M}$)

- Hadronic 4-point function ($\mathcal{H}$):
  - Project to zero external momentum transfer limit at x_{op} using the "moment method":

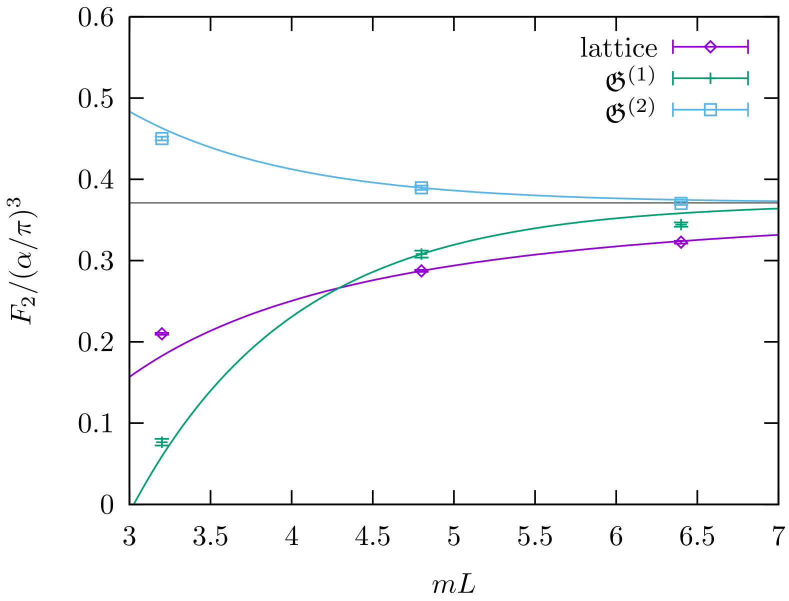
    $\vec{\mu} = \int \frac{1}{2} (\vec{x}_{\mathrm{op}} - \vec{x}_{\mathrm{ref}}) \times \vec{J}(\vec{x}_{\mathrm{op}})\, d^3\vec{x}_{\mathrm{op}}$.

    $\to$ Use the average of the two point sources as x_{ref} (RBC).

    $\to$ Use z as x_{ref} (Mainz).

Compare finite volume effects (RBC - infinite volume)

- Phys.Rev. D93 (2016) 1, 014503.
- T. Blum, N. Christ, M. Hayakawa, T. Izubuchi, L. Jin, C. Jung, C. Lehner.
- Compare the finite volume effects in different approaches in **pure QED computation**,



- Lattice: QED_L, $\mathcal{O}(1/L^2)$ finite volume effect.
- $\mathfrak{G}^{(1)}$: Infinite volume QED kernel, $\mathcal{O}(e^{-mL})$ finite volume effect.
- $\mathfrak{G}^{(2)}$: Subtracted infinite volume QED kernel, smaller $\mathcal{O}(e^{-mL})$ finite volume effect.