PDFs in small boxes

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Hampton University & Jefferson Lab

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Based on:
Briceño, JG, Hansen & Monahan, arXiv: 1805.01034 (accepted in PRD)

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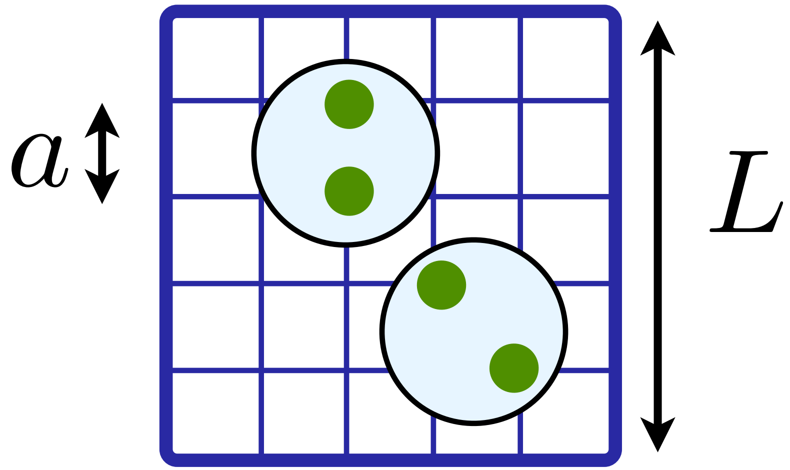
Novel idea: PDFs on the lattice

PDFs from QCD: lattice QCD is the only non-perturbative tool for QCD.

$$t_M \to -i t_E$$

Lattice QCD is defined by…

- Discretization
- Euclidean vs Minkowski
- Quark masses
- Finite volume



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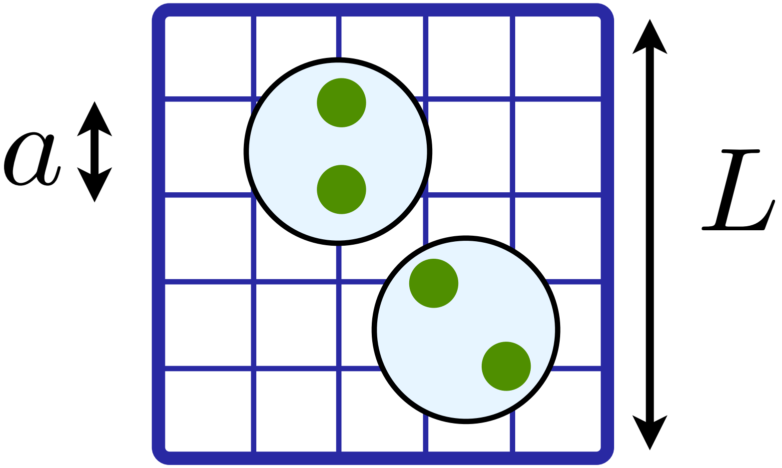
Focus of this talk…

$$t_M \to -it_E$$



Scheme to extract PDFs from the lattice

PDFs on the lattice → evaluation of matrix elements of non-local operators

$$\langle N|\mathcal{O}|N\rangle$$

There are different techniques:

- Wilson lines: $\langle N|\bar{q}\,W\,q|N\rangle_\infty$ — Ji (2013), Radyushkin (2017)
- two current operators: $\langle N|\mathcal{J}(0,\xi)\mathcal{J}(0)|N\rangle_\infty$ — Ma & Qiu (2018), Braun et al. (2008, 2018)

Lattice QCD

$$\langle N|\bar{q}\,W\,q|N\rangle_V$$

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Lattice QCD

$$\langle N|\bar{q}\,W q|N\rangle_{V}$$

$$\langle N|\mathcal{J}(0,\xi)\mathcal{J}(0)|N\rangle_{V}$$

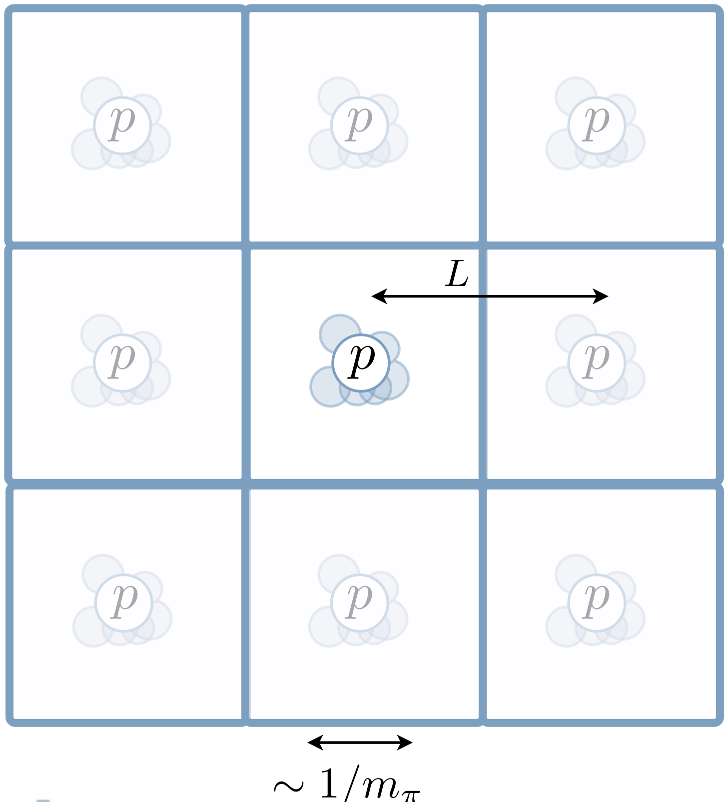
Pheno QCD

$$\langle N|\bar{q}\,W q|N\rangle_{\infty}$$

$$\langle N|\mathcal{J}(0,\xi)\mathcal{J}(0)|N\rangle_{\infty}$$

Finite volume: Infrared limit of the theory

- Finite-volume artifacts arise from the interactions with mirror images
- Assuming $L >>$ size of the hadrons ~ $1/m_\pi$
  - This is a purely infrared artifact
  - We can determine these artifact using hadrons as d.o.f.



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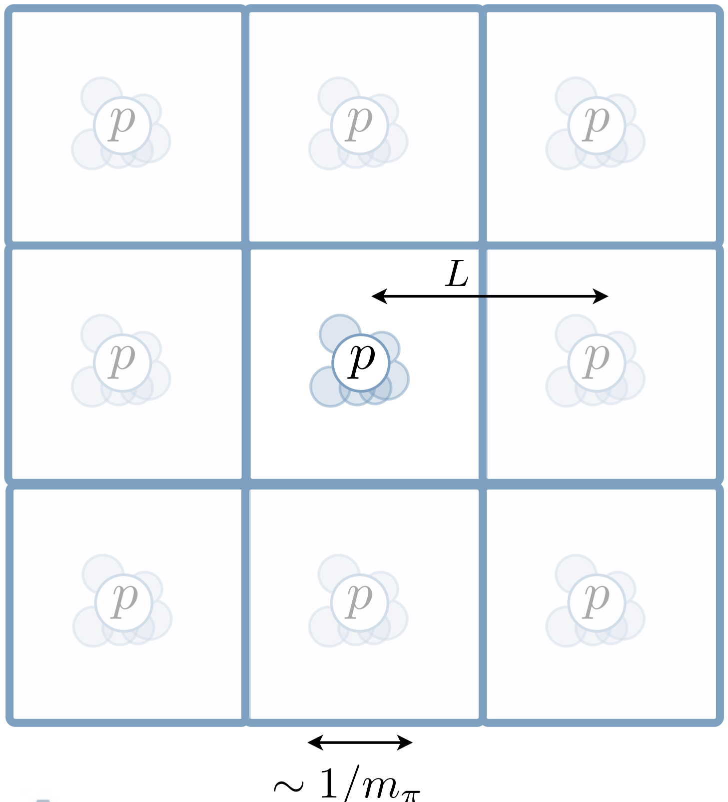


$$m_N(L) - m_N(\infty) \sim \langle N|\hat{V}|N\rangle_L \sim e^{-m_\pi L}$$

Lüscher (1985)

Finite volume effects: Matrix elements

- In general, the masses and matrix elements of stable particles have been observed to have these exponentially suppressed corrections.
- Matrix elements of non-local currents suffer from larger FV effects:

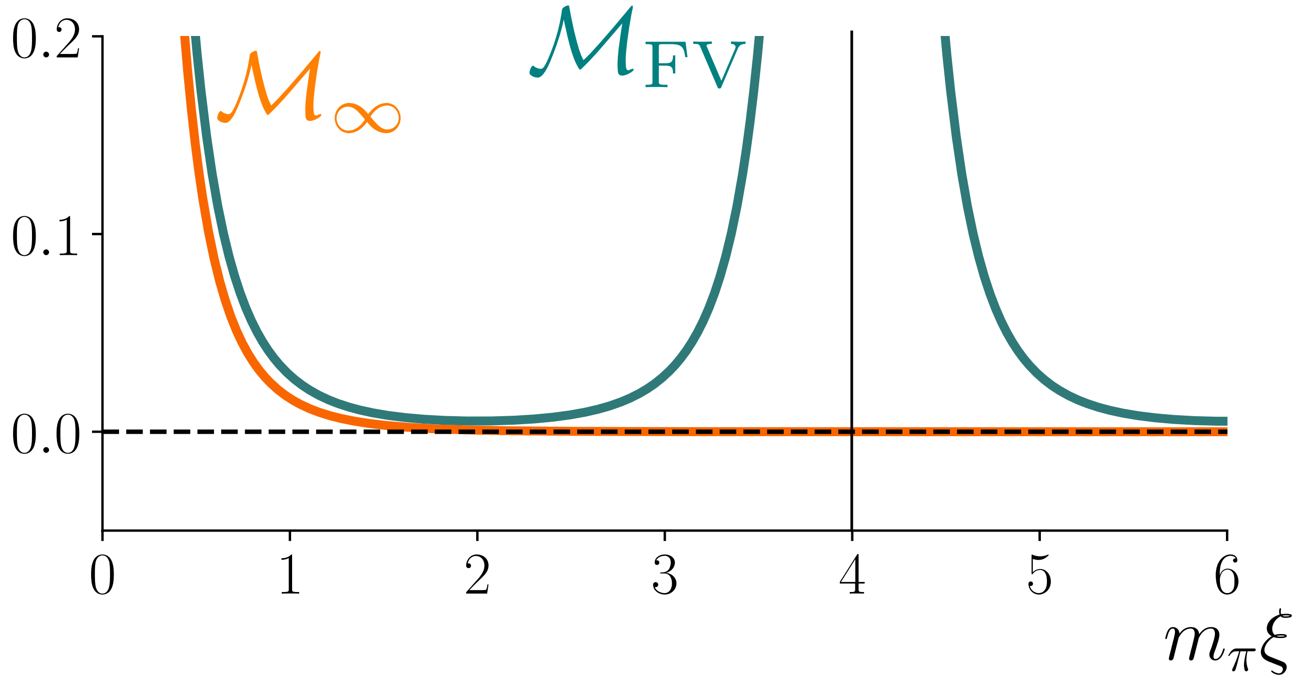
$\langle N|\mathcal{J}(0,\boldsymbol{\xi})\mathcal{J}(0)|N\rangle_\infty$: generally decays as a function of ξ

$\langle N|\mathcal{J}(0,\boldsymbol{\xi})\mathcal{J}(0)|N\rangle_V$: periodic, since

$$\mathcal{J}(t,\mathbf{x}) = \mathcal{J}(t,\mathbf{x}+L\mathbf{e}_i)$$

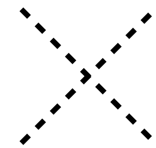
$\Rightarrow$ Expect enhanced finite volume effects to keep periodicity!

Finite volume effects: Matrix elements



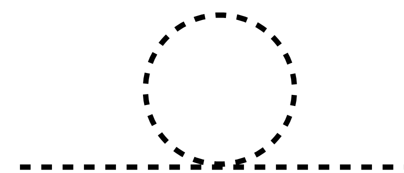
Expect enhanced finite volume effects to keep periodicity!

A simple example: mass of a pion

Consider a toy model for mesons $\mathcal{L}_M = \frac{\lambda}{4!}\varphi^4$ 

Bare propagator is volume-independent:

 $= \Delta_0(p^2) = \dfrac{i}{p^2 - m_0^2 + i\epsilon}$

so we have to have to go to loops… self-energy… 

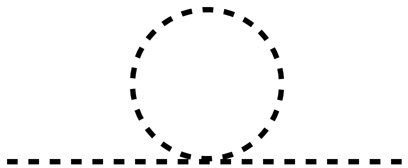
- In a finite volume, integrals over momenta become sums:

$$\textbf{1D:} \int \frac{dk_i}{2\pi} \to \sum_{k_i} \frac{\Delta k_i}{2\pi} = \sum_{k_i} \frac{2\pi \Delta n}{2\pi L} = \frac{1}{L}\sum_{k_i} \qquad \textbf{3D:} \int \frac{d^3k}{(2\pi)^3} \to \frac{1}{L^3}\sum_{k_i}$$

A simple example: self-energy of a pion

In infinite volume:

$$I_\infty = \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m_\pi^2}$$



In finite volume:

$$I_{\rm FV} = \frac{1}{L^3} \sum_{\mathbf{k}} \int \frac{dk_4}{2\pi} \frac{1}{k^2 + m_\pi^2} \overset{\text{Poisson summation}}{=} \sum_{\mathbf{n}} \int \frac{d^4k}{(2\pi)^4} \frac{e^{i\mathbf{k}\cdot\mathbf{n}L}}{k^2 + m_\pi^2}$$

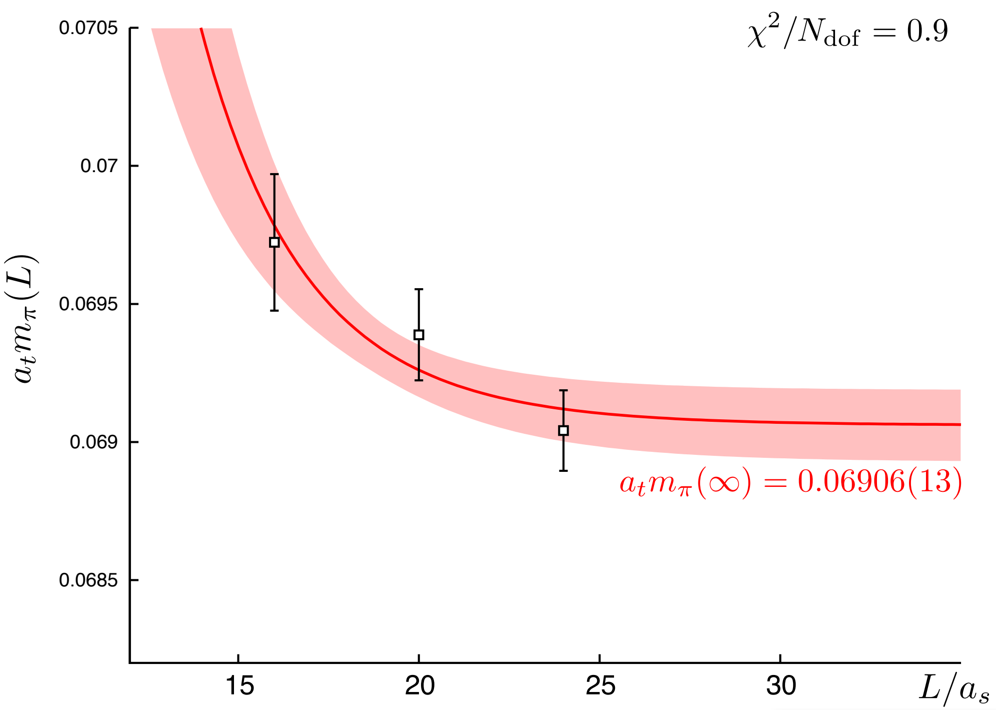
Finite/infinite volume difference: $\delta m^2(L) \sim \delta I_{\rm FV} = I_{\rm FV} - I_\infty$

$$= \sum_{\mathbf{n}\neq 0} \int \frac{d^4k}{(2\pi)^4} \frac{e^{i\mathbf{k}\cdot\mathbf{n}L}}{k^2 + m_\pi^2}$$

$$\sim K_1(Lm) \sim \frac{e^{-Lm}}{(Lm)^{3/2}}$$

A simple example: self-energy of a pion

$$m_\pi(L) = m_\pi + c\frac{e^{-m_\pi L}}{(m_\pi L)^{3/2}}$$



Dudek, Edwards & Thomas (2012)

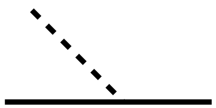
$$m_\pi \sim 390\ MeV,\ a_s \sim 0.12\ \mathrm{fm} \longrightarrow m_\pi L \sim 3.8,\ 4.7,\ 5.6$$

Our toy model

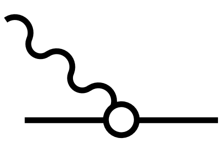
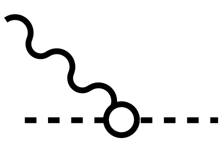
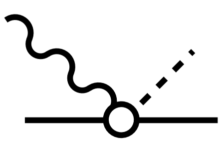
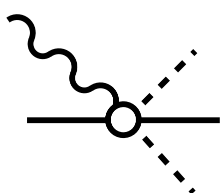
Consider a theory with two scalar particles

- a light one, φ, analogous to the pion
- a heavy one, χ, analogous to the nucleon
- momentum independent coupling

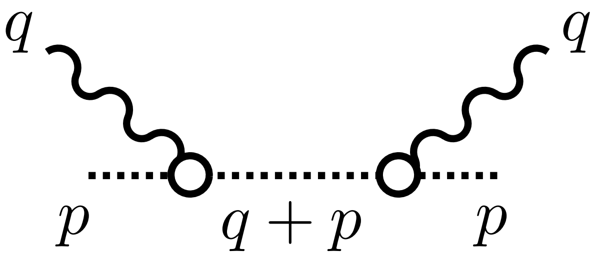
$$m_\varphi \ll m_\chi$$

 $= g$

Coupling to an external current :

 $= g_\chi$  $= g_\varphi$  $= g_{\chi\varphi}$  $= g_{\chi\varphi\varphi}$

Light external states



$$\mathcal{M}_{\infty}^{(\mathrm{LO})}(\boldsymbol{\xi},\mathbf{p}) = g_{\varphi}^{2}\int_{q_E}\frac{e^{i\mathbf{q}\cdot\boldsymbol{\xi}}}{(p_E+q_E)^2+m_{\varphi}^2}$$

Even at LO has an integral → Expect enhanced FV effects

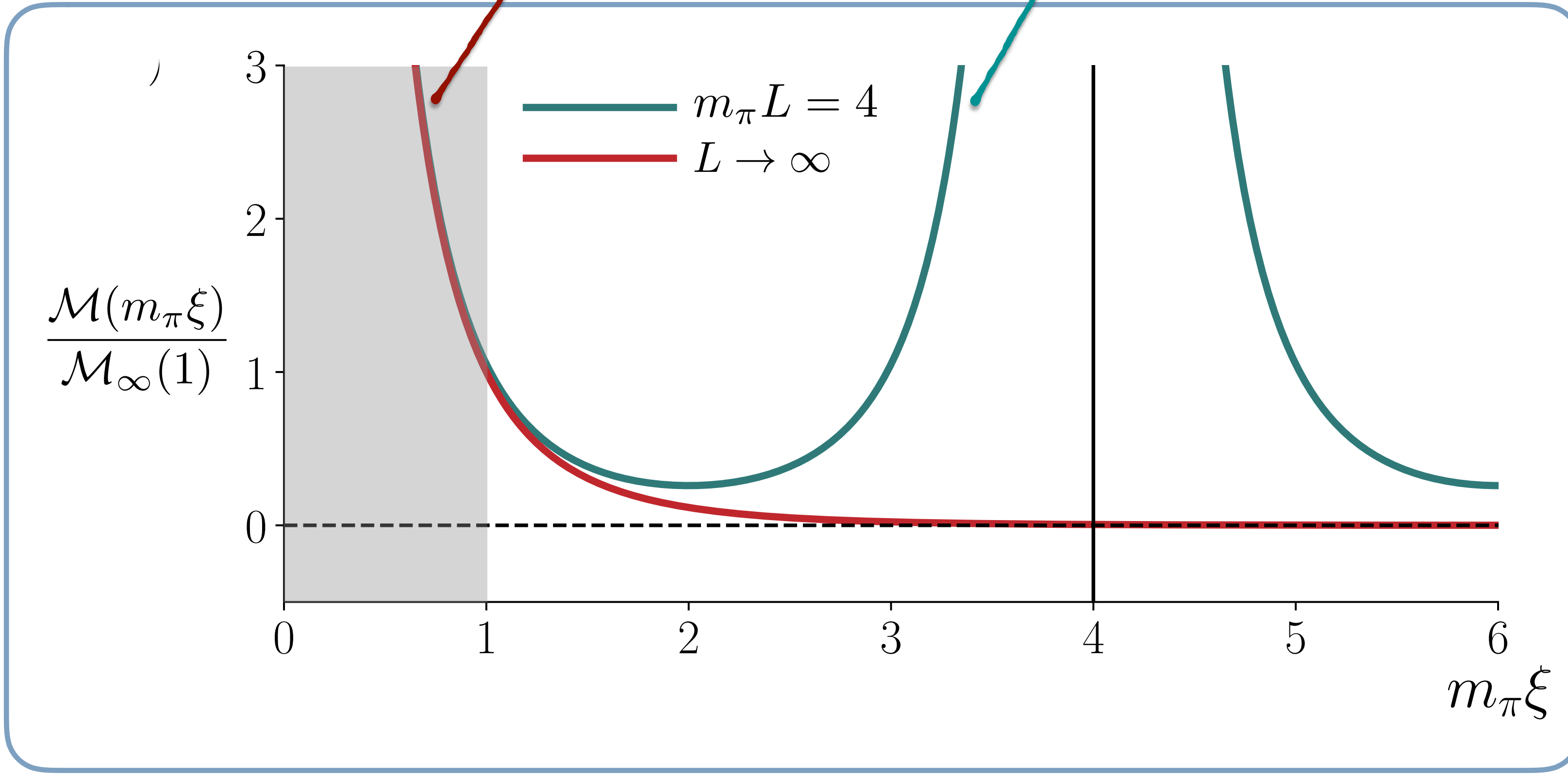
Finite volume correction: $$\delta\mathcal{M}_{L}^{(\mathrm{LO})}(\boldsymbol{\xi},\mathbf{p}) = g_{\varphi}^{2}\sum_{\mathbf{n}\neq 0}\int_{q_E}\frac{e^{i\mathbf{q}\cdot(\boldsymbol{\xi}+iL\mathbf{n})}}{(p_E+q_E)^2+m_{\varphi}^2}$$

$$\delta\mathcal{M}_{L}^{(\mathrm{LO})}(\boldsymbol{\xi},\mathbf{p}) = \frac{m_{\varphi}g_{\varphi}^{2}}{4\pi^2}e^{-i\mathbf{p}\cdot\boldsymbol{\xi}}\sum_{\mathbf{n}\neq 0}\frac{K_1\left(m_{\varphi}|\boldsymbol{\xi}+L\mathbf{n}|\right)}{|\boldsymbol{\xi}+L\mathbf{n}|} \sim \frac{m_{\varphi}g_{\varphi}^{2}}{4\pi^2}\frac{K_1\left(m_{\varphi}|L-\xi|\right)}{|L-\xi|}$$

$$\delta\mathcal{M}_{L}^{(\mathrm{LO})}(\boldsymbol{\xi},\mathbf{p}) \propto \frac{e^{-m_{\varphi}(L-\xi)}}{(L-\xi)^{3/2}}$$

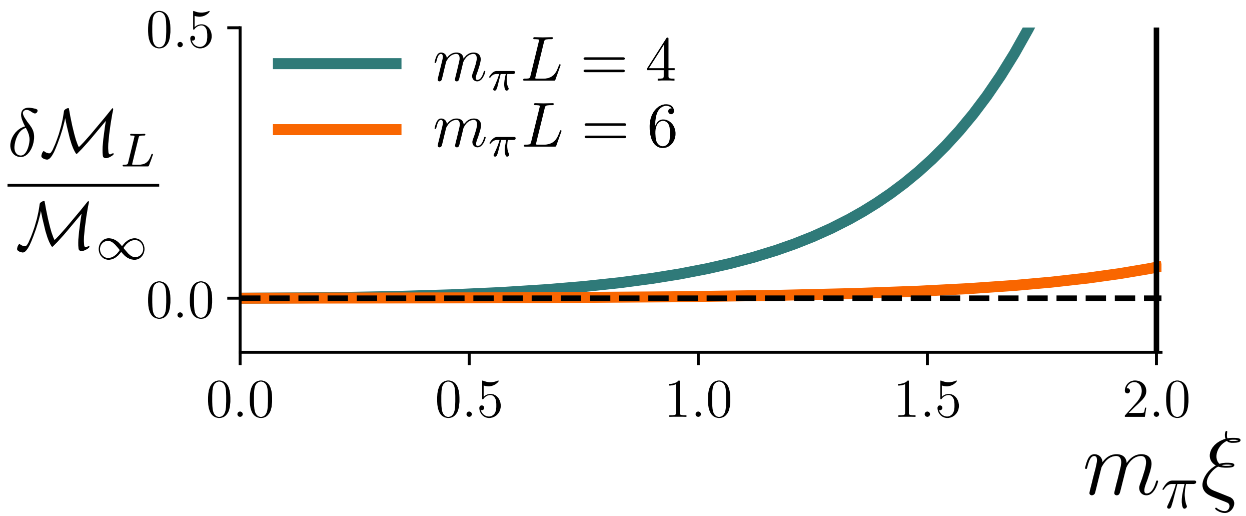
Light external states

$\mathcal{M}_{\infty}^{(\mathrm{LO})}$ $\mathcal{M}_{L}^{(\mathrm{LO})}$

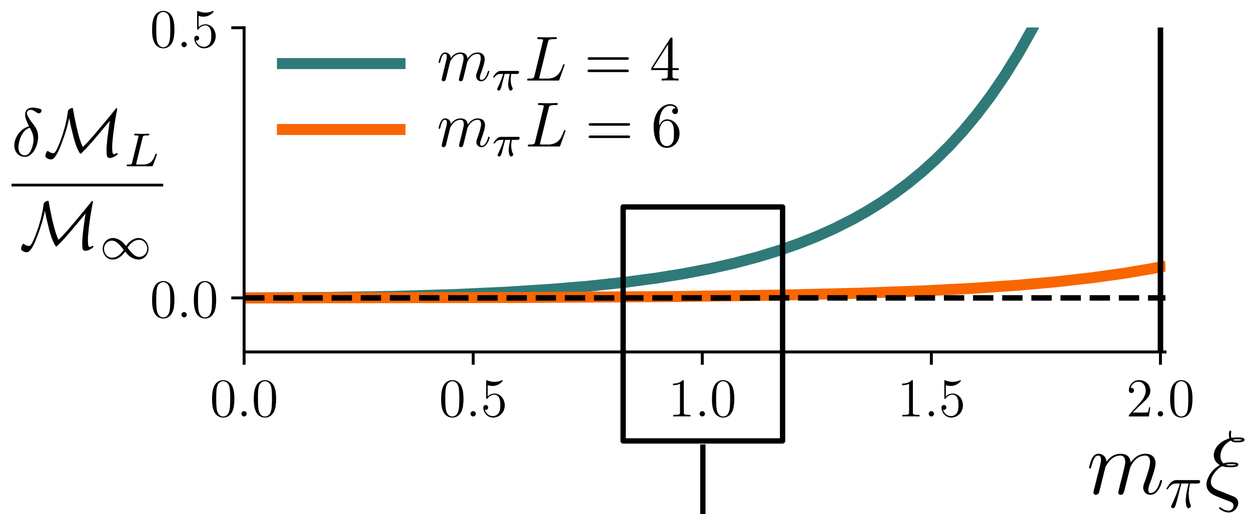


Expected behavior!

Light external states



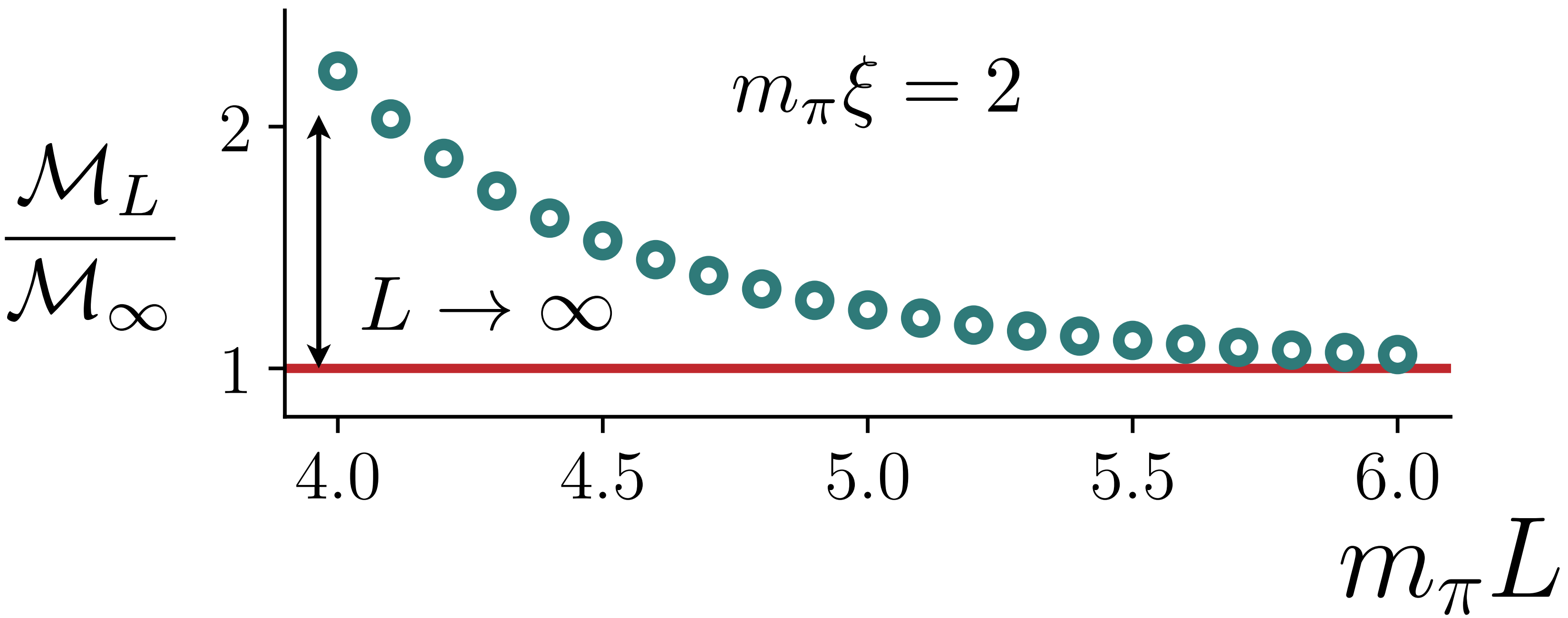
Light external states



$\sim 10\,\%$ when $\dfrac{\xi}{L} = \dfrac{1}{4}$

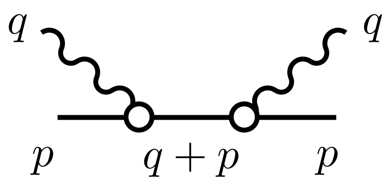
Light external states

100% systematic uncertainty!
inaccurate...despite it being arbitrarily precise!



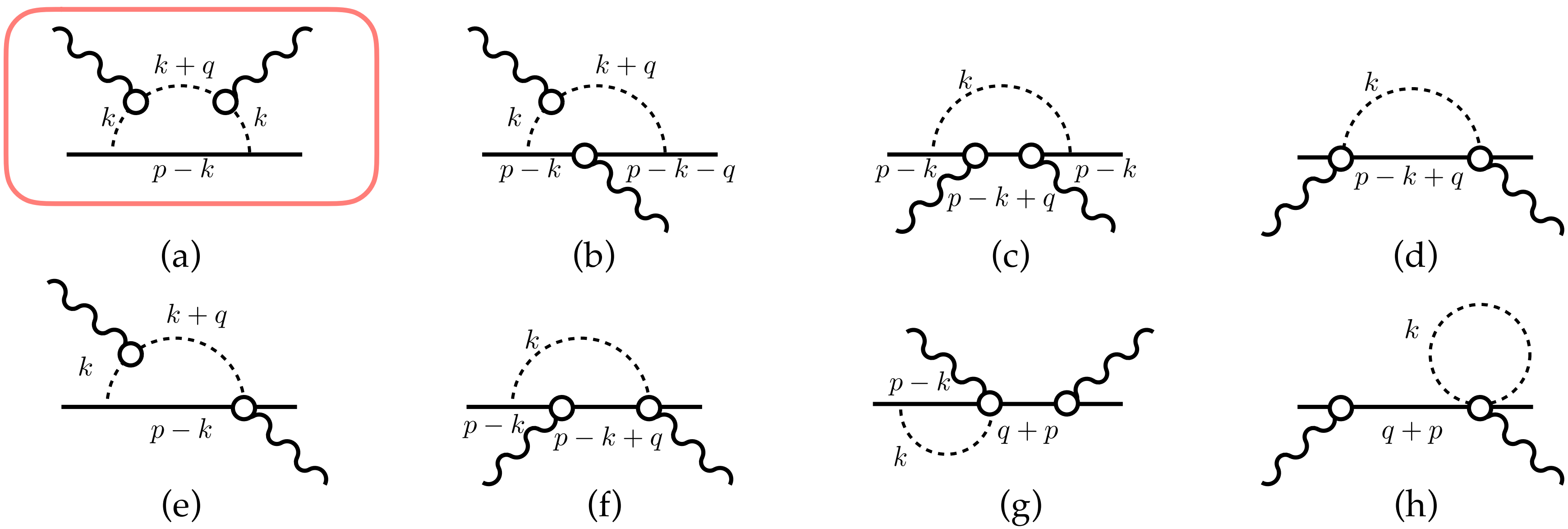
Heavy external states

Leading order



$$\delta\mathcal{M}_L^{(\mathrm{LO})}(\boldsymbol{\xi},\mathbf{p}) \propto \frac{e^{-m_\chi(L-\xi)}}{(L-\xi)^{3/2}} \ll e^{-m_\varphi(L-\xi)}$$

Next to Leading Order



In general…

We find that in general the matrix elements…

$$\langle M|\mathcal{J}(0,\boldsymbol{\xi})\mathcal{J}(0)|M\rangle_L - \langle M|\mathcal{J}(0,\boldsymbol{\xi})\mathcal{J}(0)|M\rangle_\infty = P_a(\boldsymbol{\xi},L)e^{-M(L-\xi)} + P_b(\boldsymbol{\xi},L)e^{-m_\pi L} + \cdots\,,$$

Polynomial prefactors $\propto L^m/|L-\xi|^n$

This result might be universal and have a better convergence than the EFT used, but we don't have a proof yet…

Implication for ongoing studies

parametrizes size of standard finite-volume errors

	Ref.	N_f	m_π(MeV)	L(fm)	z_{max}(fm)	$m_\pi L$	$m_\pi(L - \xi_{\text{max}})$
LP^3 nucleons	1807.07431	2+1+1	135	5.8	1.8	4	2.7
LP^3 nucleons	1803.04393	2+1+1	135	5.8	1.8	4	2.7
LP^3 mesons	1804.01483	2+1+1	310	2.9	1.8	4.6	1.7
ETMC	1807.00232	2	130	4.5	1.9	3	1.7
ETMC	1803.02685	2	130	4.5	1.9	3	1.7
Bali et al.	1807.06671	2	295	2.3	0.39	3.4	2.9
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new scale encoding errors for operators with finite size

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"finite-volume effects might be the largest underestimated systematic uncertainty…"

Summary

- First study of finite-volume artifacts in matrix elements of spatially non-local operators
  - Finite volume artifacts estimated via toy model with two scalar particles
- We considered a toy model involving two scalar particles to estimate the size of finite-volume corrections.
  - lightest particle: LO contribution dominant, effects ~ $P(\xi, L)e^{-m_\pi(L-\xi)}$
  - heaviest particle: NLO contribution dominant, effects ~ $P(\xi, L)e^{-m_\pi L}$

Thank you!

Backup slides

Finite volume effects: Matrix elements

Wilson line is not periodic:

$$W[x+\xi\mathbf{e}_i,x] \equiv U_i(x+(\xi-a)\mathbf{e}_i)\,U_i(x+(\xi-2a)\mathbf{e}_i)\times\cdots\times U_i(x+a\mathbf{e}_i)$$

Quark bilinears connected to Wilson Lines:

$$\overline{q}\big(x+(\xi+nL)\mathbf{e}_i\big)\,W\big[x+(\xi+nL)\mathbf{e}_i,x\big]\,q\big(x\big) = \overline{q}(x+\xi\mathbf{e}_i)\,W[x+\xi\mathbf{e}_i,x]\,\Big(W[x+L\mathbf{e}_i,x]^n\Big)\,q(x)$$

are no periodic. However,

q(x) and U(x) feel boundary conditions $\overset{?}{\Longrightarrow}$ expect enhanced finite volume effects for large ξ

Asymptotic behaviors

$$\delta\mathcal{M}_L^{(b)}(\boldsymbol{\xi},\mathbf{0}) = g^2 g_\varphi g_\chi \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{n}-\boldsymbol{\xi}|; M(x)\right]\right] \left[\int_0^1 \mathrm{d}y\, \mathcal{I}_2\left[|L\mathbf{m}-\boldsymbol{\xi}|; M(y)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(c)}(\boldsymbol{\xi},\mathbf{0}) = 2g^2 g_\chi^2 \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|; m_\chi\right] \left[\int_0^1 \mathrm{d}x\,(1-x)\, \mathcal{I}_3\left[|L\mathbf{m}-\boldsymbol{\xi}|; M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(d)}(\boldsymbol{\xi},\mathbf{0}) = g_{\chi\varphi}^2 \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|; m_\chi\right] \mathcal{I}_1\left[|L\mathbf{m}-\boldsymbol{\xi}|; m_\varphi\right] ,$$

$$\delta\mathcal{M}_L^{(e)}(\boldsymbol{\xi},\mathbf{0}) = g g_\varphi g_{\chi\varphi} \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|; m_\varphi\right] \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{m}-\boldsymbol{\xi}|; M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(f)}(\boldsymbol{\xi},\mathbf{0}) = g g_\chi g_{\chi\varphi} \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|; m_\chi\right] \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{m}-\boldsymbol{\xi}|; M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(g)}(\boldsymbol{\xi},\mathbf{0}) = g g_{\chi\varphi} g_\chi \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|; m_\chi\right] \left[\int_0^1 \mathrm{d}x\, \mathcal{I}_2\left[|L\mathbf{m}|; M(x)\right]\right] ,$$

$$\delta\mathcal{M}_L^{(h)}(\boldsymbol{\xi},\mathbf{0}) = \frac{1}{2} g_\chi g_{\chi\varphi\varphi} \sum_{\{\mathbf{n},\mathbf{m}\}\neq\mathbf{0}} \mathcal{I}_1\left[|L\mathbf{n}-\boldsymbol{\xi}|; m_\chi\right] \mathcal{I}_1\left[|L\mathbf{m}|; m_\varphi\right] .$$

Asymptotic behaviors

$$\delta\mathcal{M}_L^{(a)}(\boldsymbol{\xi},\mathbf{0}) \sim \frac{g^2 g_\varphi^2}{128\pi^3 m_\varphi}\left[\frac{\xi^{1/2}}{(L-\xi)^{3/2}}H_{x,3/2}(\xi) + \frac{(L-\xi)^{1/2}}{\xi^{3/2}}H_{x,3/2}(L-\xi)\right]e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(b)}(\boldsymbol{\xi},\mathbf{0}) \sim \frac{g^2 g_\varphi g_\chi}{64\pi^3 m_\varphi}\left[\frac{1}{\xi^{1/2}(L-\xi)^{1/2}}H_{1,1/2}(\xi)H_{1,1/2}(L-\xi)\right]e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(c)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g^2 g_\chi^2}{128\pi^3}\frac{m_\chi^{1/2}}{m_\varphi^{3/2}}\left[\frac{(L-\xi)^{1/2}}{\xi^{3/2}}H_{1-x,3/2}(L-\xi)\right]e^{-\xi(m_\chi-m_\varphi)}e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(d)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g_{\chi\varphi}^2 m_\chi^{1/2} m_\varphi^{1/2}}{32\pi^3}\left[\frac{1}{\xi^{3/2}(L-\xi)^{3/2}}\right]e^{-\xi(m_\chi-m_\varphi)}e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(e)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g g_\varphi g_{\chi\varphi}}{64\pi^3}\left[\frac{1}{\xi^{1/2}(L-\xi)^{3/2}}H_{1,1/2}(\xi) + \frac{1}{\xi^{3/2}(L-\xi)^{1/2}}H_{1,1/2}(L-\xi)\right]e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(f)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g g_\chi g_{\chi\varphi} m_\chi^{1/2}}{64\pi^3 m_\varphi^{1/2}}\left[\frac{1}{\xi^{3/2}(L-\xi)^{1/2}}H_{1,1/2}(L-\xi)\right]e^{-\xi(m_\chi-m_\varphi)}e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(g)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g g_{\chi\varphi} g_\chi m_\chi^{1/2}}{64\pi^3 m_\varphi^{1/2}}\left[\frac{1}{\xi^{3/2}L^{1/2}}H_{1,1/2}(L)\right]e^{-\xi m_\chi}e^{-m_\varphi L},$$

$$\delta\mathcal{M}_L^{(h)}(\boldsymbol{\xi},\mathbf{0}) = \frac{g_\chi g_{\chi\varphi\varphi} m_\varphi^{1/2} m_\chi^{1/2}}{64\pi^3}\left[\frac{1}{\xi^{3/2}L^{3/2}}\right]e^{-m_\chi\xi}e^{-m_\varphi L},$$

Heavy external states: Next to Leading Order