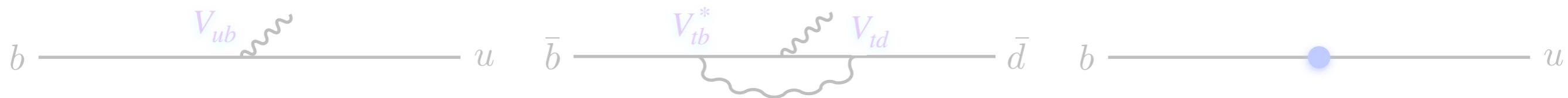


$B \rightarrow \pi \ell \nu$ and $B \rightarrow \pi \ell \ell$:

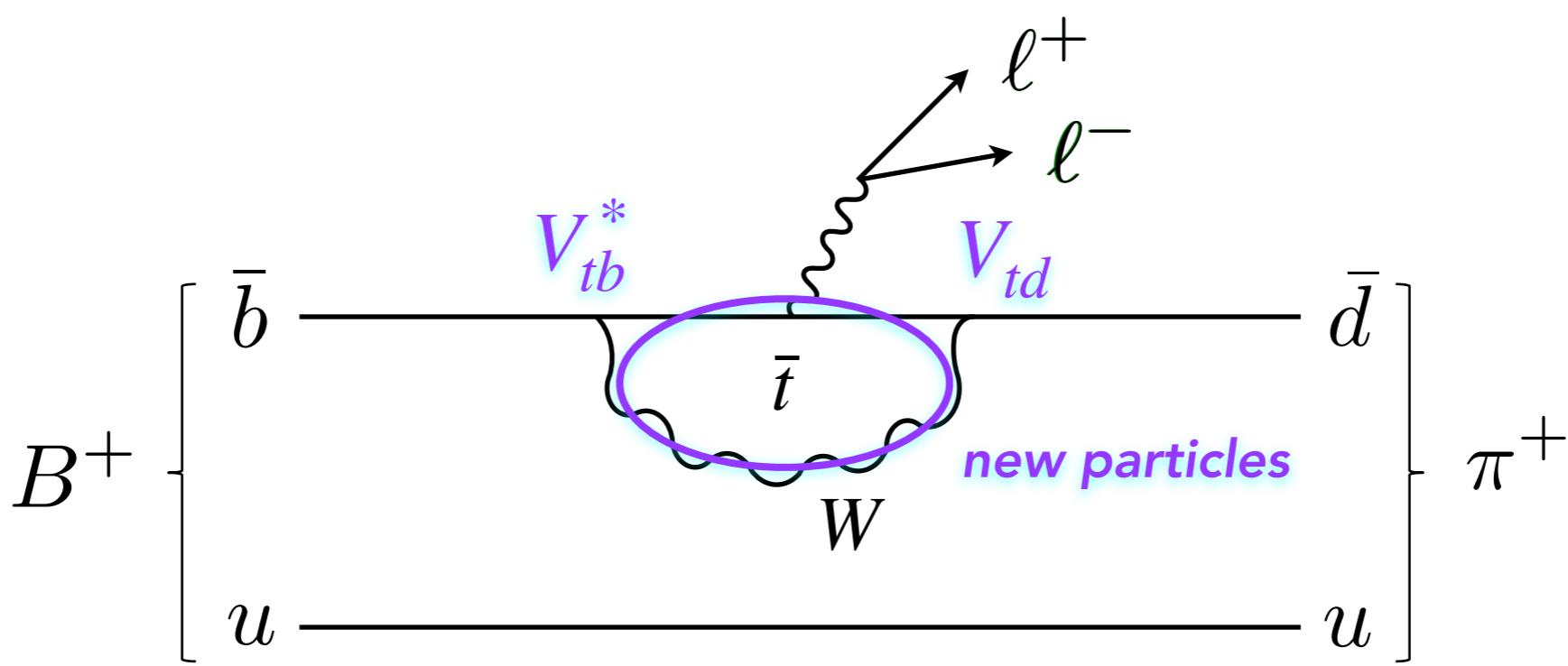
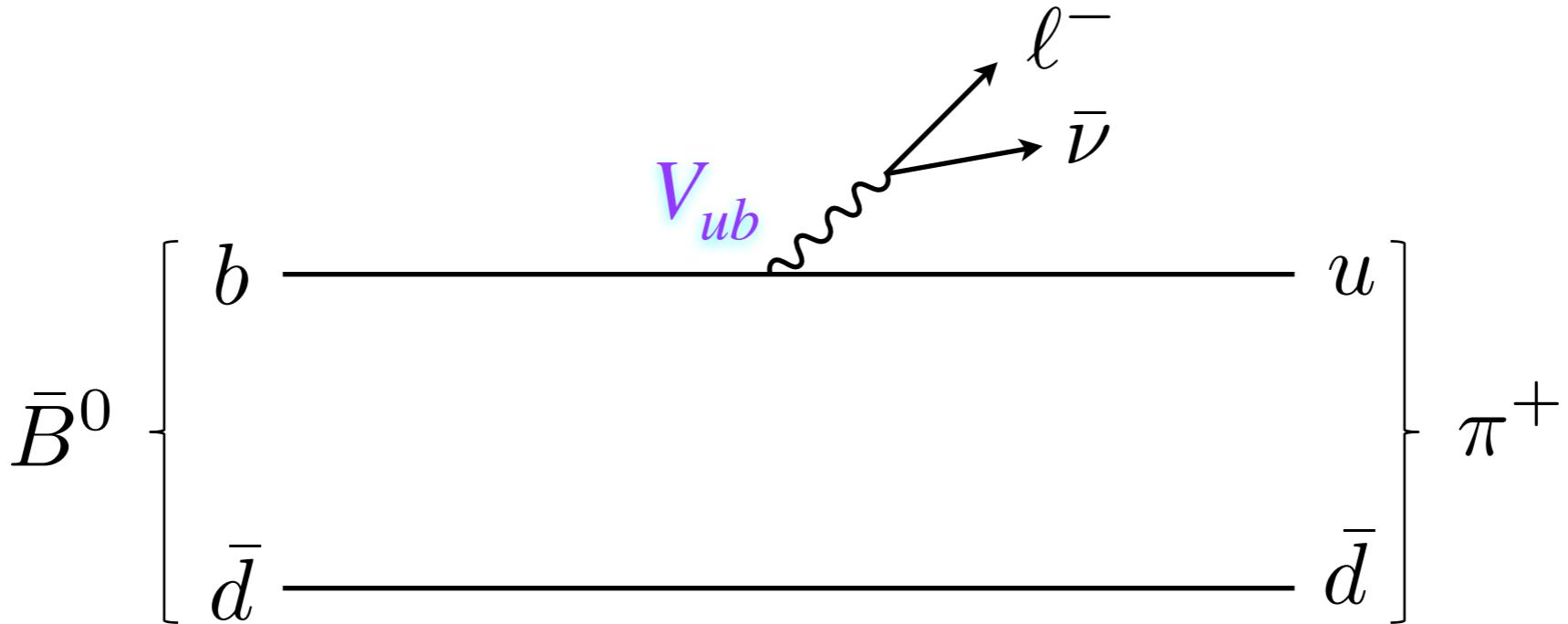
NRQCD/HISQ with $N_f = 2+1$ asqtad ensembles



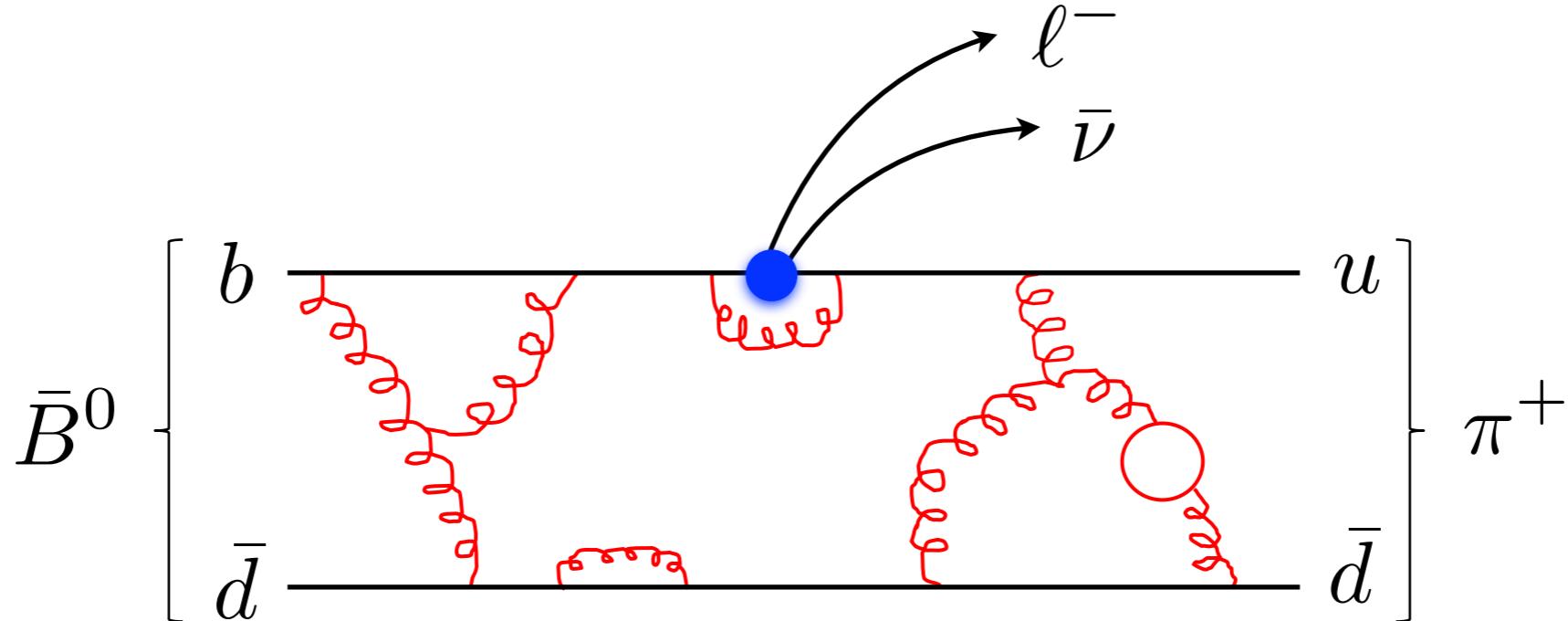
Chris Bouchard with
Chris Monahan, Junko Shigemitsu, and G. Peter Lepage

- Introduction
- simulation & correlator construction
- correlator analysis
- chiral, continuum, & kinematic extrapolations
- summary/outlook

Motivation ...



Role of lattice ...



Physics at disparate scales factorizes

$$\frac{d\Gamma}{dq^2} = \left(\sum_i C_i(V_{\text{CKM}}) \langle \pi | J_i | B \rangle \right)^2$$

- Wilson coefficients: short distance, perturbative
 - hadronic matrix elements: long distance, nonperturbative
- form factors: f_0 , f_+ , & f_T

- Introduction
- simulation & correlator construction
- correlator analysis
- chiral, continuum, & kinematic extrapolations
- summary/outlook

MILC $N_f = 2+1$ asqtad ensembles

ensemble	a / fm	M_π / MeV	$N_s^3 \times N_t$	N_{srcs}	N_{cfg}
C1 	0.12	267	$24^3 \times 64$	4	2096
C2 	0.12	348	$20^3 \times 64$	2	2242
C3 	0.12	489	$20^3 \times 64$	2	1200
F1 	0.09	313	$32^3 \times 96$	4	1896
F2 	0.09	438	$32^3 \times 96$	4	1200

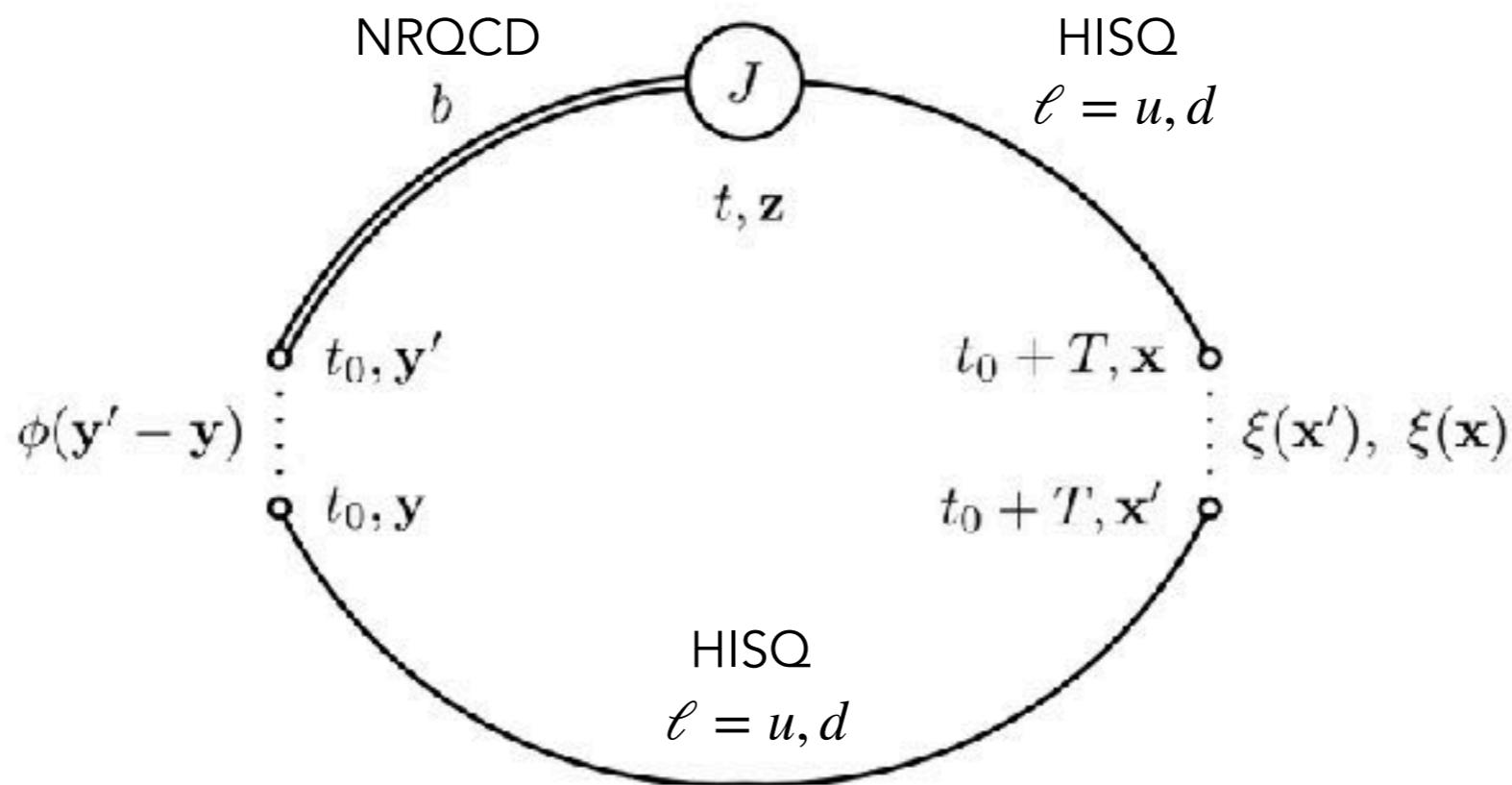
Bazavov et al, RMP 82, 1349 (2010)

Simulated momenta reaches $q^2 = 0.4 \text{ GeV}^2$

ensemble	a / fm	M_π / MeV	$p_\pi L / (2\pi)$				
			q^2 / GeV^2				
C1 	0.12	267	(0,0,0), (1,0,0), (1,1,0), (1,1,1), (2,0,0)				
			25.5,	22.9,	21.2,	19.9,	18.7
C2 	0.12	348	(0,0,0), (1,0,0), (1,1,0), (1,1,1)				
			24.8,	21.8,	19.8,	18.2	
C3 	0.12	489	(0,0,0), (1,0,0), (1,1,0), (1,1,1), (2,0,0), (3,0,0), (4,0,0), (5,0,0)				
			23.7,	21.2,	19.4,	17.9,	16.5, 11.3, 5.9, 0.4
F1 	0.09	313	(0,0,0), (1,0,0), (1,1,0), (1,1,1),				(4,0,0)
			24.9,	21.8,	19.7,	18.1,	5.8
F2 	0.09	438	(0,0,0), (1,0,0), (1,1,0), (1,1,1), (2,0,0), (3,0,0)*				
			23.9,	21.3,	19.4,	17.8,	16.5, 11.2

* Generated 300 cfgs with $p_\pi L / (2\pi) = (2,2,1)$ to search for Lorentz-violating effects relative to (3,0,0). None found.

3pt correlator construction ...



J : current insertion, with momentum

$\phi(\mathbf{y}' - \mathbf{y})$: local and Gaussian smeared NRQCD b quark

$\xi(\mathbf{x}'), \xi(\mathbf{x})$: U(1) phases for random wall HISQ sources, with momentum

T : separations of 12, 13, 14, 15 (21, 22, 23, 24) used on 0.12 fm (0.09 fm) ensembles

- Introduction
- simulation & correlator construction
- **correlator analysis**
- chiral, continuum, & kinematic extrapolations
- summary/outlook

Matching ...

- Generate 3pt correlator data for the following currents,

$$\begin{aligned}\mathcal{V}_\mu^{(0)} &= \bar{q} \gamma_\mu b , & \mathcal{V}_\mu^{(1)} &= \frac{-1}{2am_b} \bar{q} \gamma_\mu \gamma \cdot \nabla b , & \mathcal{V}_\mu^{(2)} &= \frac{-1}{2am_b} \bar{q} \gamma \cdot \overleftarrow{\nabla} \gamma_0 \gamma_\mu b , \\ \mathcal{V}_k^{(3)} &= \frac{-1}{2am_b} \bar{q} \nabla_k b , & \mathcal{V}_k^{(4)} &= \frac{-1}{2am_b} \bar{q} \overleftarrow{\nabla}_k b .\end{aligned}$$

- Match through $\mathcal{O}\left(\alpha_s \frac{\Lambda_{\text{QCD}}}{m_b}, \frac{\alpha_s}{am_b}, \alpha_s a \Lambda_{\text{QCD}}\right)$, *only partially for tensor*:

$$\langle V_0 \rangle = (1 + \alpha_s \rho_0^{(0)}) \langle \mathcal{V}_0^{(0)} \rangle + (1 + \alpha_s \rho_0^{(1)}) \langle \mathcal{V}_0^{(1),\text{sub}} \rangle + \alpha_s \rho_0^{(2)} \langle \mathcal{V}_0^{(2)} \rangle$$

$$\langle V_k \rangle = (1 + \alpha_s \rho_k^{(0)}) \langle \mathcal{V}_k^{(0)} \rangle + (1 + \alpha_s \rho_k^{(1)}) \langle \mathcal{V}_k^{(1),\text{sub}} \rangle + \alpha_s \sum_{i=2}^4 \rho_k^{(i)} \langle \mathcal{V}_k^{(i)} \rangle$$

$$\langle T_{k0} \rangle = (1 + \alpha_s \rho_T^{(0)}) \langle \mathcal{T}_{k0}^{(0)} \rangle + \langle \mathcal{T}_{0k}^{(1),\text{sub}} \rangle$$

with power-law subtraction $\langle J^{(1),\text{sub}} \rangle = \langle J^{(1)} \rangle - \alpha_s \xi_J \langle J^{(0)} \rangle$.

- Correlator data are combined with matching uncertainty embedded via priors for matching coefficients. Monahan et al, PRD 87, 034017 (2013) Gulez et al, PRD 73, 074502 (2006); D75, 119906(E) (2007)

Matching ...

- Generate 3pt correlator data for the following currents,

$$\mathcal{V}_\mu^{(2)} = \frac{-1}{2am_b} \bar{q} \gamma \cdot \overleftarrow{\nabla} \gamma_0 \gamma_\mu b ,$$

momentum dependent :

$$\mathcal{V}_k^{(4)} = \frac{-1}{2am_b} \bar{q} \overleftarrow{\nabla}_k b .$$

- Match through $\mathcal{O}\left(\alpha_s \frac{\Lambda_{\text{QCD}}}{m_b}, \frac{\alpha_s}{am_b}, \alpha_s a \Lambda_{\text{QCD}}\right)$, *only partially for tensor* :

$$\langle V_0 \rangle = (1 + \alpha_s \rho_0^{(0)}) \langle \mathcal{V}_0^{(0)} \rangle + (1 + \alpha_s \rho_0^{(1)}) \langle \mathcal{V}_0^{(1),\text{sub}} \rangle + \alpha_s \rho_0^{(2)} \langle \mathcal{V}_0^{(2)} \rangle$$

$$\langle V_k \rangle = (1 + \alpha_s \rho_k^{(0)}) \langle \mathcal{V}_k^{(0)} \rangle + (1 + \alpha_s \rho_k^{(1)}) \langle \mathcal{V}_k^{(1),\text{sub}} \rangle + \alpha_s \sum_{i=2}^4 \rho_k^{(i)} \langle \mathcal{V}_k^{(i)} \rangle$$

$$\langle T_{k0} \rangle = (1 + \alpha_s \rho_T^{(0)}) \langle \mathcal{T}_{k0}^{(0)} \rangle + \langle \mathcal{T}_{0k}^{(1),\text{sub}} \rangle$$

with power-law subtraction $\langle J^{(1),\text{sub}} \rangle = \langle J^{(1)} \rangle - \alpha_s \xi_J \langle J^{(0)} \rangle$.

- Correlator data are combined with matching uncertainty embedded via priors for matching coefficients. Monahan et al, PRD 87, 034017 (2013) Gulez et al, PRD 73, 074502 (2006); D75, 119906(E) (2007)

Correlator fitting ...

- Bayesian, using `lsqfit` Lepage, github.com/gplepage/lsqfit
- simultaneous fit to $(2\text{pt} + 3\text{pts})_p$

$$C_{2pt}^\pi(p; t) = \sum_n |Z_n^\pi(p)|^2 (-1)^{nt} \left(e^{-E_n^\pi(p)t} + e^{-E_n^\pi(p)(N_t - t)} \right)$$

$$C_{3pt}^{B\pi}(p; t) = \sum_{n,m} Z_n^\pi(p) A_{nm}(p) Z_m^B (-1)^{mt+n(T-t)} e^{-E_m^b t} e^{-E_n^\pi(p)(T-t)}$$

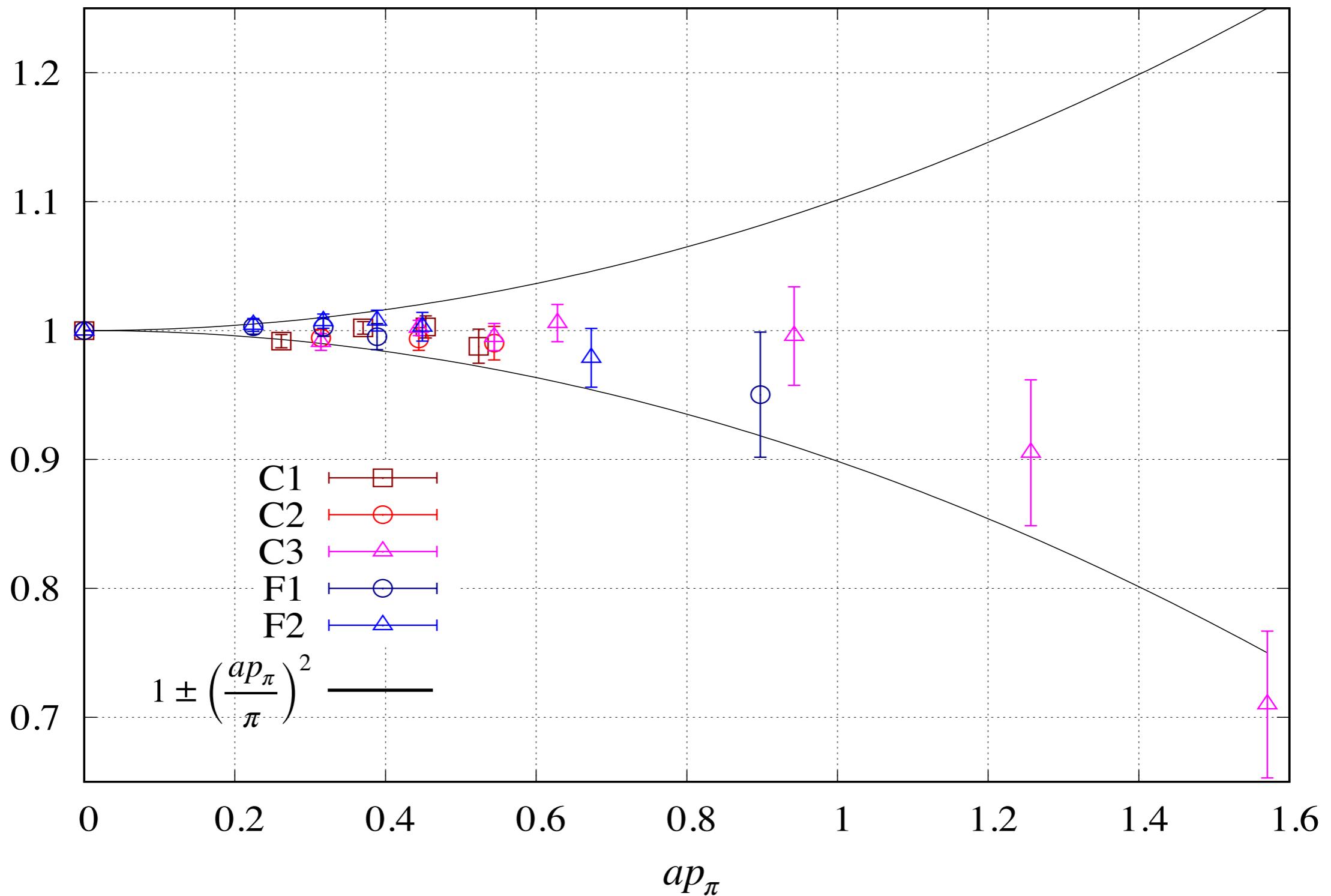
- posteriors from $(2\text{pt} + 3\text{pts})_{(0,0,0)}$ are priors in $(2\text{pt} + 3\text{pts})_{(1,0,0)}$
- posteriors from $(2\text{pt} + 3\text{pts})_{(1,0,0)}$ are priors in $(2\text{pt} + 3\text{pts})_{(1,1,0)} \dots$
- $E^\pi(p)$ and $Z^\pi(p)$ ground state priors set by

$$\text{prior}[E^\pi(p)] = \sqrt{E^\pi(0)^2 + p^2} \left[1 + c_1 \left(\frac{ap}{\pi} \right)^2 + c_2 \left(\frac{ap}{\pi} \right)^4 \right]$$

$$\text{prior}[Z^\pi(p)] = Z^\pi(0) \sqrt{\frac{E^\pi(0)}{E^\pi(P)}} \left[1 + c_3 \left(\frac{ap}{\pi} \right)^2 + c_4 \left(\frac{ap}{\pi} \right)^4 \right]$$

Dispersion relation ...

$$(E_\pi^2 - M_\pi^2)_{\text{fit}} / (p_\pi^2)_{\text{sim}}$$



- Introduction
- simulation & correlator construction
- correlator analysis
- chiral, continuum, & kinematic extrapolations
- summary/outlook

Matching (higher order effects) ...

- Matrix elements written in terms of intermediate form factors convenient for ChPT

$$\langle V_0 \rangle = \sqrt{2M_B} f_{\parallel}, \quad \langle V_k \rangle = \sqrt{2M_B} p_{\pi}^k f_{\perp}, \quad \langle T_{k0} \rangle = \frac{2M_B p_{\pi}^k}{M_B + M_{\pi}} f_T.$$

- Uncertainty from omitted higher order matching effects are accounted for after correlator fitting by inflating uncertainties:

$$f_{\parallel,\perp,T} \rightarrow f_{\parallel,\perp,T} \left(1 + \mu_{\parallel,\perp,T} + \tilde{\mu}_{\parallel,\perp,T} \frac{p_{\pi}}{p_{\pi}^{\max}} \right)$$

where the coefficients

$$\mu_{\parallel,\perp,T} = 0 \pm 0.04$$

$\mu_{\parallel,\perp,T}$: generic p_{π} -independent effects

$$\tilde{\mu}_{\parallel} = 0 \pm 0.02$$

$\tilde{\mu}_{\parallel,\perp,T}$: generic p_{π} -dependent effects

$$\tilde{\mu}_{\perp} = 0 \pm 0.03$$

$\tilde{\mu}_T = 0 \pm 0.05$

are **priors** in subsequent fit, with values set by observed size of $J^{(i)} / J^{(0)}$.

Chiral and continuum extrapolations ...

- Restrict to subset of data: (0,0,0), (1,0,0), (1,1,0), (1,1,1).
- Use SU(2) hard pion ChPT Bijnens and Jemos, NPB 840, 54 (2010); B 844, 182(E) (2011)
Bijnens and Jemos, NPB 846, 145 (2011)

$$f_{\parallel} = \frac{\kappa_{\parallel}}{f_{\pi}} (1 + [\log]) (1 + \mathcal{C}_{\parallel} + \mathcal{E}_{\parallel} + \mathcal{D}_{\parallel})$$

$$f_{\perp,T} = \frac{g_{BB^*\pi} \kappa_{\perp,T}}{f_{\pi}(E_{\pi} + \Delta^*)} (1 + [\log]) (1 + \mathcal{C}_{\perp,T} + \mathcal{E}_{\perp,T} + \mathcal{D}_{\perp,T})$$

where the hard pion ChPT log is

$$[\log] = \left(\frac{3 + 9g_{BB^*\pi}^2}{8} \right) \frac{M_{\pi}^2}{(4\pi f_{\pi})^2} \log \left(\frac{M_{\pi}^2}{(4\pi f_{\pi})^2} \right) .$$

- Factorization of chiral logarithmic corrections and kinematic dependence.

$$f_{\parallel} = \frac{\kappa_{\parallel}}{f_{\pi}} \left(1 + [\text{logs}] \right) \left(1 + \mathcal{C}_{\parallel} + \mathcal{E}_{\parallel} + \mathcal{D}_{\parallel} \right)$$

$$f_{\perp,T} = \frac{g_{BB^*\pi} \kappa_{\perp,T}}{f_{\pi}(E_{\pi} + \Delta^*)} \left(1 + [\text{logs}] \right) \left(1 + \mathcal{C}_{\perp,T} + \mathcal{E}_{\perp,T} + \mathcal{D}_{\perp,T} \right)$$

Using the following expansion parameters,

$$\chi_{l,\text{val}} = \left(\frac{m_l}{m_c} \right)_{\text{HISQ}} \quad \chi_{s,\text{val}} = \left(\frac{m_s}{m_c} \right)_{\text{HISQ}} \quad \chi_{\text{sea}} = \left(\frac{m_l}{m_c} \right)_{\text{asqtad}}$$

$$\chi_E = \frac{\sqrt{2}E_{\pi}}{4\pi f_{\pi}} \quad \chi_{a^2} = \left(\frac{a}{r_1} \right)^2 \quad \chi_b = am_b - 2.26$$

the analytic terms are given by:

$$\mathcal{C} = c_1 \chi_{l,\text{val}} + c_2 \chi_{s,\text{val}} + c_3 \chi_{\text{sea}} + c_4 \chi_{l,\text{val}}^2 + c_5 \chi_{l,\text{val}} \chi_E + c_6 \chi_{l,\text{val}} \chi_E^2 + c_7 \chi_{a^2} \chi_E + c_8 \chi_{a^2} \chi_E^2$$

$$\mathcal{E} = e_1 \chi_E + e_2 \chi_E^2 + e_3 \chi_E^3 + e_4 \chi_E^4$$

$$\mathcal{D} = d_1 \chi_{a^2} (1 + \ell_1 \chi_{\text{val}} + \ell_2 \chi_{\text{val}}^2) (1 + b_1 \chi_b + b_2 \chi_b^2) + d_2 \chi_{a^2}^2 (1 + \ell_3 \chi_{\text{val}} + \ell_4 \chi_{\text{val}}^2) (1 + b_3 \chi_b + b_4 \chi_b^2)$$

- hard pion SU(2) ChPT with

$$\frac{a\vec{p}_\pi \cdot L}{2\pi} \in \{(0,0,0), (1,0,0), (1,1,0), (1,1,1)\}$$

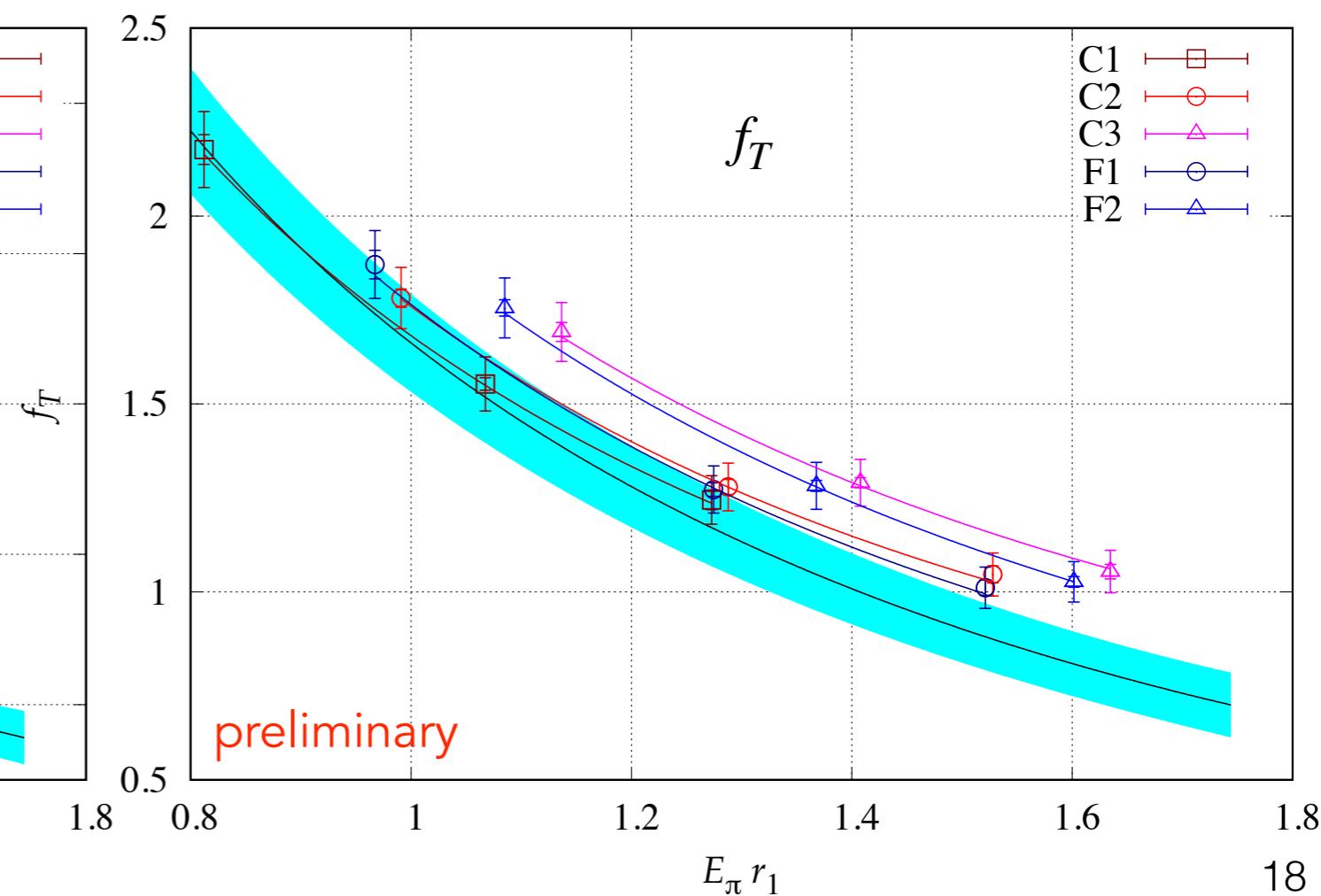
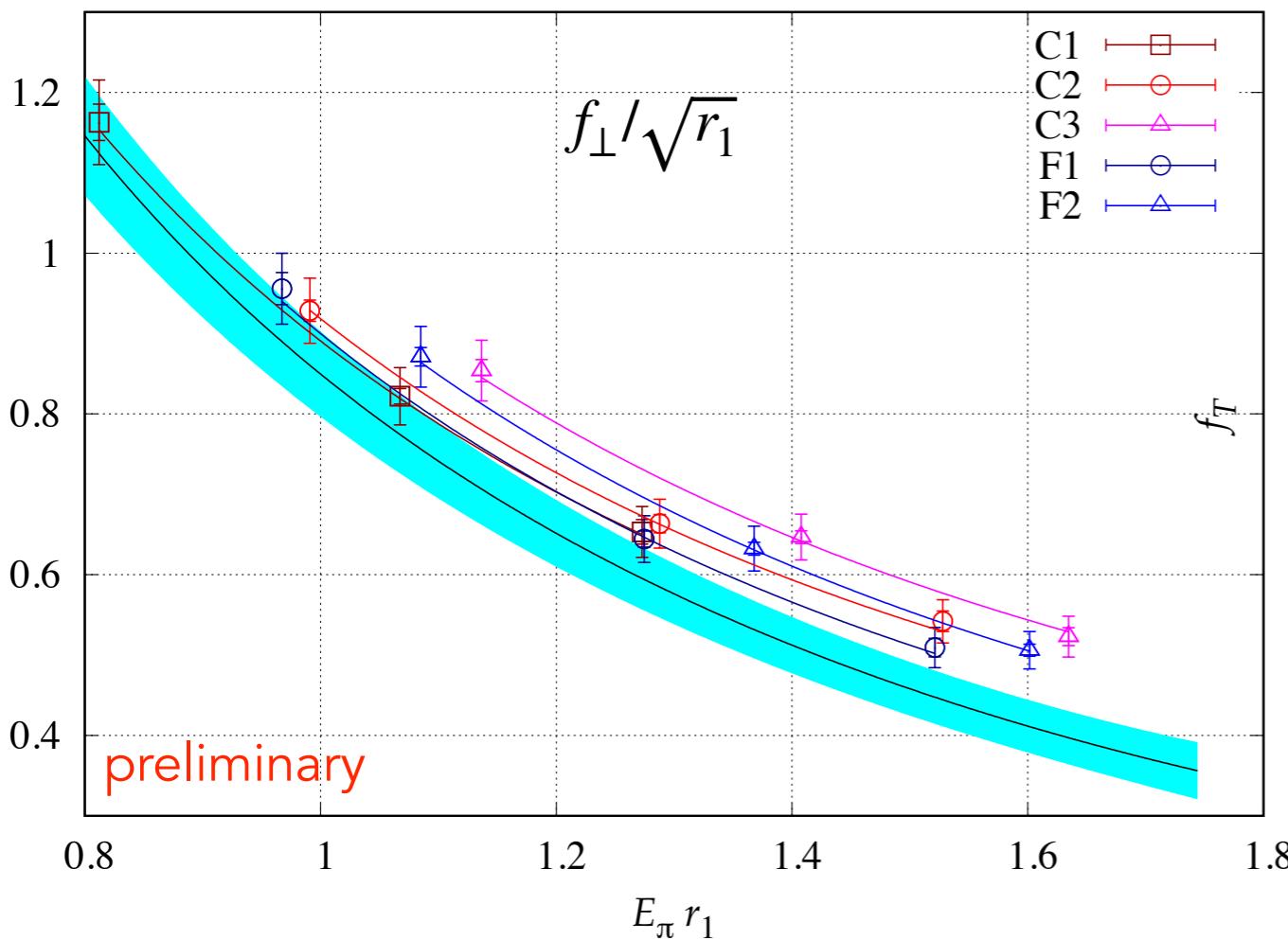
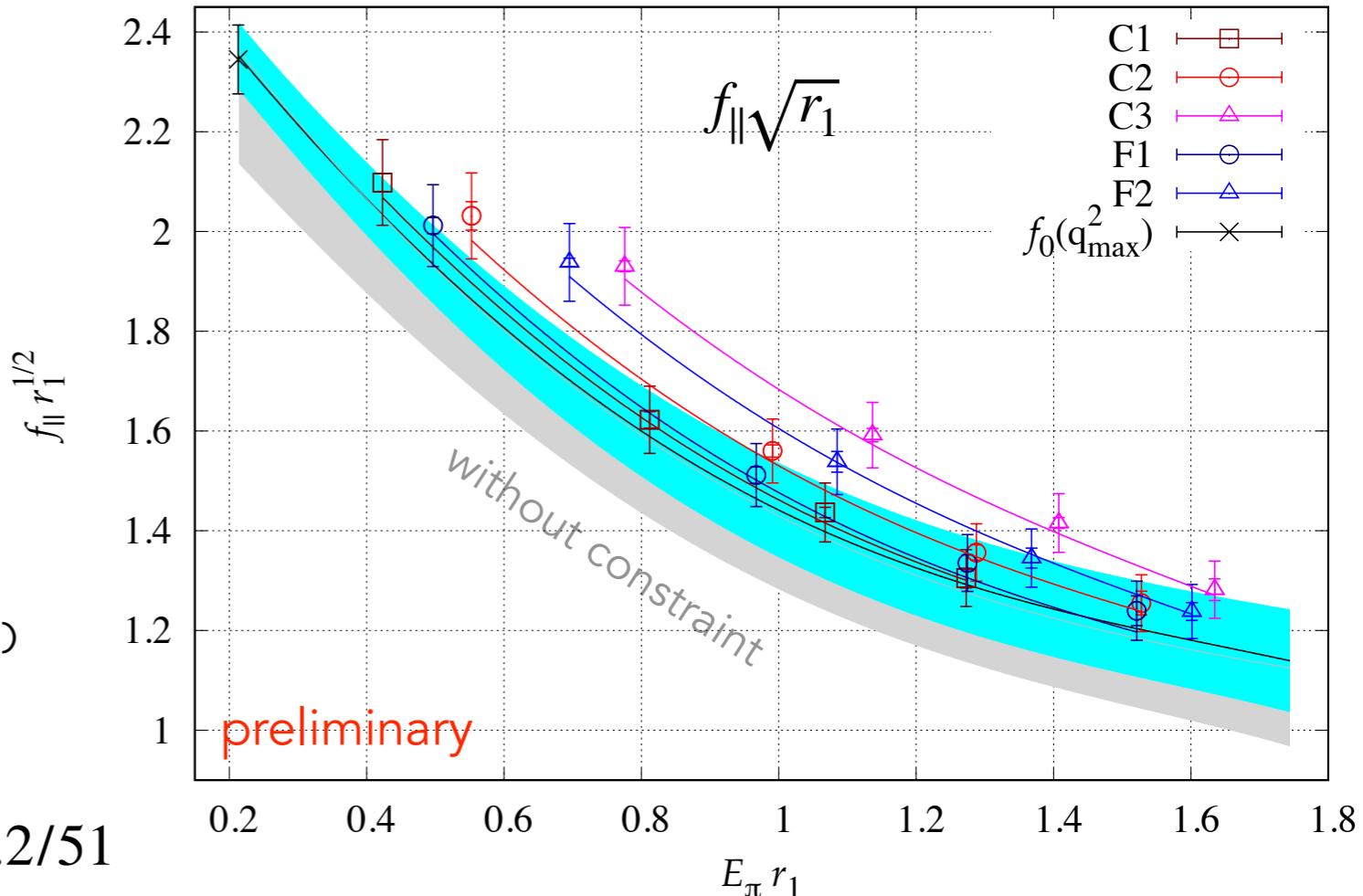
- analytic terms through NNLO
- impose constraint

$$f_0(q_{\max}^2) = 1.120(33)$$

Colquhoun et al, PRD 93, 034502 (2016)

translated into a constraint on f_{\parallel}

- simultaneous, Bayesian fit: $\chi^2/\text{dof} = 24.2/51$



Kinematic Extrapolation ...

Bourrely, Lellouch, & Caprini, PRD 79, 013008 (2009); D82, 099902(E) (2010)

- Use synthetic data from chiral and continuum extrapolation in standard BCL z -expansion

$$f_0(q^2)P_0(q^2) = \sum_{j=0}^J A_j^{(0)} z(q^2)^j \quad f_{+,T}(q^2)P_{+,T}(q^2) = \sum_{j=0}^{J-1} A_j^{(+,T)} \left(z(q^2)^j - (-1)^{j-J} \frac{j}{J} z(q^2)^J \right)$$

where Blaschke factors account for possible poles

$$P_i(q^2) = 1 - \frac{q^2}{M_{i,pole}^2} .$$

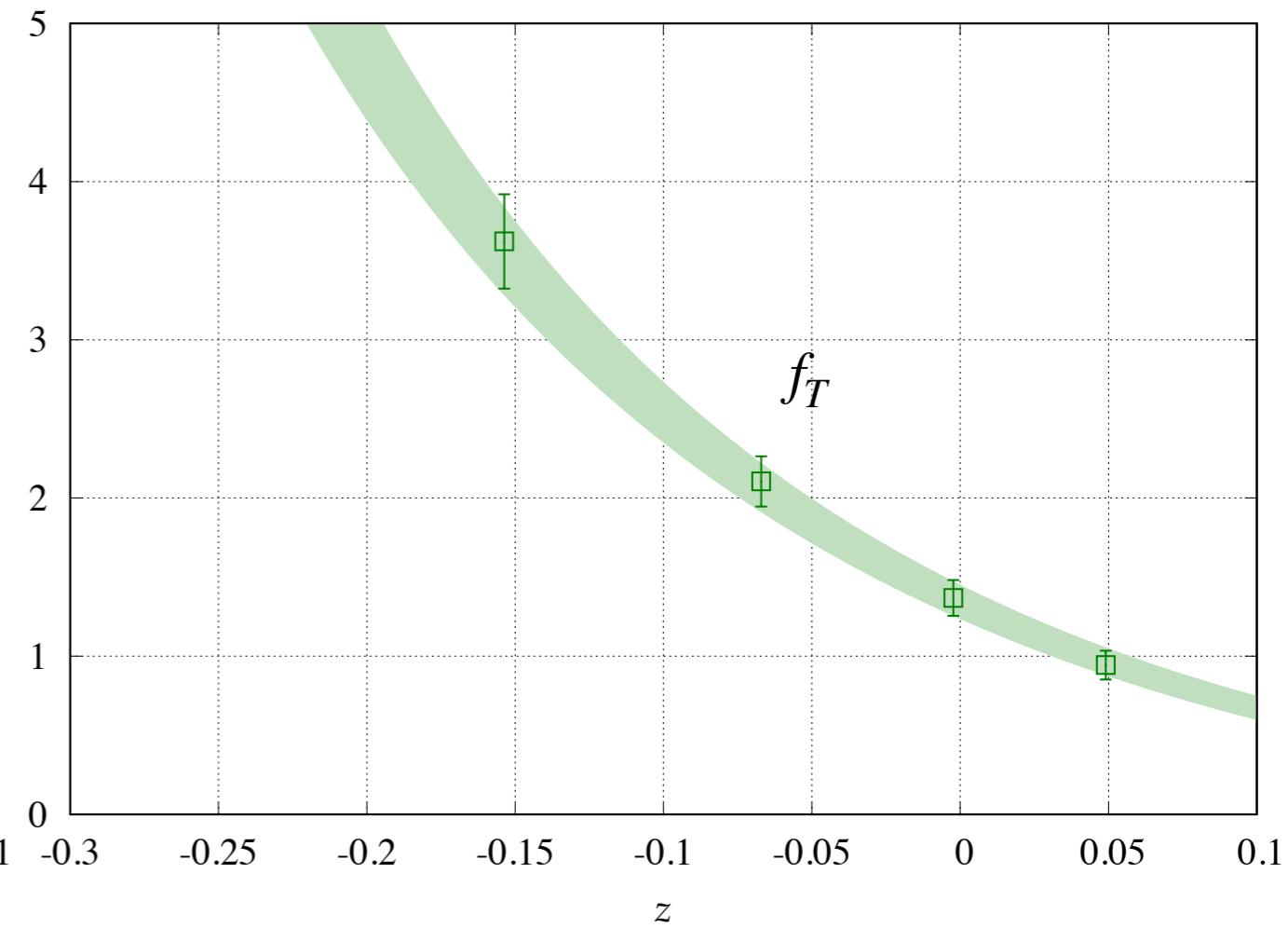
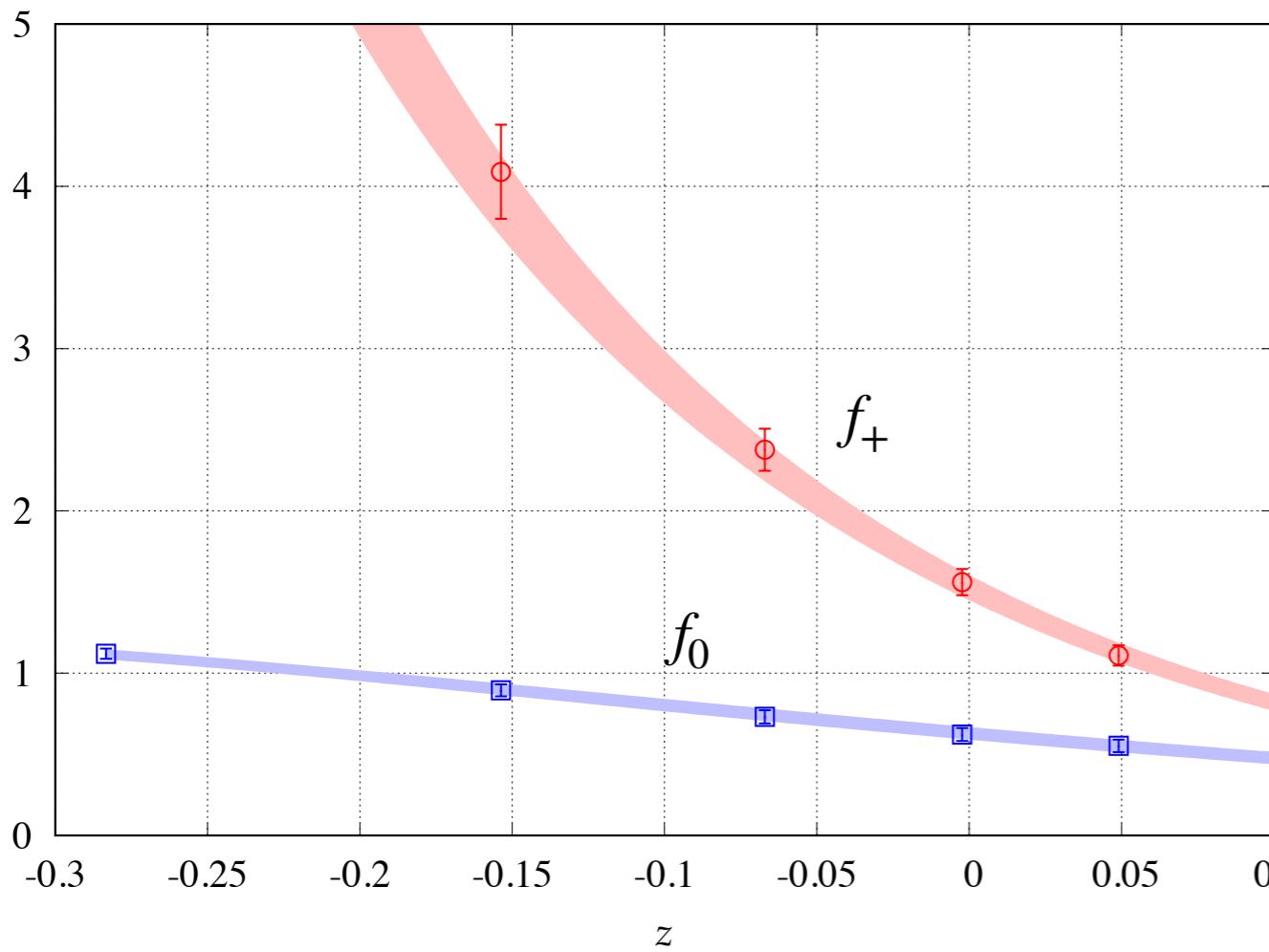
- Additional data points impose the kinematic

$$\text{data } [f_0(0) - f_+(0)] = 0 \pm 10^{-5} ,$$

and the heavy-quark constraint

Beneke et al, NPB 643, 431 (2002)
Hill, PRD 73, 014012 (2006)

$$\text{data } \left[f_T(q^2) - M_B(M_B + M_\pi) \left(\frac{f_+(q^2) - f_0(q^2)}{q^2} \right) \right] \Big|_{q^2=0} = 0 \pm \mathcal{O}\left(\frac{\Lambda}{m_b}\right) .$$

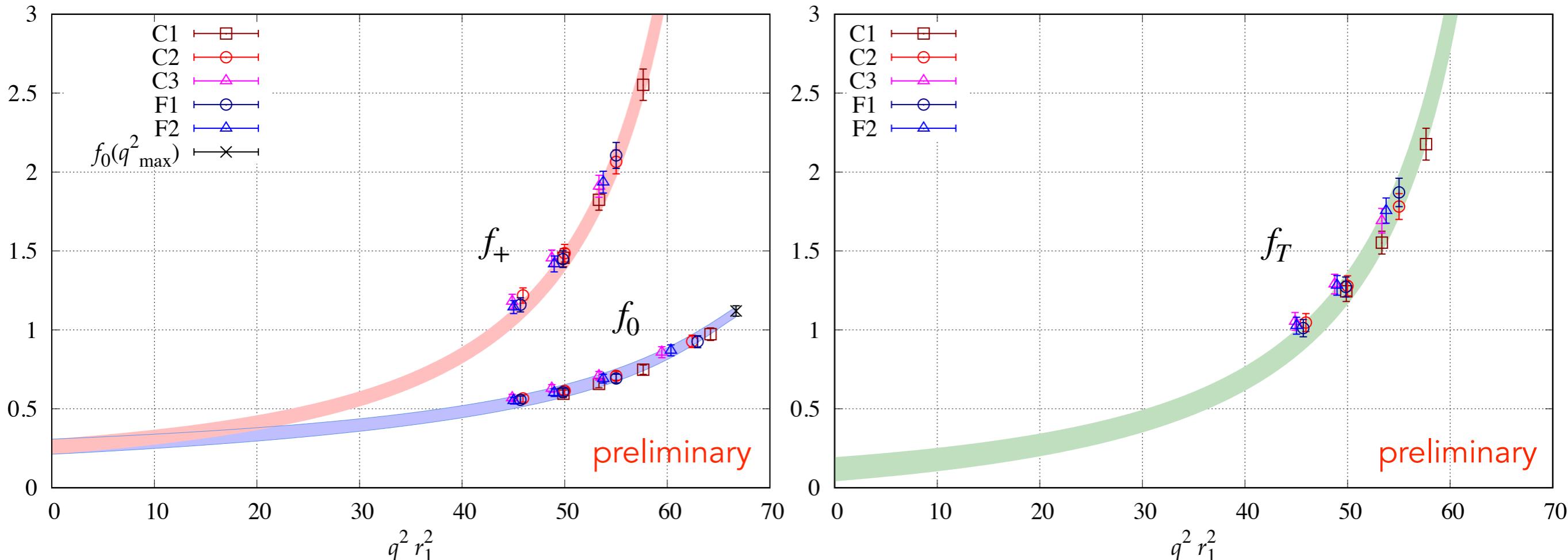


$$f_0(q^2)P_0(q^2) = \sum_{j=0}^J A_j^{(0)} z(q^2)^j$$

$$f_{+,T}(q^2)P_{+,T}(q^2) = \sum_{j=0}^{J-1} A_j^{(+,T)} \left(z(q^2)^j - (-1)^{j-J} \frac{j}{J} z(q^2)^J \right)$$

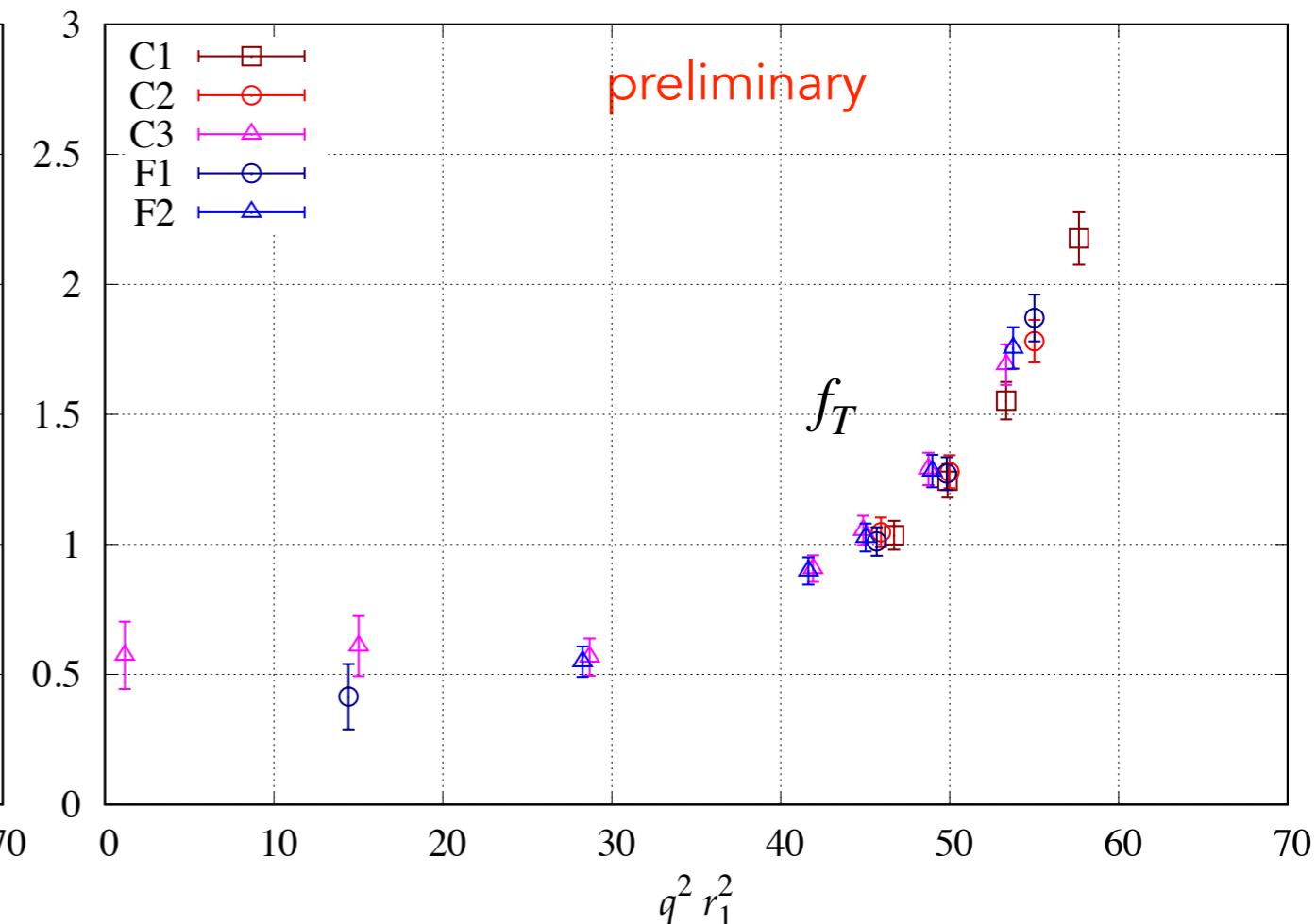
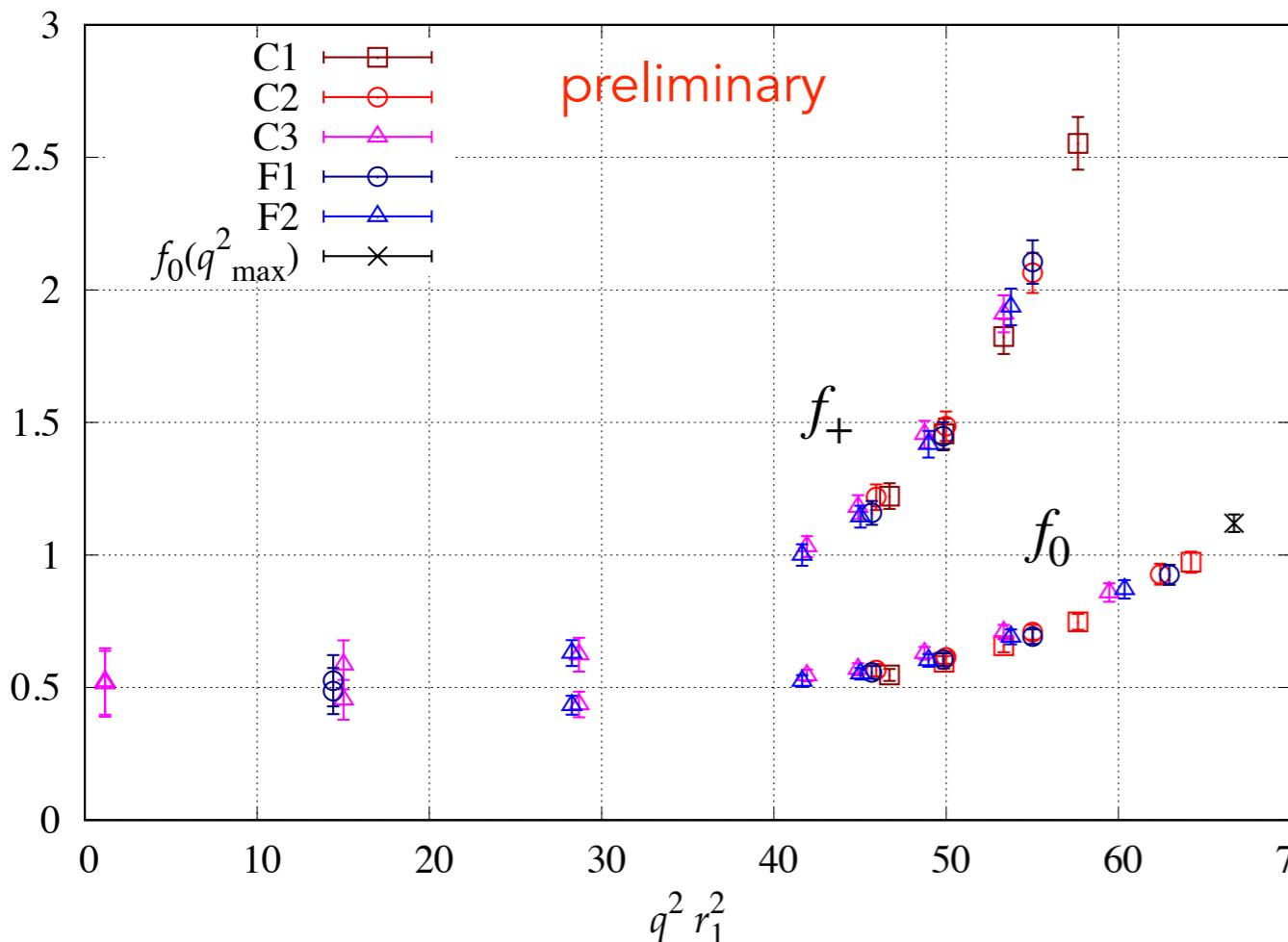
- BCL z expansion embeds large- q^2 constraint on $f_{+,T}$
- prior[$A_j^{(0,+T)}$] = 0 ± 1
- simultaneous, Bayesian fit though $O(z^4)$: $\chi^2/\text{dof} = 9.8/14$

Results via standard method ...



- data for $|(\mathbf{0}, \mathbf{0}, \mathbf{0})| \leq \frac{p_\pi L}{2\pi} \leq |(\mathbf{1}, \mathbf{1}, \mathbf{1})|$
- chiral and continuum extrapolation via SU(2) hard pion ChPT with discretization effects
- kinematic extrapolation via BCL z expansion with constraints
 - kinematic $f_0(0) = f_+(0)$
 - large q^2 scaling of $f_{+,T}(q^2)$
 - heavy quark relation for $f_T(0)$

Next step ...



- include large momentum data
- kinematics: trade E_π for z

$$f(E_\pi) = (1 + [\text{logs}]) \mathcal{K}(E_\pi) \longrightarrow f(z) = (1 + [\text{logs}]) \mathcal{K}(z)$$

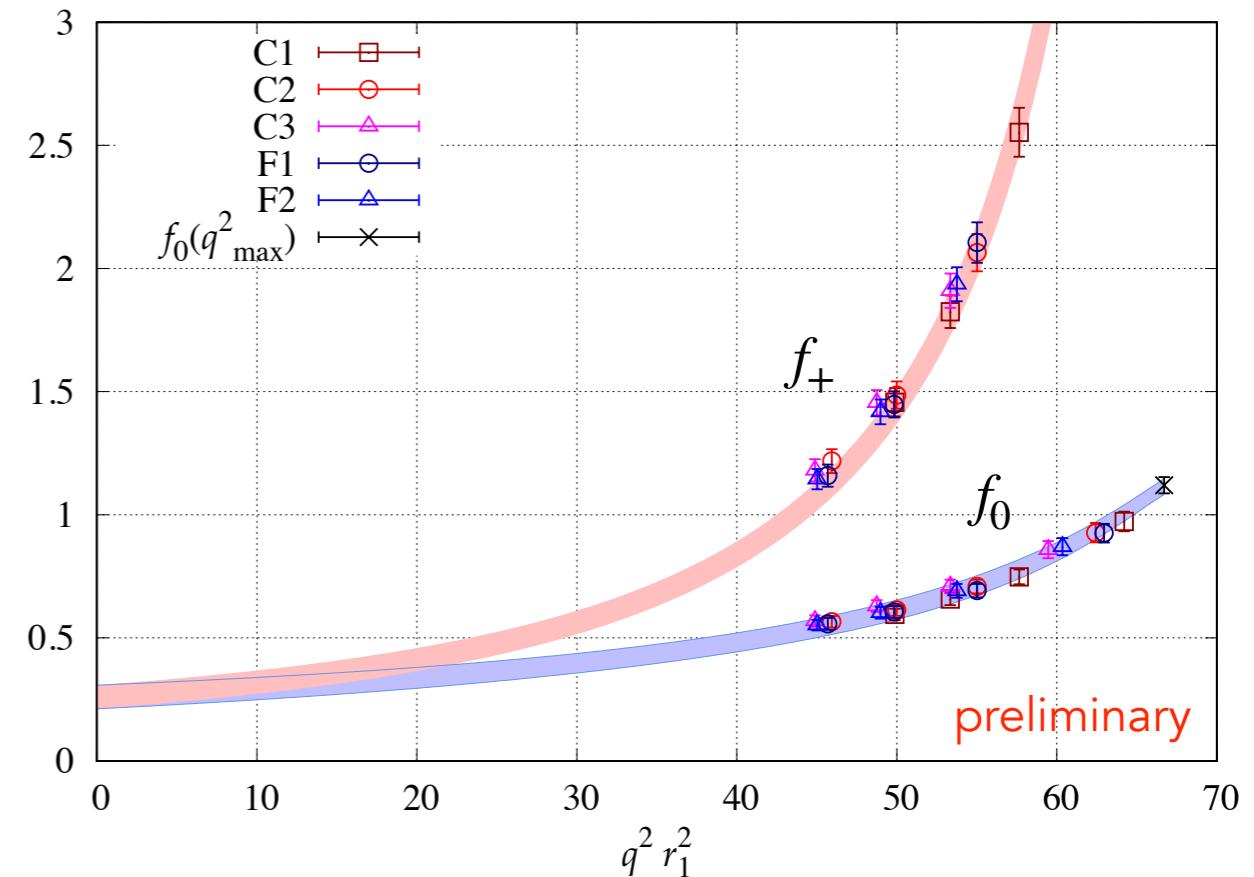
Bouchard, Lepage, Monahan, Na, and Shigemitsu, PRD 90, 054506 (2014)

- work in progress

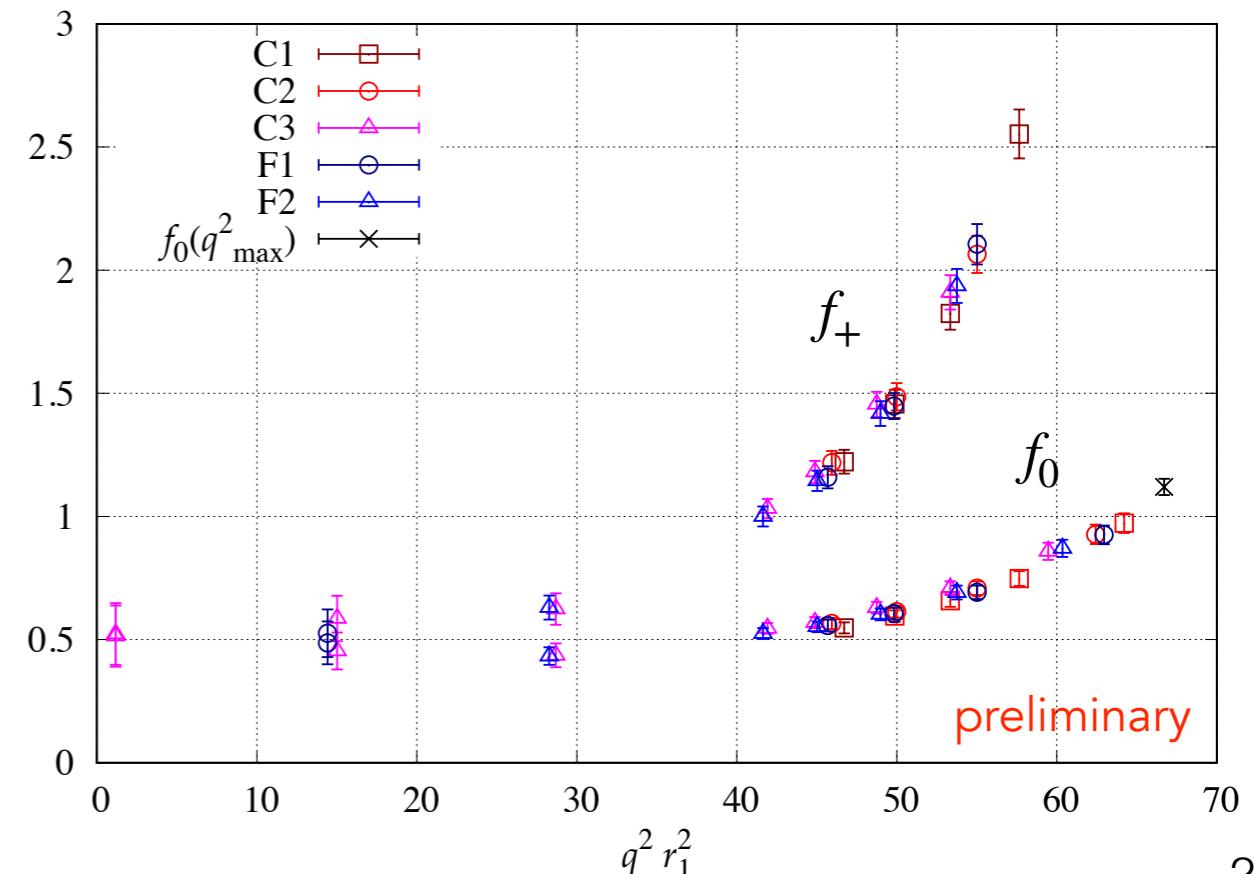
- Introduction
- simulation & correlator construction
- correlator analysis
- chiral, continuum, & kinematic extrapolations
- summary/outlook

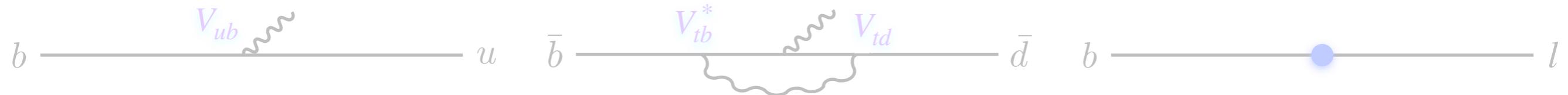
Summary/outlook ...

- preliminary results for $f_{0,+T}$
 - using subset of data at "low" momenta
 - SU(2) hard pion ChPT
 - BCL z expansion



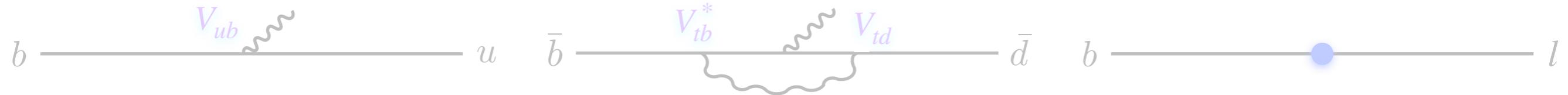
- including "large" momentum data
 - lattice data for $q^2 = 0.4 \text{ GeV}^2$
 - hard pion ChPT trading E for z



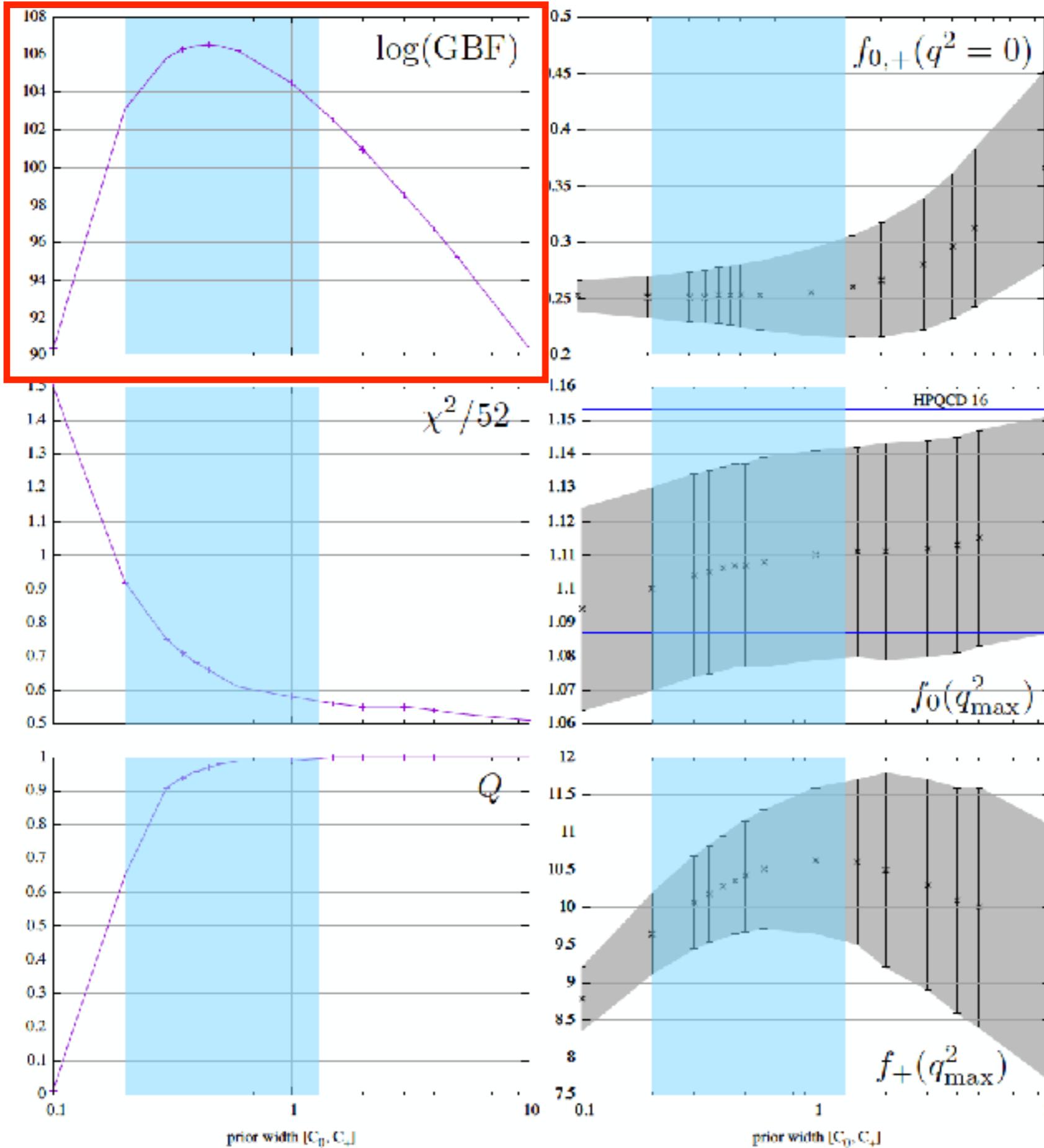


Thank you.

Back up slides ...



Empirical Bayes study of z -expansion coefficients ...



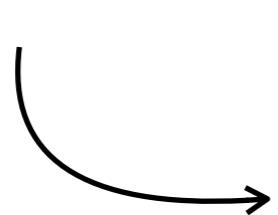
Empirical Bayes study:

- vary prior width for both $A_n(0, +)$
- GBF peaks at width of 0.45
- for $\log(\text{GBF})$ within 3 of the max, width ranges from: 0.2 - 1.1

Hard pion ChPT motivated modified z expansion ...

- hard pion ChPT

$$f(E_\pi) = (1 + [\text{logs}]) \mathcal{K}(E_\pi)$$

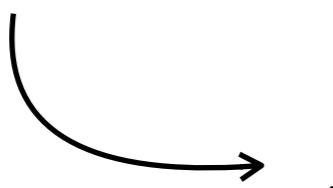


$$f_{\parallel} = \frac{\kappa_{\parallel}}{f_\pi} (1 + [\text{logs}]) (1 + \mathcal{C}_{\parallel} + \mathcal{E}_{\parallel} + \mathcal{D}_{\parallel})$$

$$f_{\perp,T} = \frac{g_{BB^*\pi} \kappa_{\perp,T}}{f_\pi(E_\pi + \Delta^*)} (1 + [\text{logs}]) (1 + \mathcal{C}_{\perp,T} + \mathcal{E}_{\perp,T} + \mathcal{D}_{\perp,T})$$

- hard pion ChPT modified z expansion

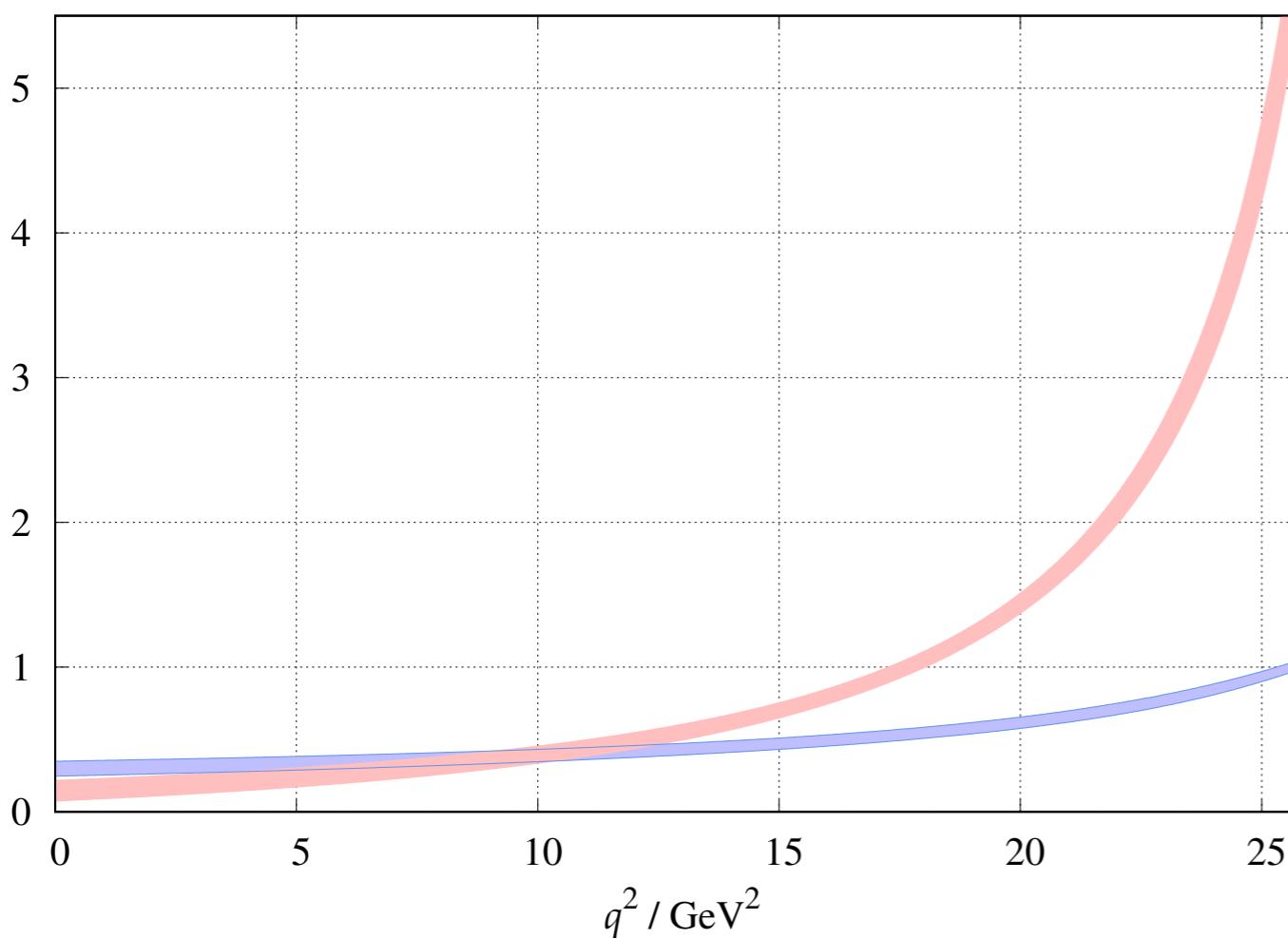
$$f(z) = (1 + [\text{logs}]) \mathcal{K}(z)$$



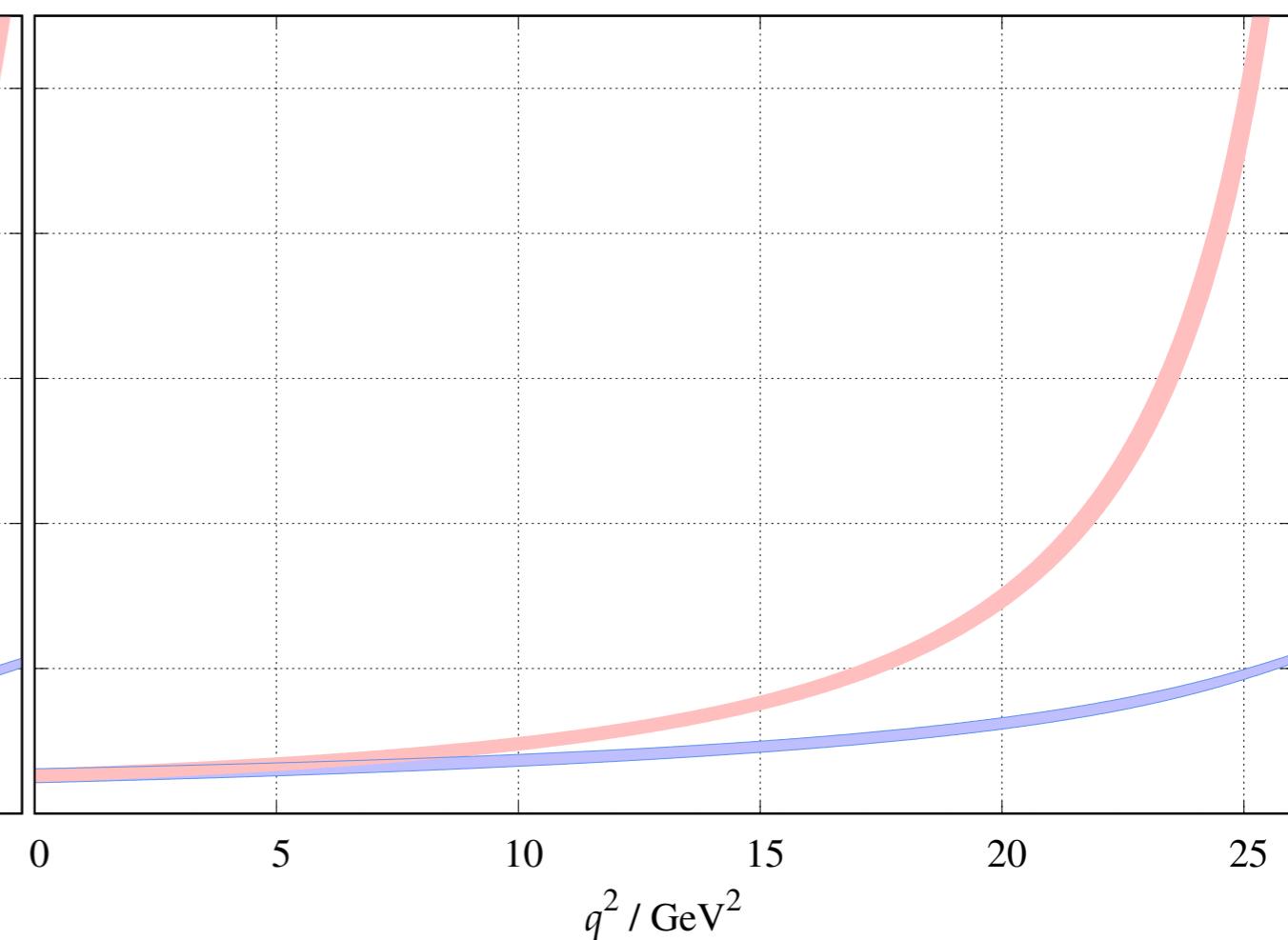
$$f_{0,+T}(z) = \frac{1}{P_{0,+T}(q^2)} (1 + [\text{logs}]) \sum_j C_j^{(0,+T)} (1 + \mathcal{C}_j^{(0,+T)} + \mathcal{D}_j^{(0,+T)}) z^j$$

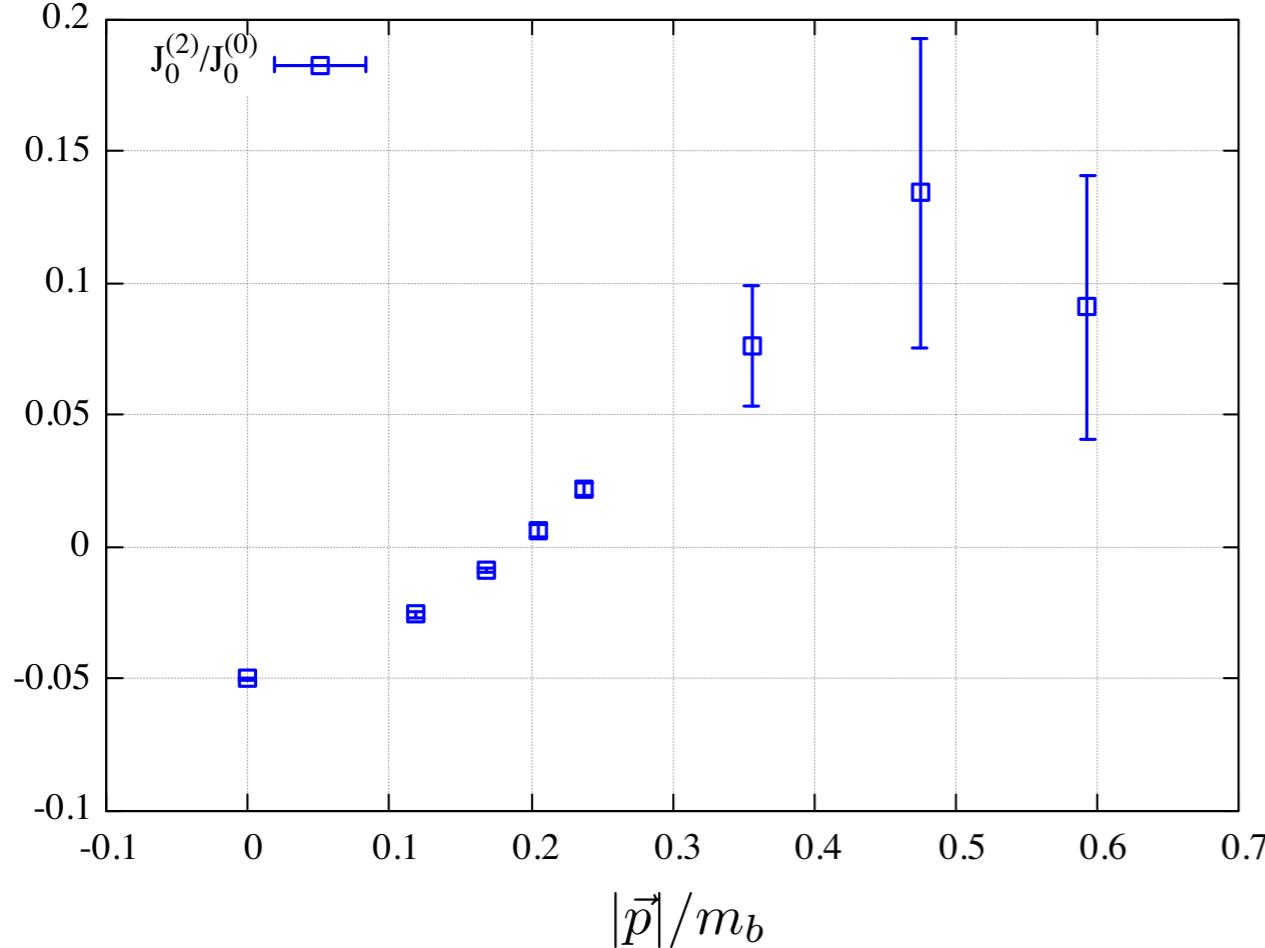
kinematic or heavy quark constraint ...

without



with





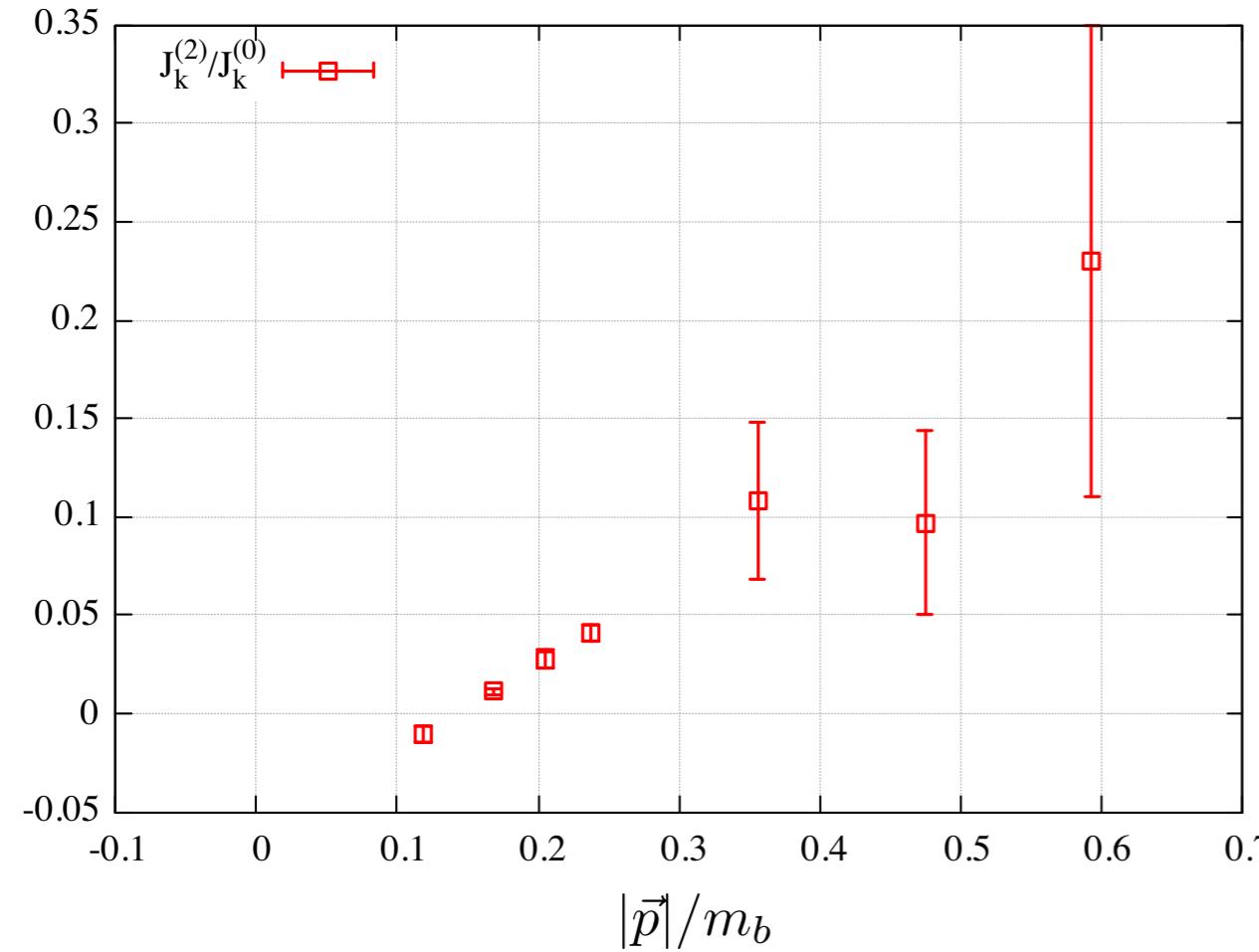
ensemble C3

$$am_b = 2.650$$

$$\alpha_s = 0.3047$$

$$a|\vec{p}|_{\max} = \pi/2$$

$$\begin{aligned}\tilde{m}_{\parallel} &\sim \mathcal{O}\left(\frac{J_0^{(2)}}{J_0^{(0)}}\right) \times \mathcal{O}\left(\alpha_s \frac{|\vec{p}|^2}{m_b^2}, \alpha_s^2 \frac{|\vec{p}|}{m_b}\right) \\ &\sim 0.15 \times (0.11, 0.055) \\ \implies \text{prior}[\tilde{m}_{\parallel}] &= 0(0.02)\end{aligned}$$

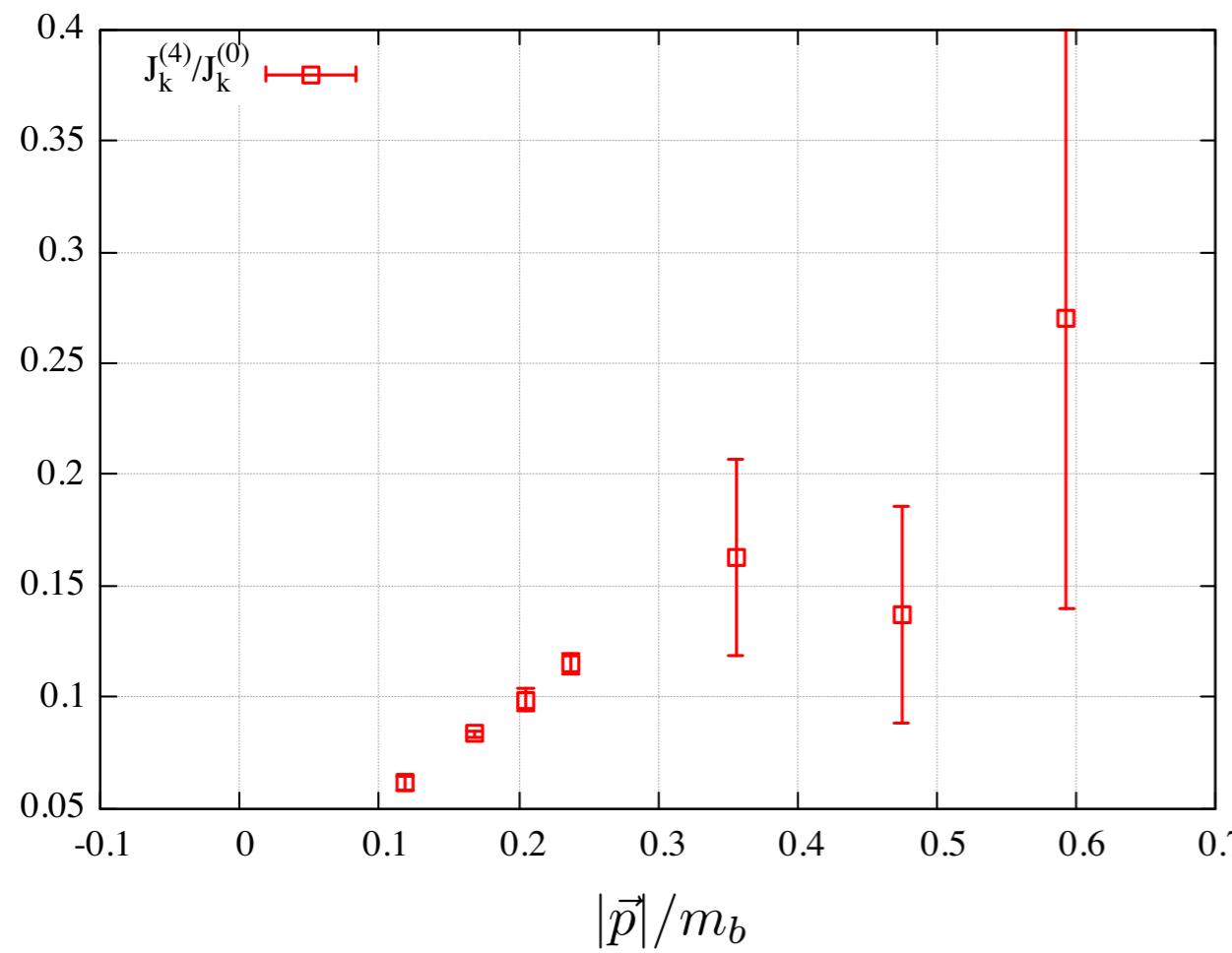


ensemble C3

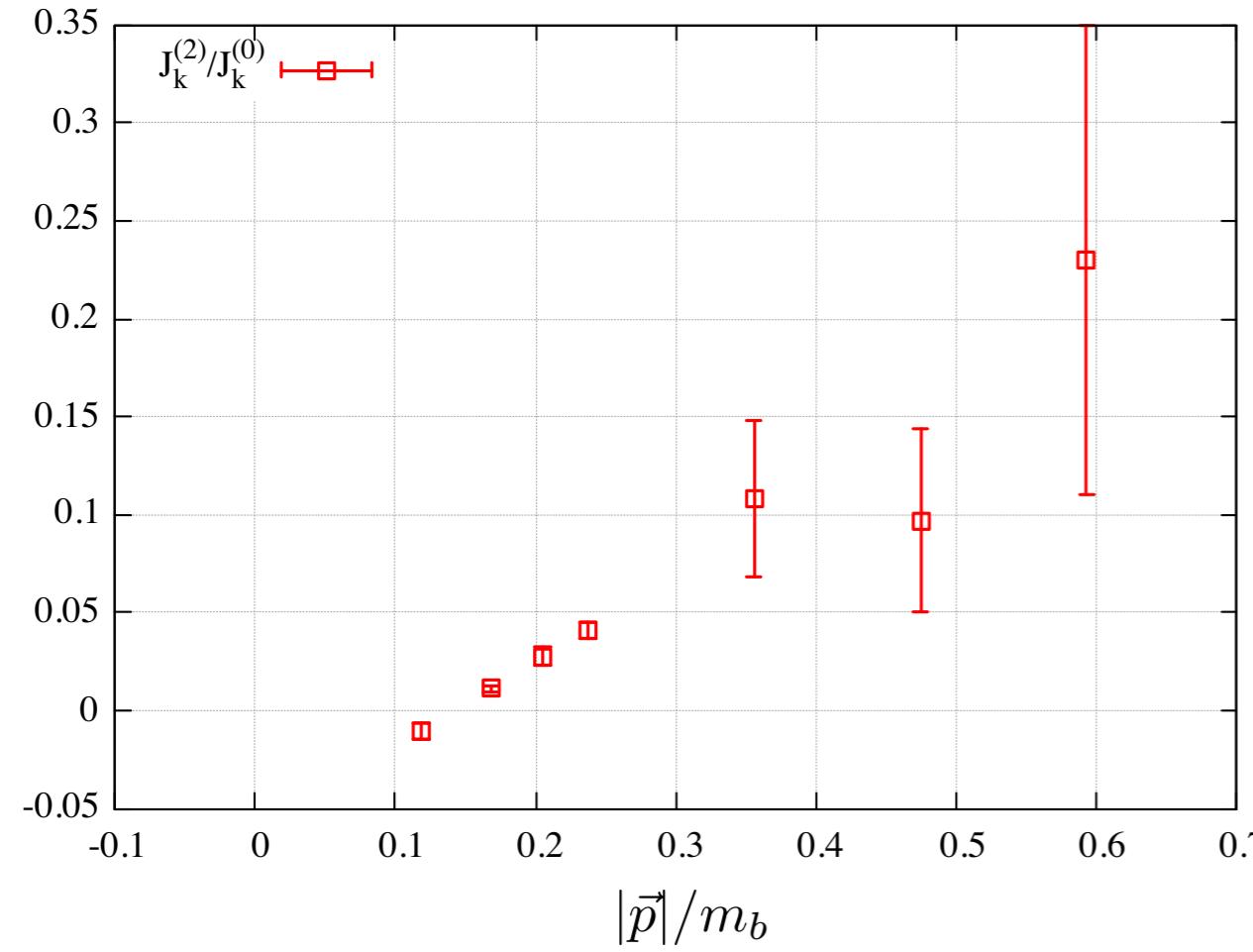
$$am_b = 2.650$$

$$\alpha_s = 0.3047$$

$$a|\vec{p}|_{\max} = \pi/2$$



$$\begin{aligned}
 \tilde{m}_\perp &\sim \mathcal{O}\left(\frac{J_k^{(2)}}{J_k^{(0)}}, \frac{J_k^{(4)}}{J_k^{(0)}}\right) \times \\
 &\quad \mathcal{O}\left(\alpha_s \frac{|\vec{p}|^2}{m_b^2}, \alpha_s^2 \frac{|\vec{p}|}{m_b}\right) \\
 &\sim (0.25, 0.27) \times (0.11, 0.055) \\
 \implies \text{prior}[\tilde{m}_\perp] &= 0(0.03)
 \end{aligned}$$

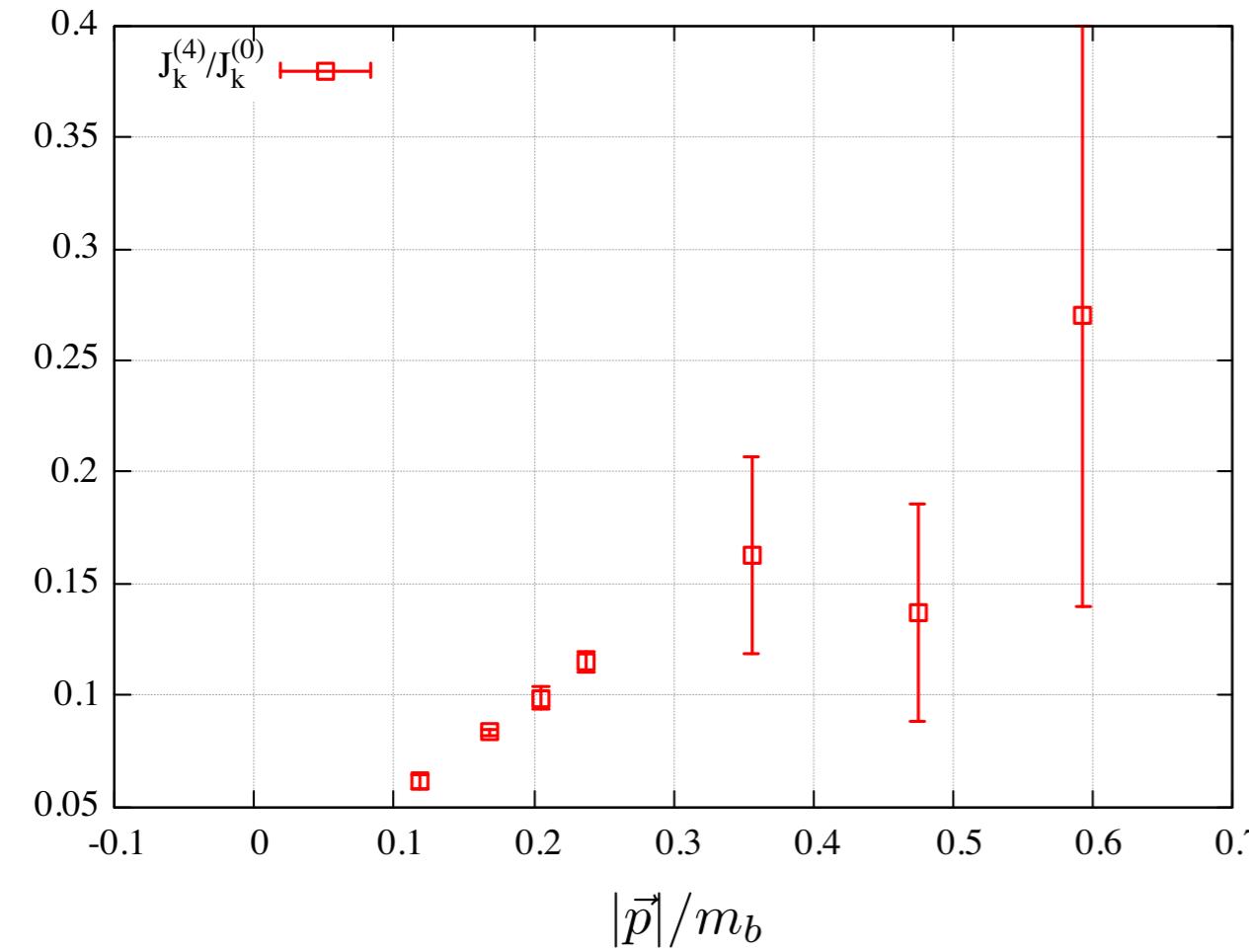


ensemble C3

$$am_b = 2.650$$

$$\alpha_s = 0.3047$$

$$a|\vec{p}|_{\max} = \pi/2$$



$$\tilde{m}_T \sim \mathcal{O}\left(\frac{J_k^{(2)}}{J_k^{(0)}}, \frac{J_k^{(4)}}{J_k^{(0)}}\right) \times$$

$$\mathcal{O}\left(\alpha_s \frac{|\vec{p}|}{m_b}\right)$$

$$\sim (0.25, 0.27) \times 0.18$$

$$\implies \text{prior}[\tilde{m}_T] = 0(0.05)$$

