

A Lattice Study of Renormalons in Asymptotically Free Sigma Models

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Motivation

- ▶ Perturbative expansion for observable O in QFTs

$$\langle O \rangle = \sum_{n=0}^{\infty} a_n g^n, \quad g \dots \text{coupling}$$

- ▶ In general divergent (asymptotic series) with $a_n \sim n!$
- ▶ Assign a unique value to $\langle O \rangle$?
- ▶ Borel summation
 - ▶ If a_n are alternating in sign ($a_n \sim (-1)^n n!$)
 - ▶ Otherwise: Pole in the Borel transformation \rightarrow Renormalon
- ▶ Renormalon gives rise to ambiguity $\sim e^{-\frac{1}{g}}$
(Resurgence, OPE, ...)
- ▶ Interesting physics in the behaviour of expansion coefficients

Sigma Models in 2D

- The $\text{CP}(N-1)$ Model

- Lattice action:

$$\mathcal{S} = -2\beta N \sum_{x,\nu} \text{Re} (n^\dagger(x) U_\nu(x) n(x+\nu))$$

- Fields $n \in \mathbb{C}^N$, constraint: $n^\dagger n = 1$, links $U_\nu \in U(1)$ (auxiliary)
- The Principal Chiral Model (PCM(N))

- Lattice action

$$\mathcal{S} = -2\beta N \sum_{x,\nu} \text{Re} \text{Tr} (U(x) U(x+\nu)^\dagger)$$

- Fields $U \in \mathbb{C}^{N \times N}$, constraint: $U \in SU(N)$

Sigma Models in 2D

- ▶ Features
 - ▶ Asymptotic freedom $CP(N-1)$, $PCM(N)$
 - ▶ Non-perturbative mass gap $CP(N-1)$, $PCM(N)$
 - ▶ Confinement $CP(N-1)$, $PCM(N)$
 - ▶ Instantons $CP(N-1)$
 - ▶ ...
- ▶ Study interesting physics in a (relatively) simple setting

Numerical Stochastic Perturbation Theory (NSPT)

- ▶ Ansatz: Expand fields in coupling g up to order M

(Di Renzo et al. *Nucl. Phys.* B426, 675–685) [hep-lat/9405019]

$$\phi = \phi_0 + \phi_1 g + \phi_2 g^2 + \cdots + \phi_M g^M = \sum_{n=0}^M \phi_n g^n$$

- ▶ Define **sum** and **product** for finite series

$$\phi + \psi = \sum_{n=0}^M (\phi_n + \psi_n) g^n, \quad \phi \cdot \psi = \sum_{n=0}^M \left(\sum_{l=0}^n \phi_l \psi_{l-n} \right) g^n$$

- ▶ Define functions via (finite) Taylor series
- ▶ Numerical cost $\sim M^2$

Numerical Stochastic Perturbation Theory (NSPT)

- ▶ MC with accept/reject step not expandable in g !
- ▶ Introduce additional dimension: “stochastic time” τ
- ▶ Langevin equation describes τ -dependence

$$\frac{\partial \phi(x, \tau)}{\partial \tau} = -\frac{\delta \mathcal{S}[\phi(x, \tau)]}{\delta \phi(x, \tau)} + \eta(x, \tau)$$

- ▶ Gaussian noise $\eta(x, \tau)$ with

$$\langle \eta(x, \tau) \rangle_\eta = 0, \quad \langle \eta(x, \tau) \eta(x', \tau') \rangle_\eta = 2\delta(x-x')\delta(\tau-\tau')$$

Numerical Stochastic Perturbation Theory (NSPT)

- ▶ Expectation values $\langle \cdots \rangle_\eta$ with respect to the noise

$$\langle \cdots \rangle_\eta = \frac{\int D[\eta] (\cdots) e^{-\frac{1}{4} \int dx d\tau \eta^2(x, \tau)}}{\int D[\eta] e^{-\frac{1}{4} \int dx d\tau \eta^2(x, \tau)}}$$

- ▶ Asymptotic ($\tau \rightarrow \infty$) distribution $P(\phi) \propto e^{-S[\phi]}$
- ▶ For expectation values of operators:

$$\lim_{\tau \rightarrow \infty} \langle O(\phi(x, \tau)) \rangle_\eta = \langle O(\phi) \rangle = \frac{1}{Z} \int D[\phi] e^{-S[\phi]} O(\phi)$$

- ▶ **Stochastic Quantisation**
 - ▶ no accept/reject step \Rightarrow compatible with expansion in g

Numerical Stochastic Perturbation Theory (NSPT)

- ▶ Discretise stochastic time for numerical treatment:

$$\tau \rightarrow \tau_k = k\epsilon$$

- ▶ Euler approximation ($\mathcal{O}(\epsilon)$) of the Langevin equation:

$$\phi(x, \tau_k + \epsilon) = \phi(x, \tau_k) - \epsilon \frac{\delta \mathcal{S}[\phi(x, \tau_k)]}{\delta \phi(x, \tau_k)} - \sqrt{\epsilon} \eta(x, \tau_k)$$

- ▶ Asymptotic $P(\phi) \propto e^{-\bar{\mathcal{S}}[\phi]}$ with $\bar{\mathcal{S}}[\phi] = \mathcal{S}[\phi] + \mathcal{O}(\epsilon)$

- ▶ Systematic error \rightarrow extrapolation to $\epsilon = 0$

- ▶ Runs for several ϵ values necessary

- ▶ Use Runge-Kutta ($\mathcal{O}(\epsilon^2)$) approximation

(Bali et al. PRD 87, 094517 [1303.3279])

Langevin for constrained fields

- ▶ ϕ subject to constraint \mathcal{C} such that $\mathcal{C}(\phi) = 0$
- ▶ In general: $\mathcal{C}(\phi(x, \tau_k)) = 0 \not\Rightarrow \mathcal{C}(\phi(x, \tau_k + \epsilon)) = 0$
- ▶ Large constraint violations for long Langevin trajectories
- ▶ Modify Langevin update
(E.g., Batrouni et al. *PRD* 32, 2736)
- ▶ Langevin update with modifications respects constraints
(up to numerical errors)

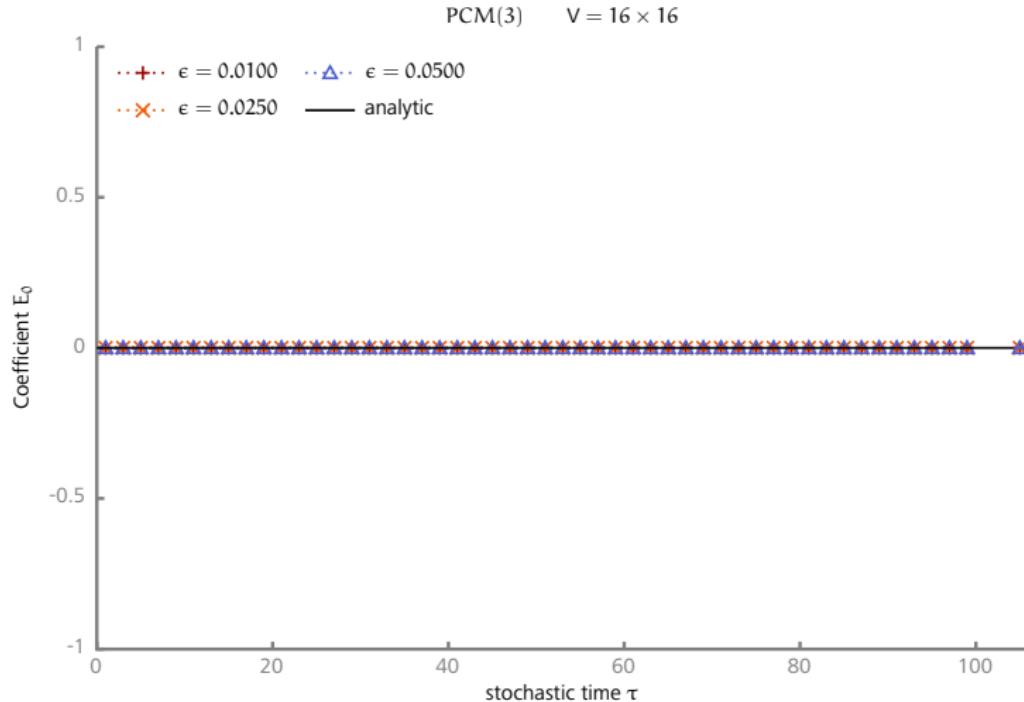
Numerical Results

- ▶ Expand around vacuum
 - ▶ **CP(N-1)**: $n_0(x) \equiv \frac{1}{\sqrt{N}} (1, \dots, 1)^\dagger, \quad u_0(x, v) \equiv 1$
 - ▶ **PCM(N)**: $u_0(x) \equiv 1$
- ▶ Compute coefficients of the energy density E

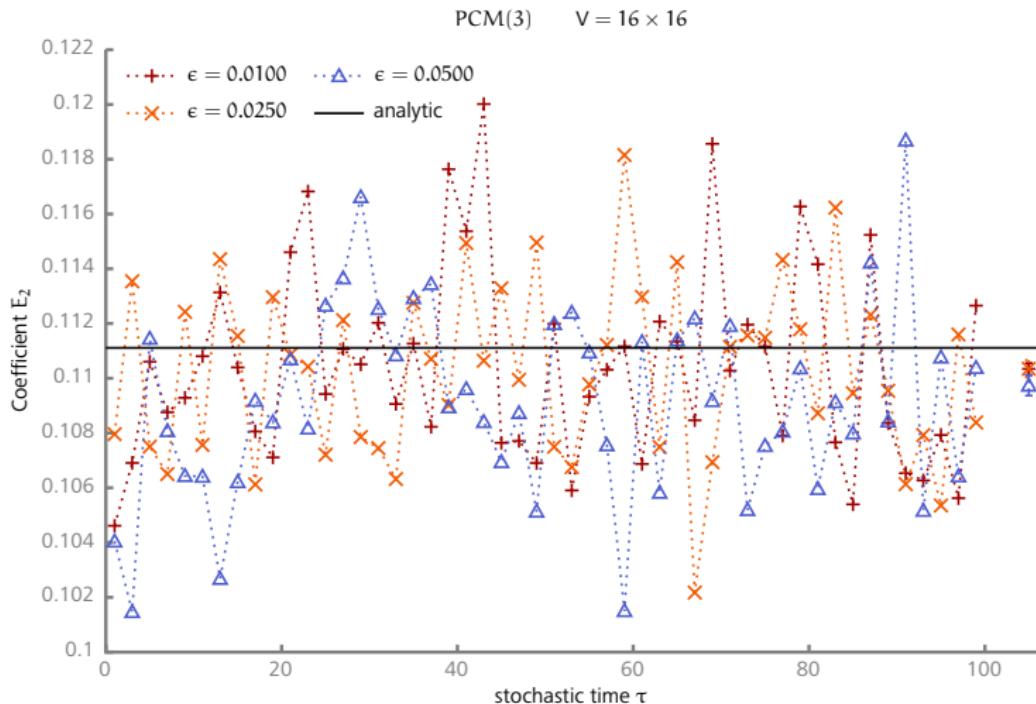
$$E = \sum_{n=0}^{\infty} E_{2n} g^{2n} \quad (\text{Odd terms vanish})$$

- ▶ Analytic expressions known for leading coefficients

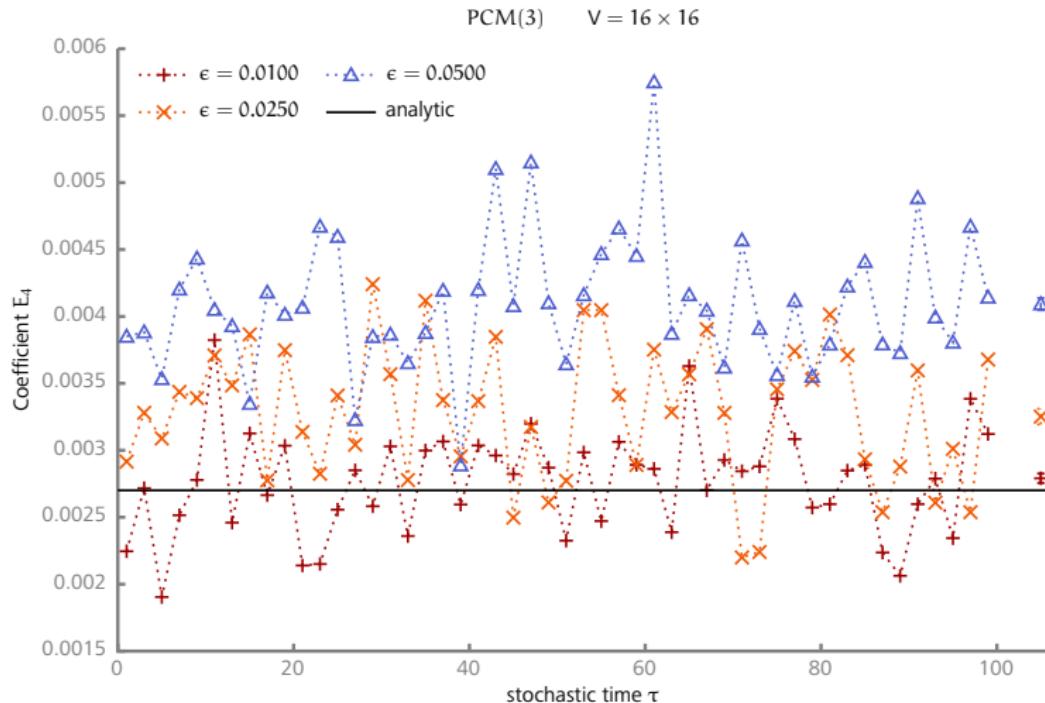
Energy Density Coefficients PCM(3)



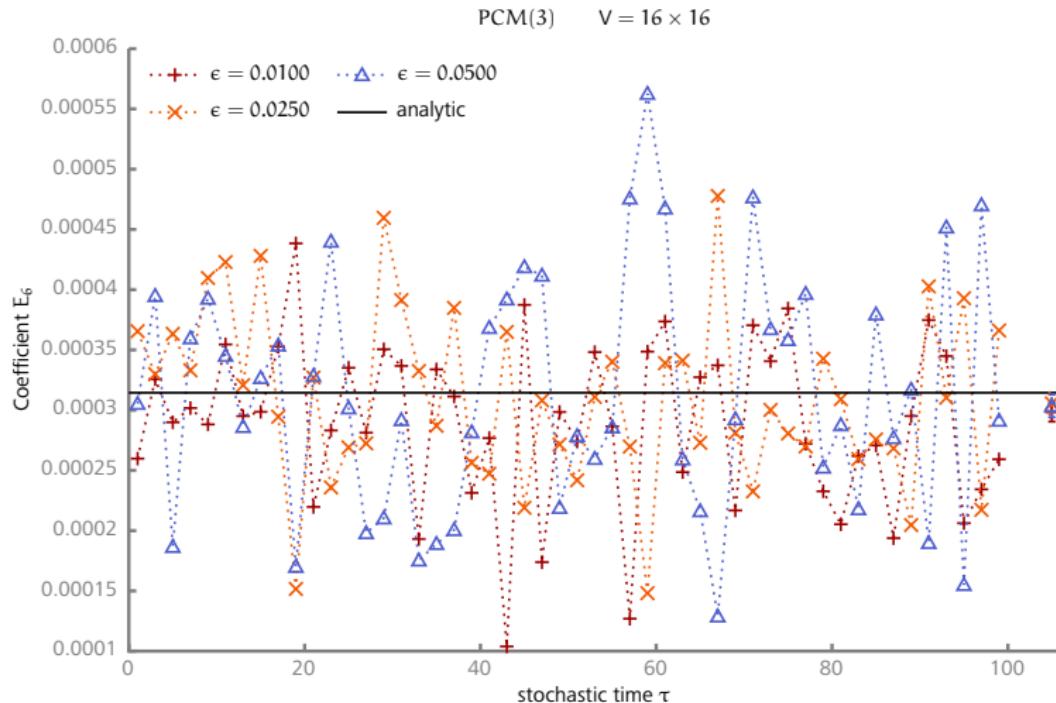
Energy Density Coefficients PCM(3)



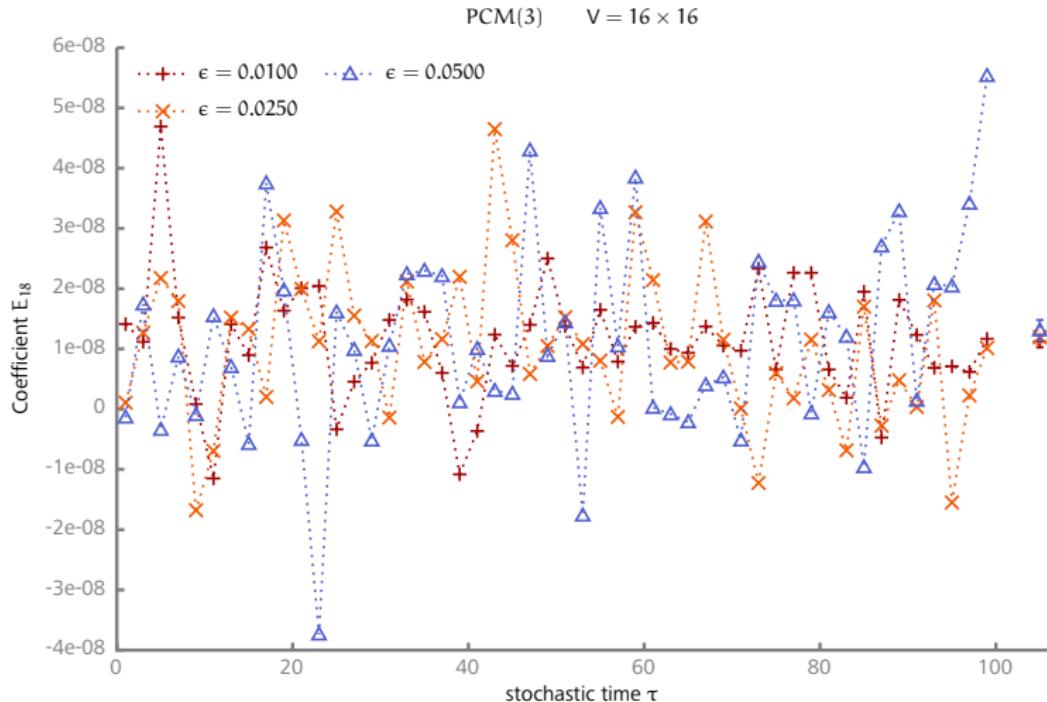
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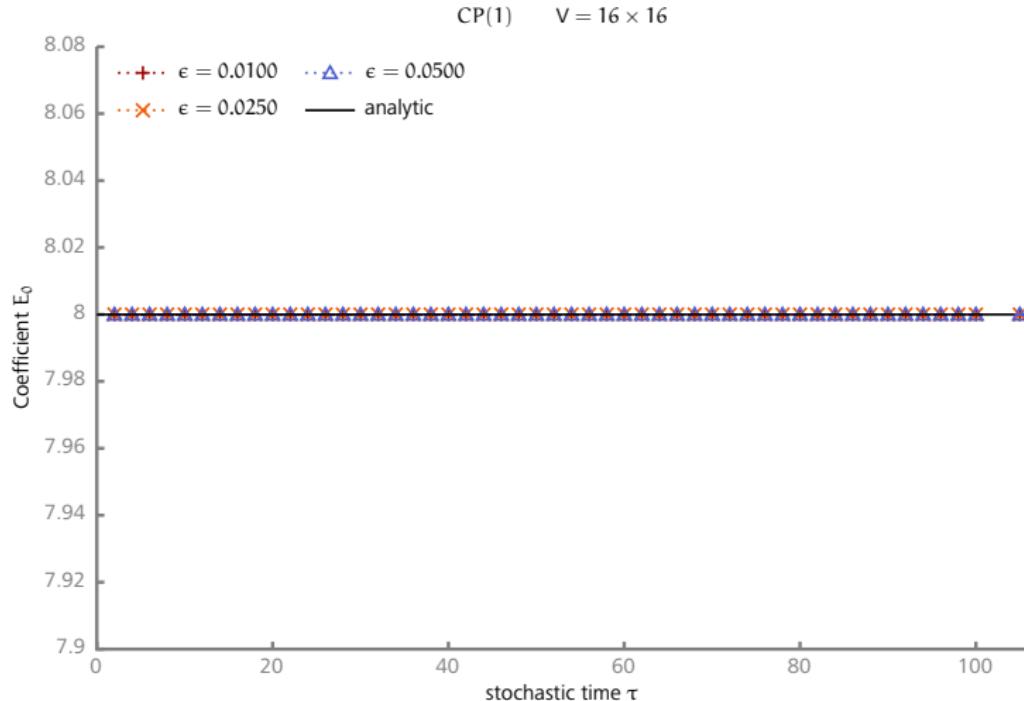
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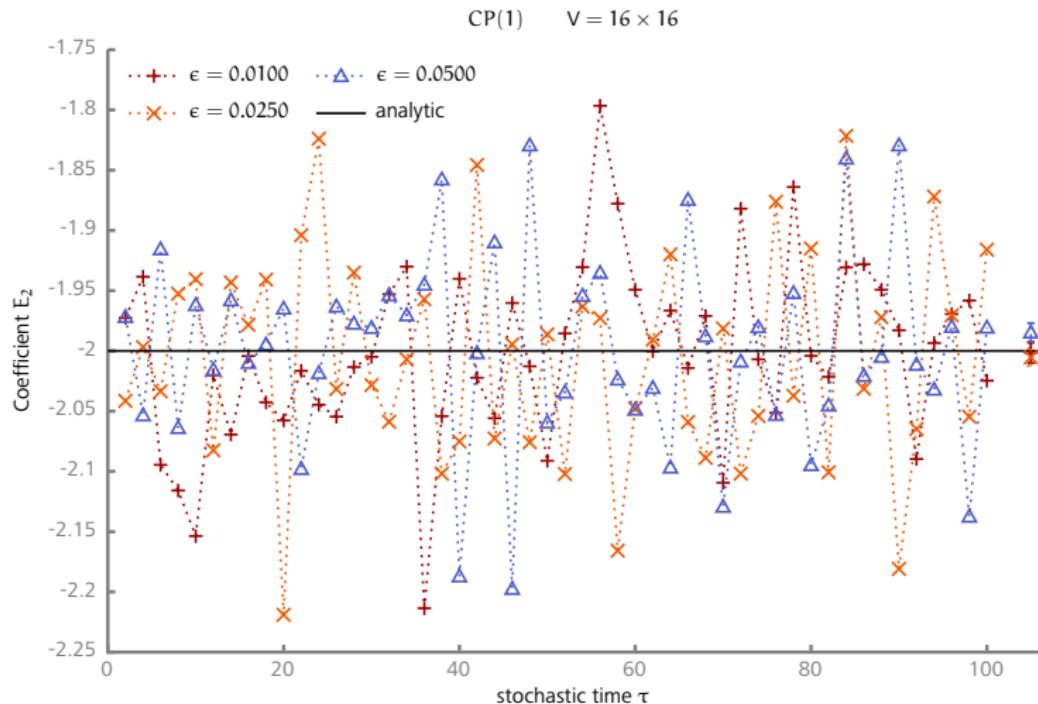
Energy Density Coefficients PCM(3)



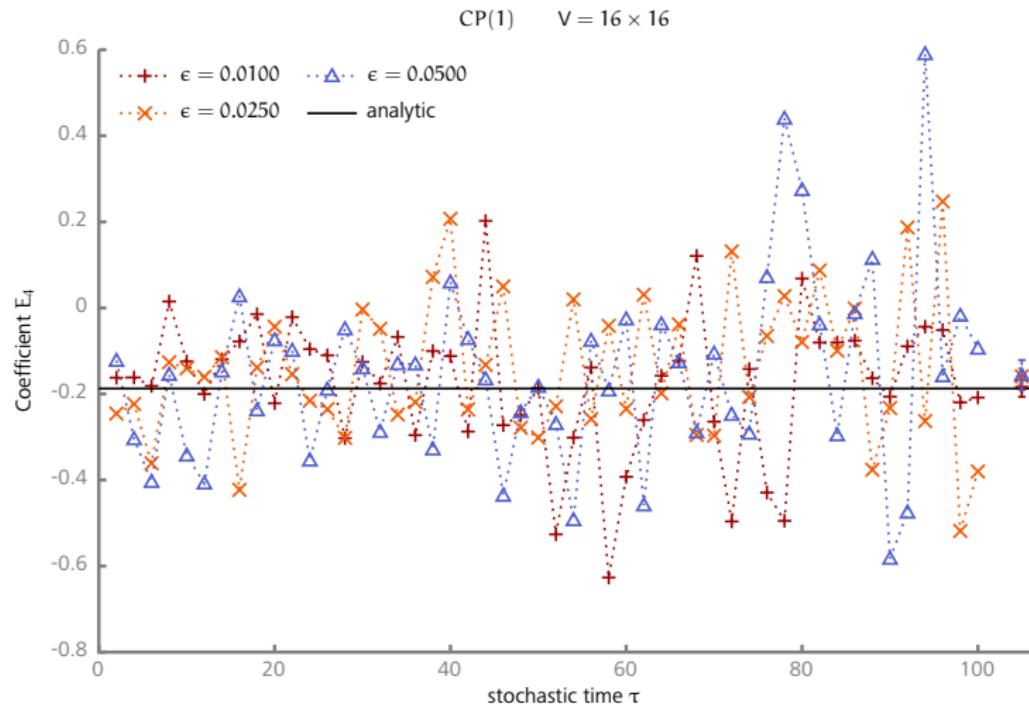
Energy Density Coefficients CP(1)



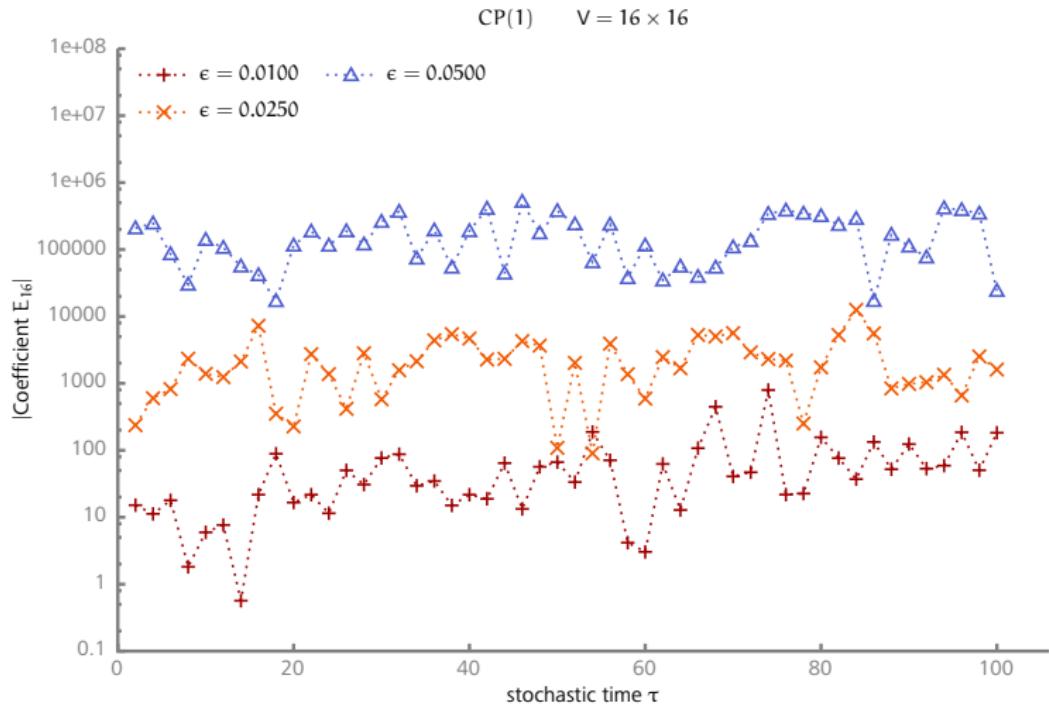
Energy Density Coefficients CP(1)



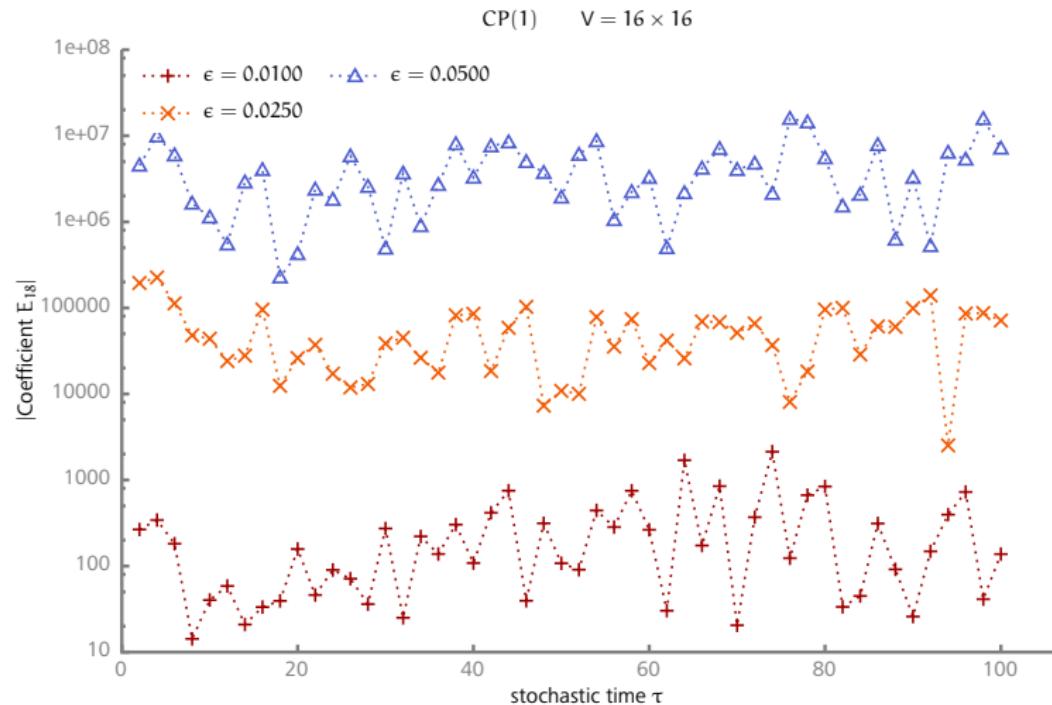
Energy Density Coefficients CP(1)



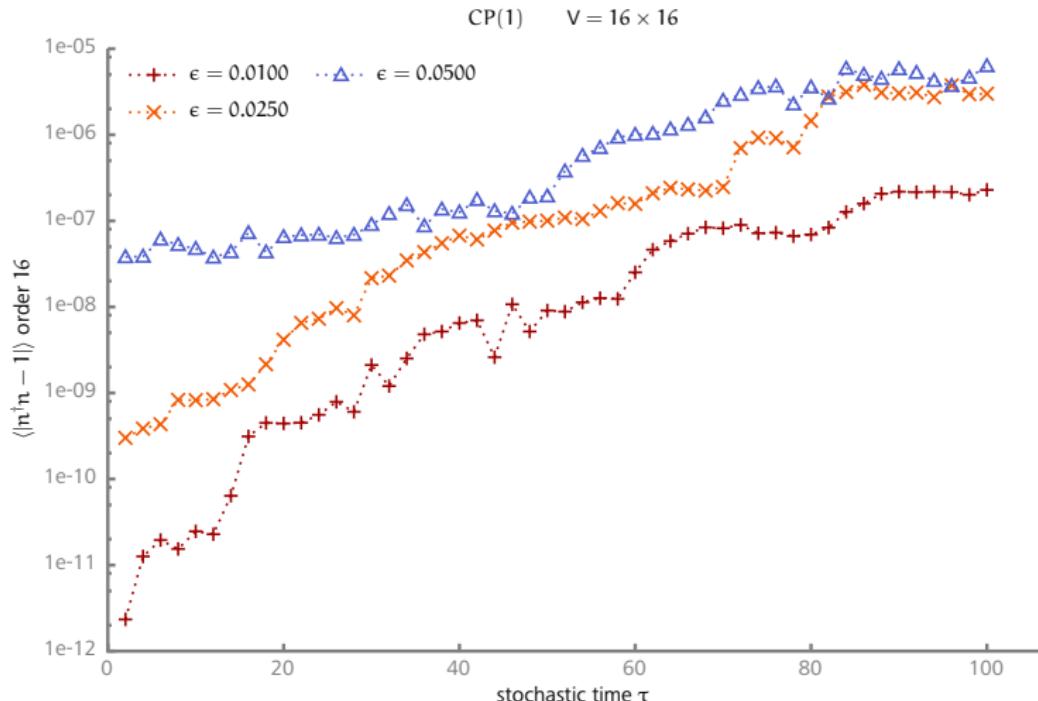
Energy Density Coefficients CP(1)



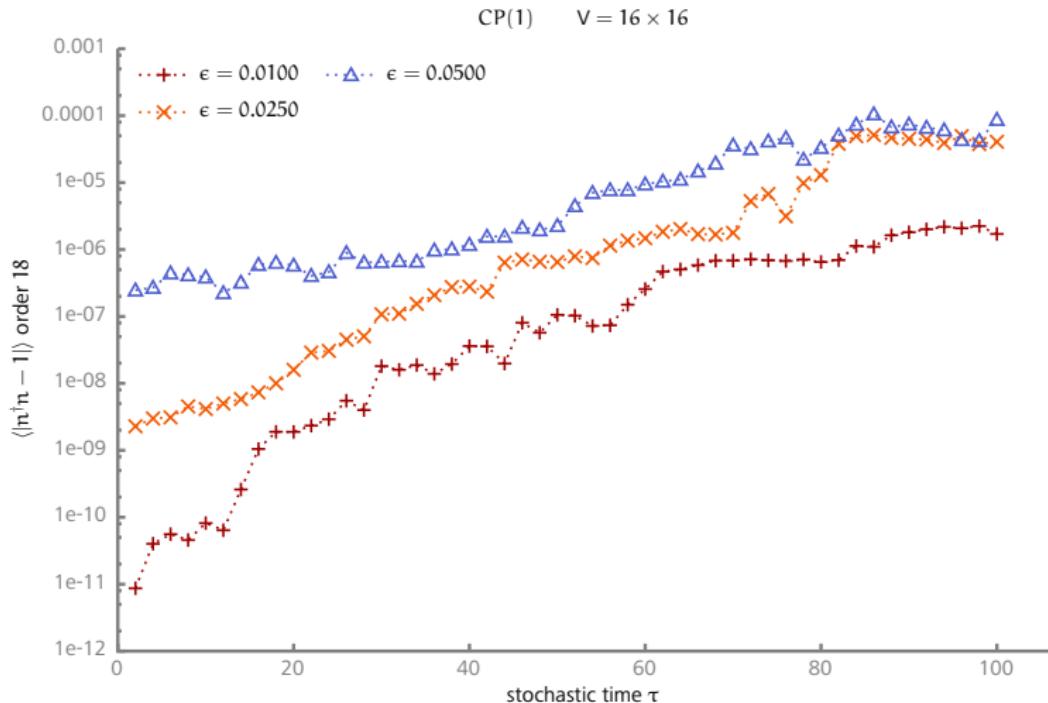
Energy Density Coefficients CP(1)



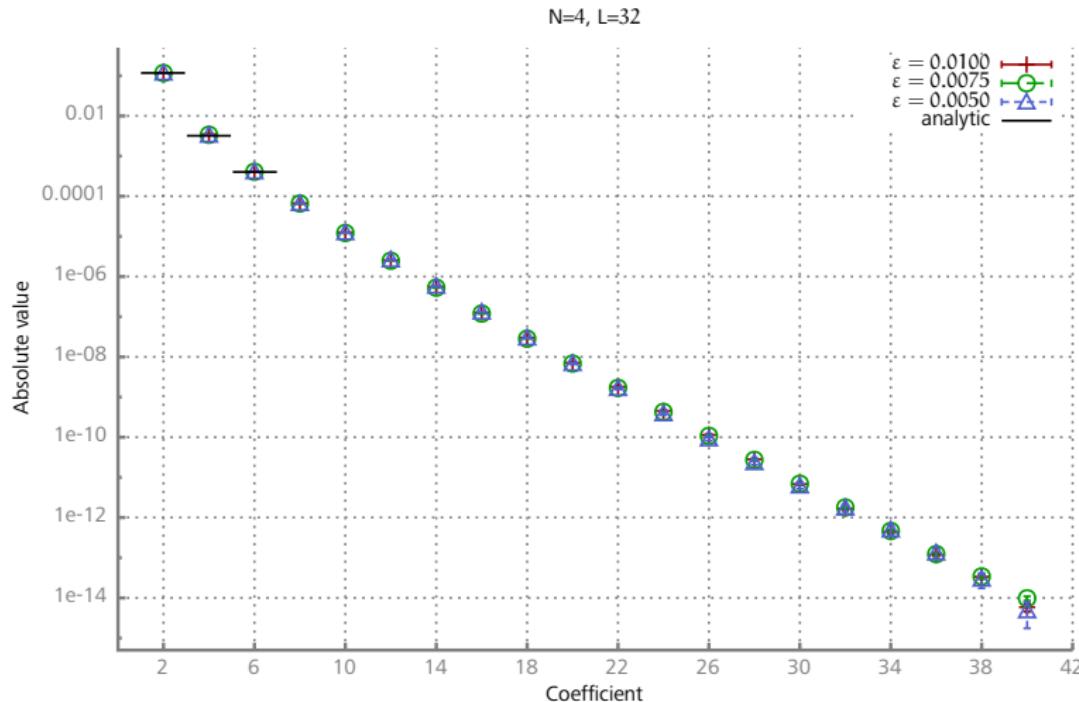
Constraint Violation CP(1)



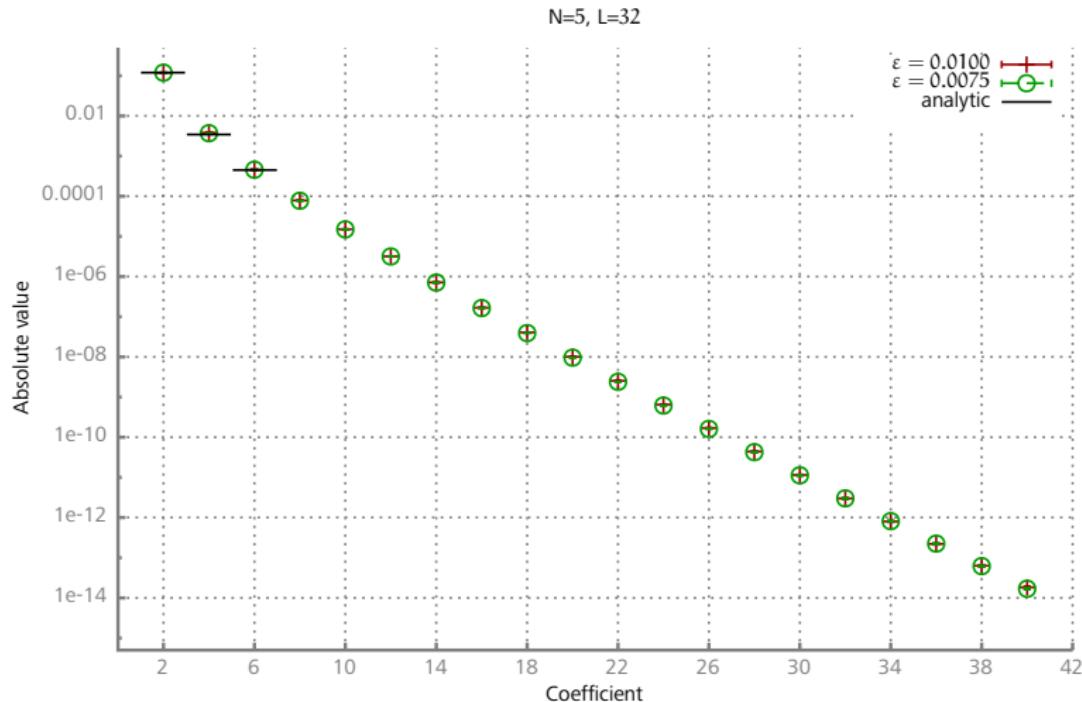
Constraint Violation CP(1)



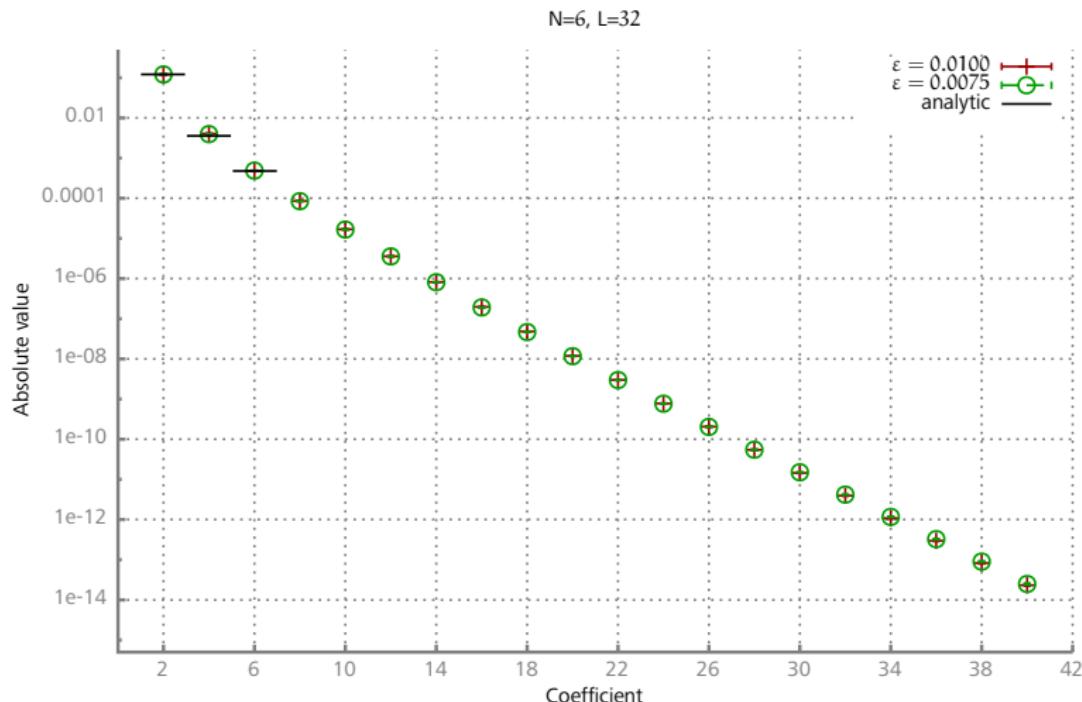
- ▶ Good agreement with analytic (low order) coefficients
(even for finite ϵ)
- ▶ Numerical issues for $CP(N-1)$
 - ▶ Field constraint violated in higher orders
 - ▶ Results reliable?
- ▶ Focus on $PCM(N)$ for further calculations
 - ▶ Larger lattice $V = 32 \times 32$
 - ▶ Smaller ϵ
 - ▶ Higher orders
 - ▶ N dependence
- ▶ Expected large n behavior: $E_n \sim a^n n!$ ($a \propto \beta_0$)

PCM(N) coefficients

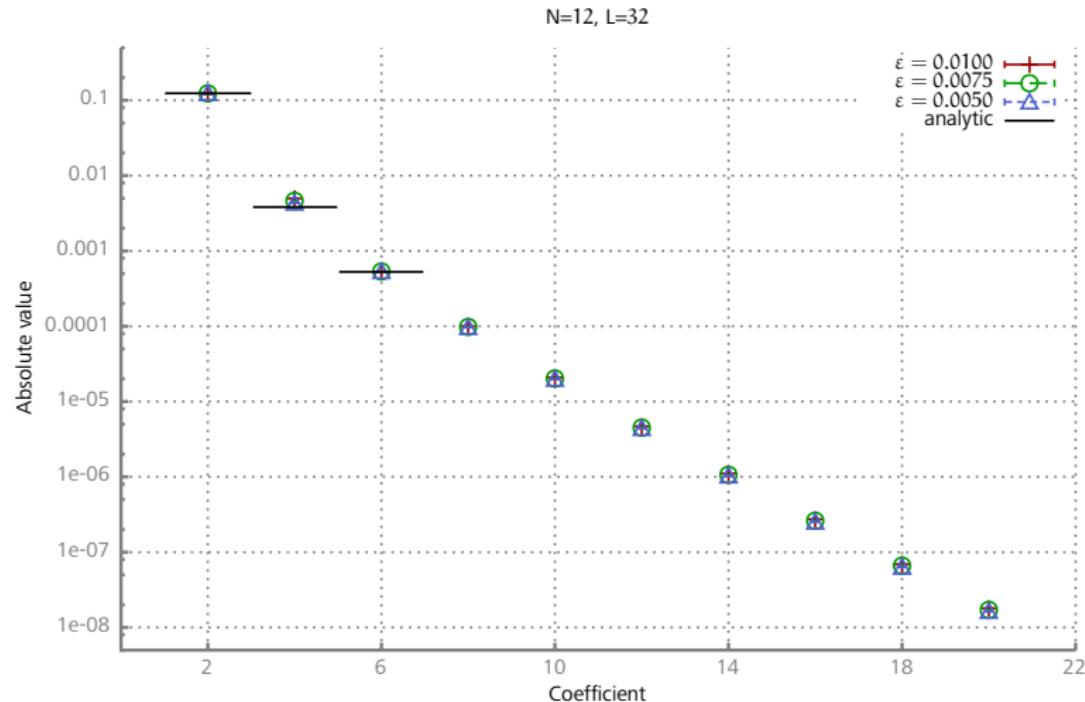
PCM(N) coefficients



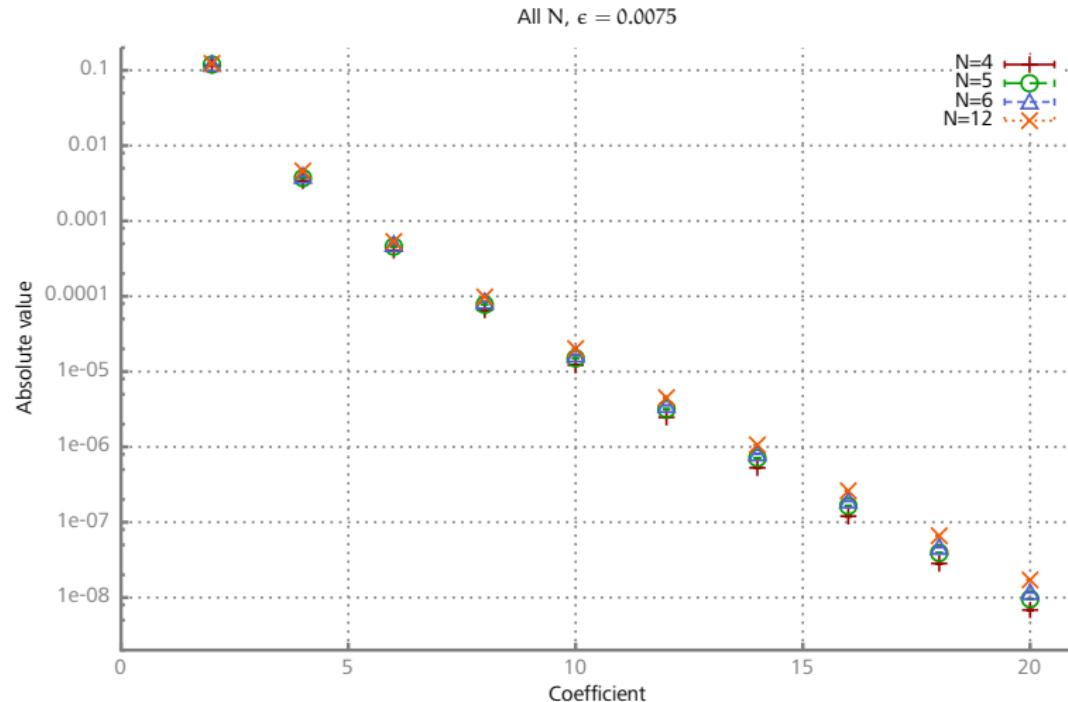
PCM(N) coefficients



PCM(N) coefficients



PCM(N) comparison



Summary

- ▶ NSPT for $CP(N - 1)$ and $PCM(N)$
- ▶ First results for expansion of energy density
- ▶ Good agreement with analytic coefficients (where available)
- ▶ Numerical issues spoil high order calculations in $CP(N - 1)$
- ▶ No renormalon observed in $PCM(N)$ up to order 40 (20) for $N \leq 6$ ($N = 12$)
- ▶ Expansion coefficients $E_n(N)$ decrease slower for larger N

Outlook

- ▶ Better control of numerical errors in $CP(N - 1)$
- ▶ Twisted boundary conditions in $PCM(N)$

Backup Slides

Langevin for constrained fields

- ▶ Find a Lie Group G such that

$$\forall g \in G : \mathcal{C}(\phi) = 0 \quad \Rightarrow \mathcal{C}(g\phi) = 0$$

- ▶ Define **Lie derivative** (Λ^α generators of G)

$$f(e^{i\epsilon\Lambda^\alpha}\phi) = \left(1 + \epsilon \nabla^\alpha + \mathcal{O}(\epsilon^2)\right) f(\phi)$$

- ▶ New Langevin equation ($\nabla_x = \Lambda^\alpha \nabla_x^\alpha$, $\eta(x, \tau) = \Lambda^\alpha \eta^\alpha(x, \tau)$)

$$\frac{\partial \phi(x, \tau)}{\partial \tau} = -i(\nabla_x \mathcal{S}[\phi(x, \tau)] + \eta(x, \tau)) \phi(x, \tau),$$

Discretised constrained Langevin

- Discretised constrained Langevin can be written as

$$\phi(x, \tau_i + \epsilon) = e^{-if(x, \tau_i)^\alpha \Lambda^\alpha} \phi(x, \tau_i)$$

- To order $\mathcal{O}(\epsilon)$ ("Euler")

$$f(x, \tau_i)^\alpha = \epsilon \nabla_x^\alpha \mathcal{S}[\phi(x, \tau_i)] + \sqrt{\epsilon} \eta^\alpha(x, \tau)$$

- By construction $e^{-if(x, \tau_i)^\alpha \Lambda^\alpha} \in G$ and

$$\mathcal{C}(\phi(x, \tau_i)) = 0 \quad \Rightarrow \quad \mathcal{C}(\phi(x, \tau_i + \epsilon)) = 0$$

Constraint Violation in NSPT

- ▶ Constraints have to hold order by order

$$n^\dagger n = \sum_{l=0}^M \left(\sum_{k=0}^l n_k^\dagger n_{l-k} \right) \beta^{-\frac{l}{2}} \stackrel{!}{=} 1$$

- ▶ Since $n_0^\dagger n_0 = 1$ for our expansion

$$\sum_{k=0}^l n_k^\dagger n_{l-k} \stackrel{!}{=} 0 \quad \forall l > 0$$

- ▶ Analogous formula for gauge field U_ν and matrices U

Constraint Violation in NSPT

- ▶ Numerical errors accumulate and lead to constraint violation
- ▶ “Normalise” the fields during Langevin runs?

$$\sum_{k=0}^l n_k^\dagger n_{l-k} = n_0^\dagger n_l + n_l^\dagger n_0 + \sum_{k=1}^{l-1} n_k^\dagger n_{l-k} = 0$$

$$\Rightarrow n_l \rightarrow \frac{1}{\alpha} n_l \quad \text{with} \quad \alpha = -\frac{\sum_{k=1}^{l-1} n_k^\dagger n_{l-k}}{2 \operatorname{Re}(n_0^\dagger n_l)}$$

- ▶ Numerically unstable!

Constraint Violation in NSPT

- ▶ “Normalisation” is a lot easier for U and U_ν

$$A = \log(U) \Rightarrow \exp(A) \in SU(N)$$

- ▶ This gives constraints for A ...

$$A \stackrel{!}{=} -A^\dagger \quad \text{and} \quad \text{Tr} A \stackrel{!}{=} 0$$

- ▶ ... which are simple to implement order by order

$$A_l \hookrightarrow \frac{A_l - A_l^\dagger}{2} \quad \text{and} \quad A_l \hookrightarrow A_l - \frac{\text{Tr} A_l}{N} \mathbb{1}$$