

Topological Susceptibility to High Temperatures via Reweighting

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arXiv: 1806.01162, with Guy D. Moore and Daniel Robaina



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Outline

- ▶ Introduction
 - ▶ Why topology at high temperature?
 - ▶ Why is it complicated?
- ▶ Reweighting
 - ▶ basic idea
 - ▶ implementation
- ▶ Quenched Results on Topological Susceptibility up to $4.1 T_c$
- ▶ Conclusions

Introduction

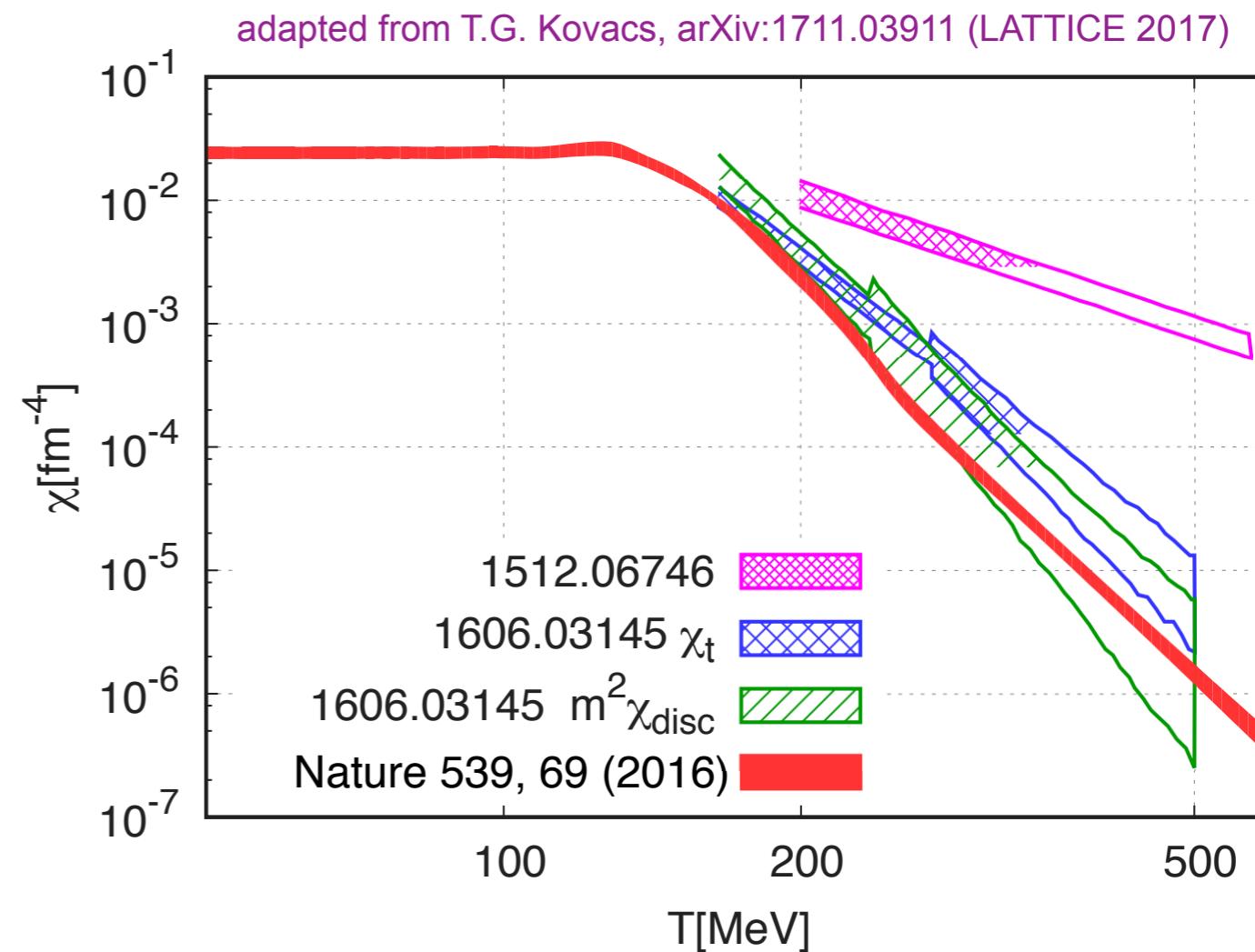
$$\chi_{\text{top}} = \int d^4x \langle q(x)q(0) \rangle = \frac{T}{L^3} \langle Q^2 \rangle$$

$$Q = \int d^4x \ q(x) \in \mathbb{Z}, \quad q(x) = \frac{1}{64\pi^2} \epsilon_{\mu\nu\rho\sigma} F_{\mu\nu}^a(x) F_{\rho\sigma}^a(x)$$

- properties of QCD axion are sensitive to χ_{top} up to $\sim 7 T_c$

G. Moore, arXiv:1709.09466; V. Klaer and G. Moore, JCAP 1711, 049 (2017)

- very disputed quantity



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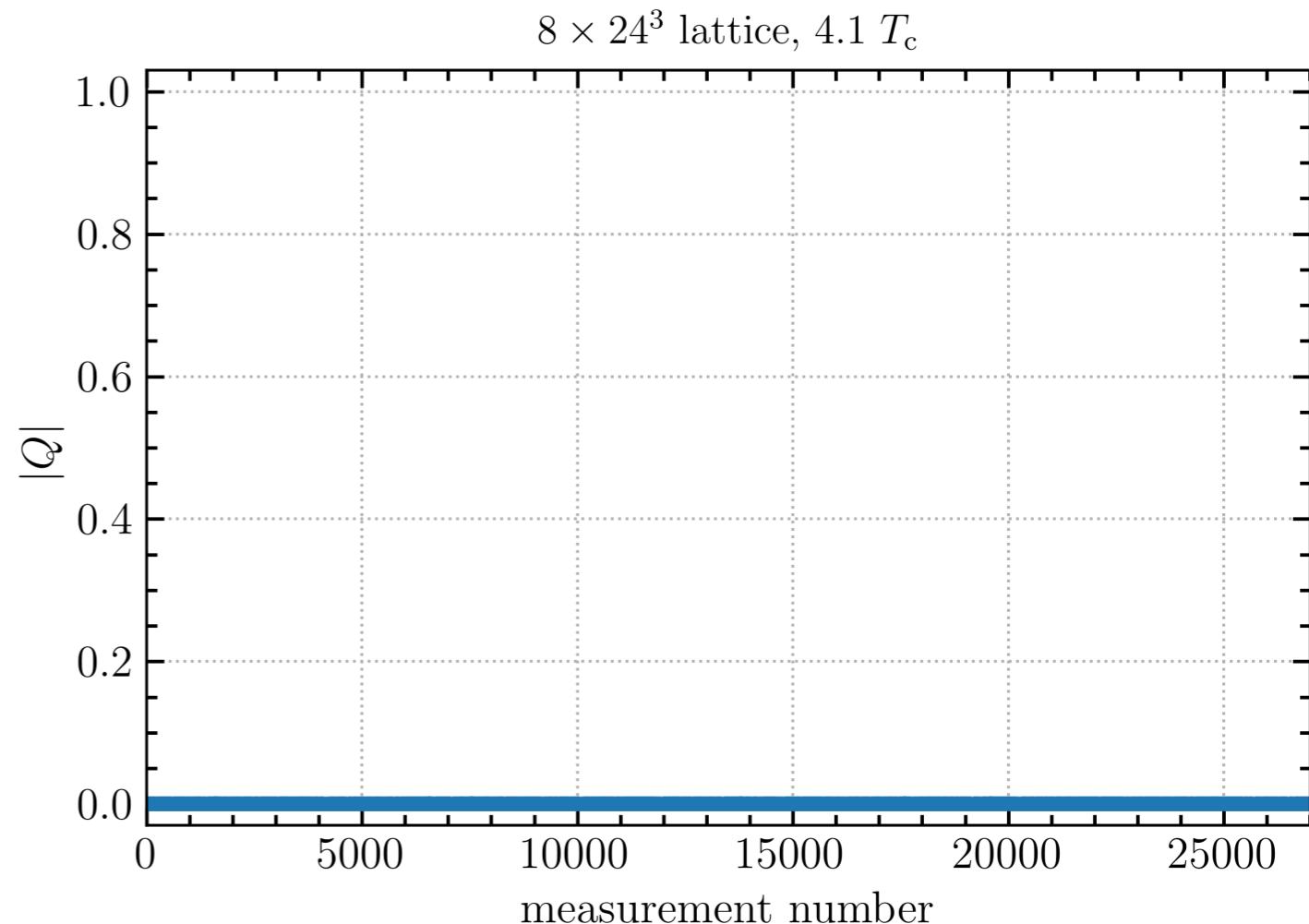
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- very suppressed at high temperatures
 - virtually no configs with $|Q| \neq 0$



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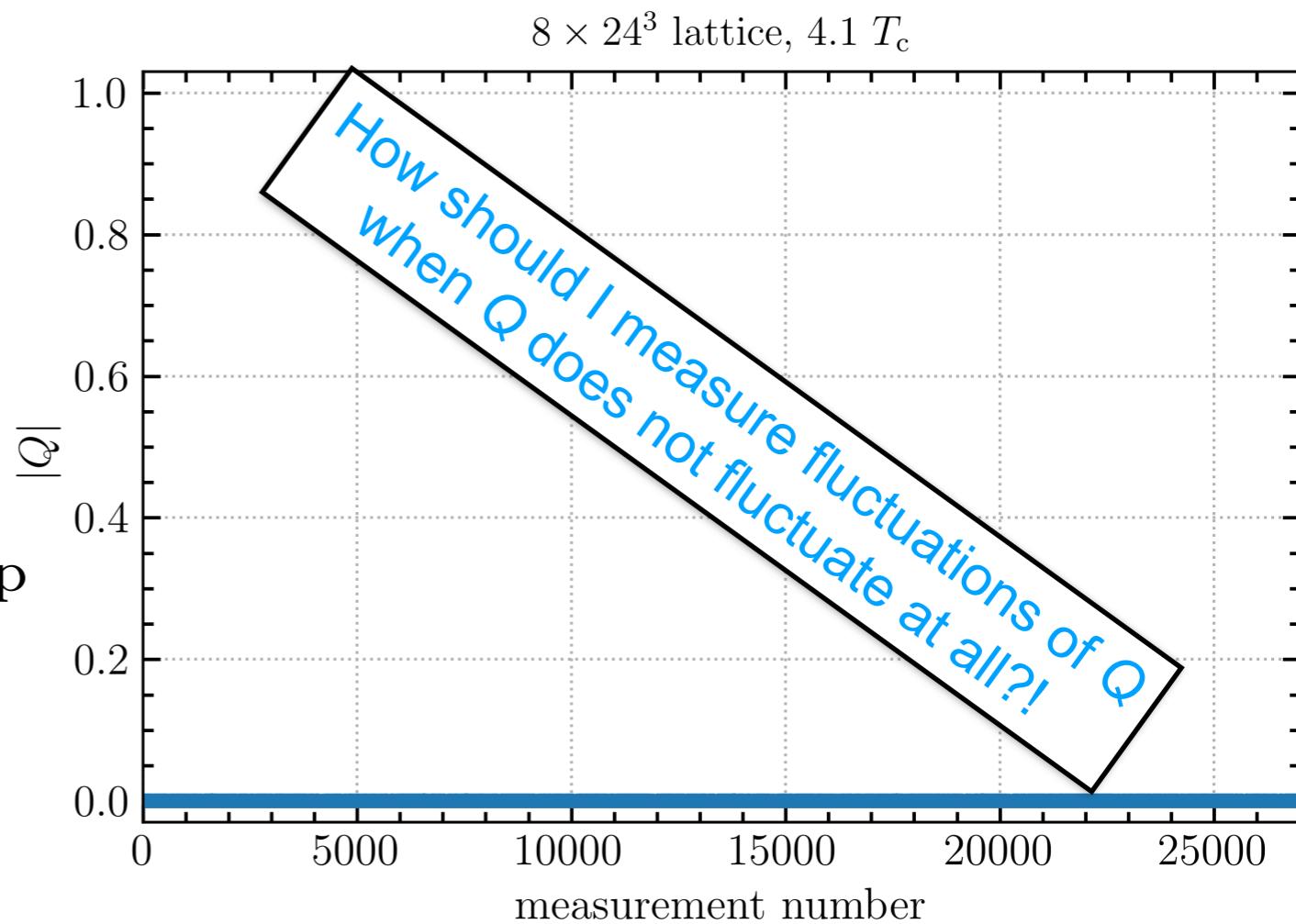
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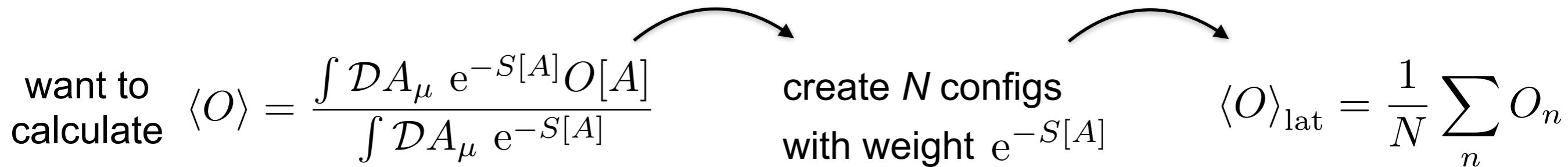
- very disputed quantity
- very suppressed at high temperatures
 - virtually no configs with $|Q| \neq 0$
- two possibilities to determine χ_{top}
 - run forever to get good statistics
 - have a clever idea to artificially enhance the number of instanton configs



Reweighting Basic Idea

B. A. Berg and T. Neuhaus, PLB **267**, 249 (1991); Kajantie, Laine, Rummukainen, Shaposhnikov, Nucl. Phys., B**466**, 189 (1996); M. Laine and K. Rummukainen, Nucl. Phys. **B535**, 423 (1998); F. Wang and D.P. Landau, PRL **86**, 2050 (2001)

Standard Lattice QCD:



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Standard Lattice QCD:

want to calculate $\langle O \rangle = \frac{\int \mathcal{D}A_\mu e^{-S[A]} O[A]}{\int \mathcal{D}A_\mu e^{-S[A]}}$

create N configs with weight $e^{-S[A]}$

$\langle O \rangle_{\text{lat}} = \frac{1}{N} \sum_n O_n$

Reweighting:

$\langle O \rangle = \frac{\int \mathcal{D}A_\mu e^{-S[A]+W(\xi)} e^{-W(\xi)} O[A]}{\int \mathcal{D}A_\mu e^{-S[A]+W(\xi)} e^{-W(\xi)}}$

$e^{-S[A]+W(\xi)}$

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$$e^{-S[A]+W(\xi)}$$
$$\langle O \rangle_{\text{lat}} = \frac{\sum_n O_n e^{-W(\xi_n)}}{\sum_n e^{-W(\xi_n)}}$$

holds for any **reweighting function** W and any **reweighting variable** ξ

if chosen correctly, the number of instantons can be significantly enhanced!

Reweighting Reweighting Variable

natural choice for reweighting variable: $Q = \sum_x -\frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \text{tr} \left(\hat{F}_{\mu\nu}(x) \hat{F}_{\rho\sigma}(x) \right)$

we use $\mathcal{O}(a^2)$ improved field-strength tensor

$$\hat{F}_{\text{clov}} = \frac{1}{4} \begin{array}{|c|c|}\hline \bullet & \\ \hline & \bullet \\ \hline \end{array} \quad \Rightarrow \quad \hat{F}_{\text{imp}} = \frac{5}{12} \begin{array}{|c|c|}\hline \bullet & \\ \hline & \bullet \\ \hline \end{array} - \frac{1}{24} \left(\begin{array}{|c|c|c|}\hline \bullet & & \\ \hline & \bullet & \\ \hline & & \bullet \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline & \bullet & \\ \hline \bullet & & \\ \hline & & \bullet \\ \hline \end{array} \right)$$

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BUT: UV fluctuations spoil topological charge → **gradient flow**

$$\partial_t B_\mu(x, t) = D_\nu[B] F_{\nu\mu}[B], \quad B_\mu(x, 0) = A_\mu(x)$$

gauge fields are “smeared”, UV fluctuations are removed, Q pushed towards integers

on the lattice: *Wilson Flow* R. Narayanan and H. Neuberger, JHEP **03**, 64 (2006); M. Lüscher, Comm.Math.Phys. **293**, 899 (2010)

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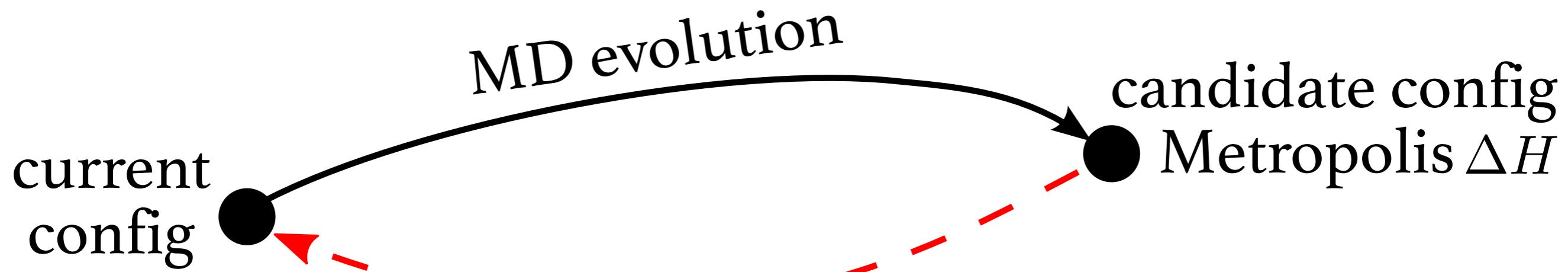
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reweighting variable: $Q' \equiv (Q \text{ after small amount of flow})$

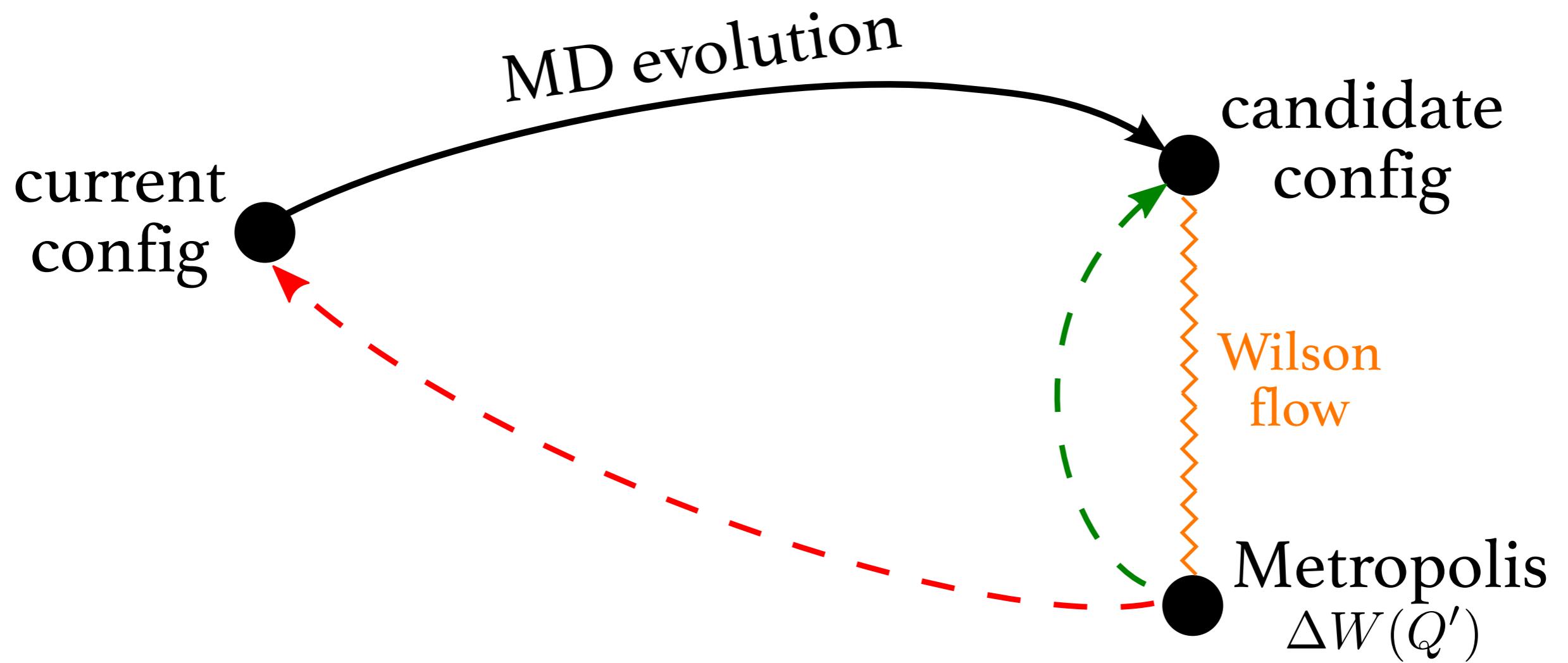
Reweighting Implementation

current
config ●

Reweighting Implementation



Reweighting Implementation



Reweighting Reweighting Function

We build W in an extra Markov chain as we go

similar to M. Laine and K. Rummukainen, Nucl. Phys. **B535**, 423 (1998); F. Wang and D.P. Landau, PRL **86**, 2050 (2001)

when it is built, we keep it fixed and start a new simulation for the measurements

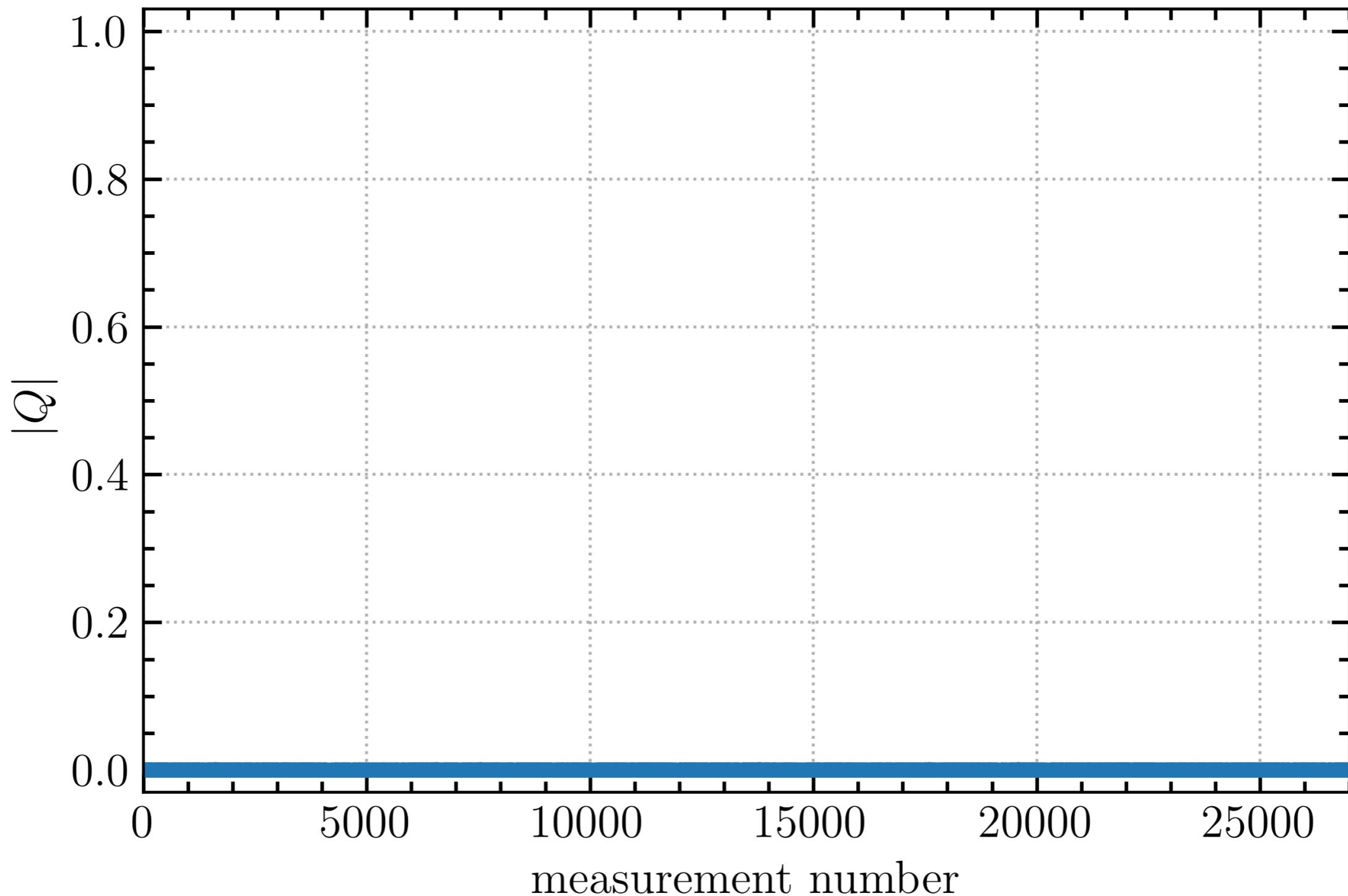
idea for building W :

weight $\propto e^{W(Q')}$ \rightarrow make W small at those Q' where the algorithm spends the most time

- start with flat, piecewise linear function on discretized values constrained to $|Q'| \in [0, 1]$
- do modified HMC update as described above and measure Q'
- lower W function close to Q'
- repeat until W “converges” (we lower W less and less)

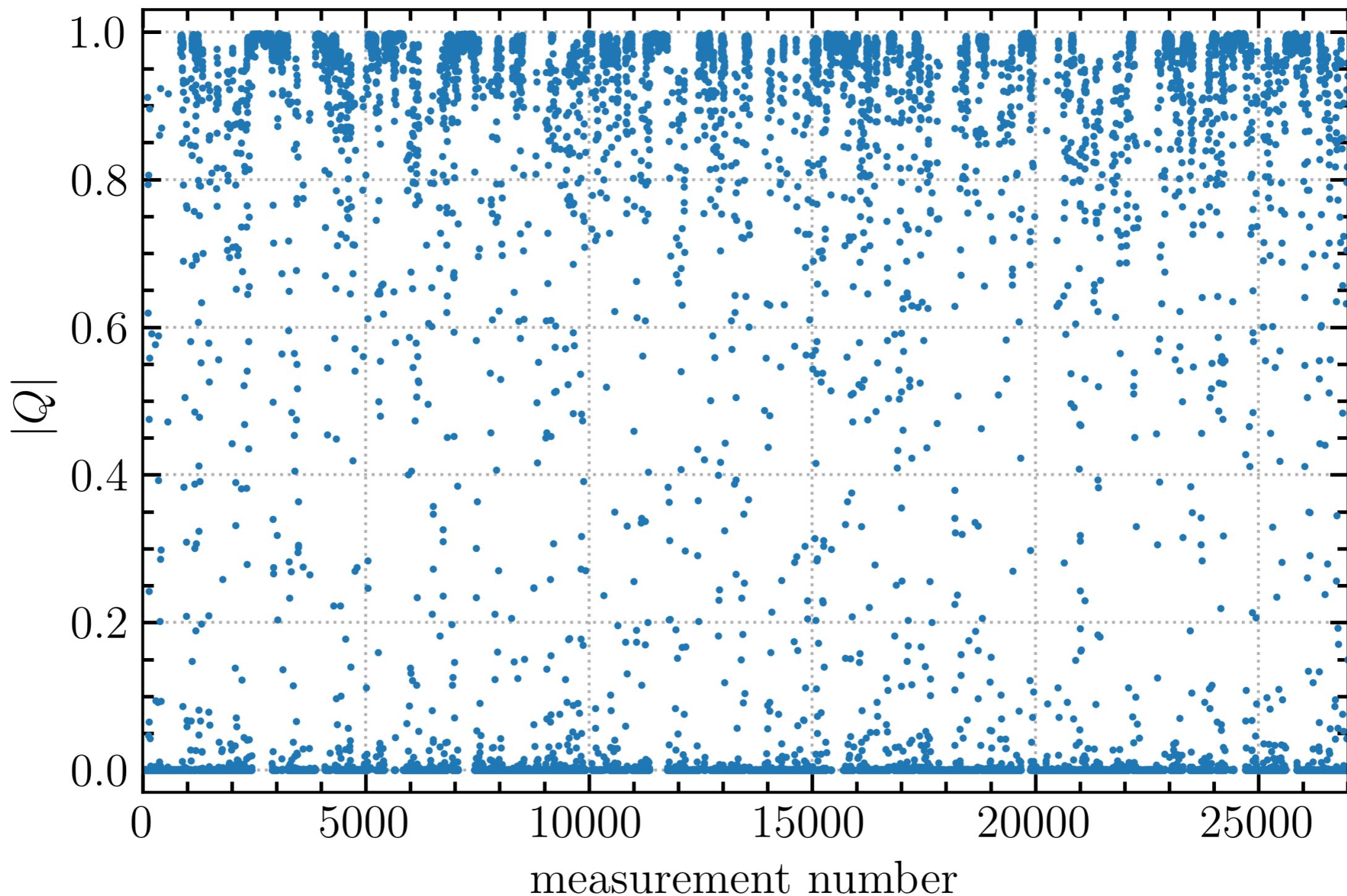
Reweighting It Works!

8×24^3 lattice, $4.1 T_c$



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Setting up the Calculation

- ▶ two high temperatures in quenched approximation
- ▶ three lattice spacings each → continuum limit
- ▶ finite volume study at the higher temperature

| T/T_c | $6/g_0^2$ * | β/a | L/a | #measurements | #back-and-forth |
|---------|-------------|-----------|-------|---------------|-----------------|
| 2.5 | 6.507 | 6 | 16 | 61,759 | 591 |
| | 6.722 | 8 | 16 | 96,068 | 263 |
| | 6.903 | 10 | 24 | 66,840 | 195 |
| 4.1 | 6.883 | 6 | 16 | 70,699 | 313 |
| | | | 8 | 50,992 | 94 |
| | | | 12 | 50,390 | 82 |
| | 7.135 | 8 | 16 | 52,900 | 145 |
| | | | 24 | 74,900 | 168 |
| | | | 32 | 72,800 | 151 |
| | 7.325 | 10 | 24 | 82,663 | 104 |

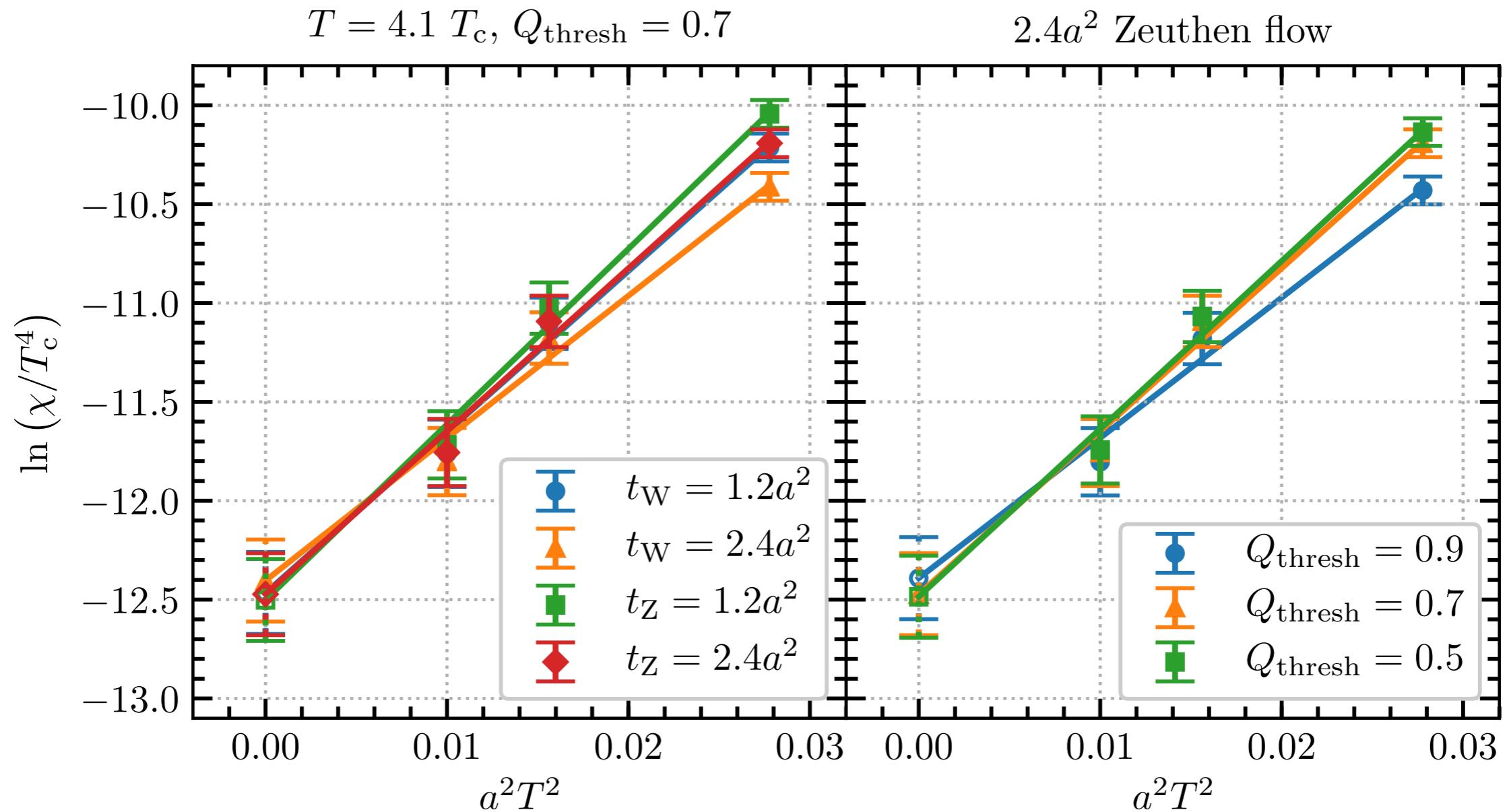
*

M. Caselle, A. Nada, and M. Panero, arXiv:1801.03110; L. Giusti and H. B. Meyer, PRL **106**, 131601 (2011)

Results

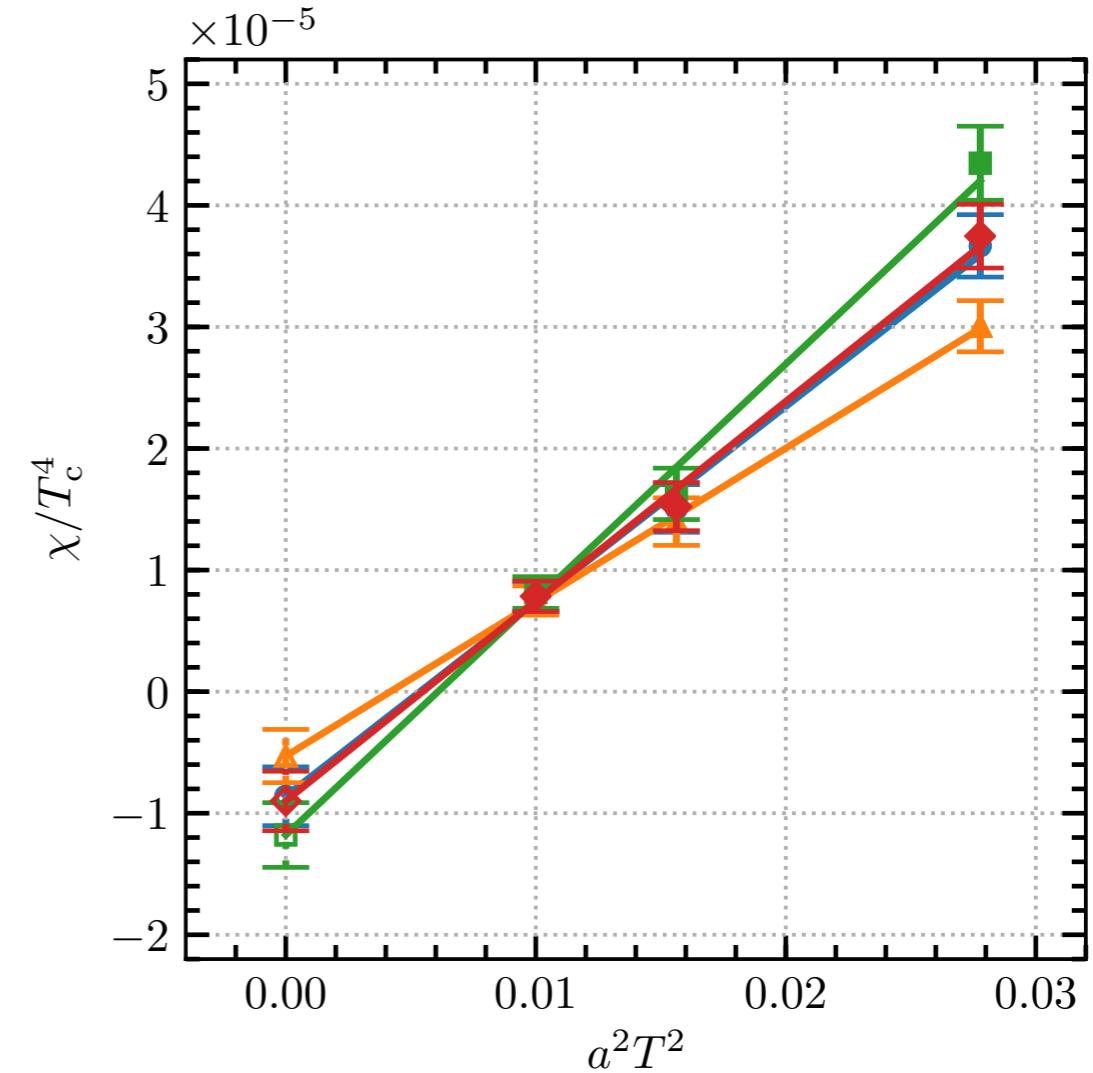
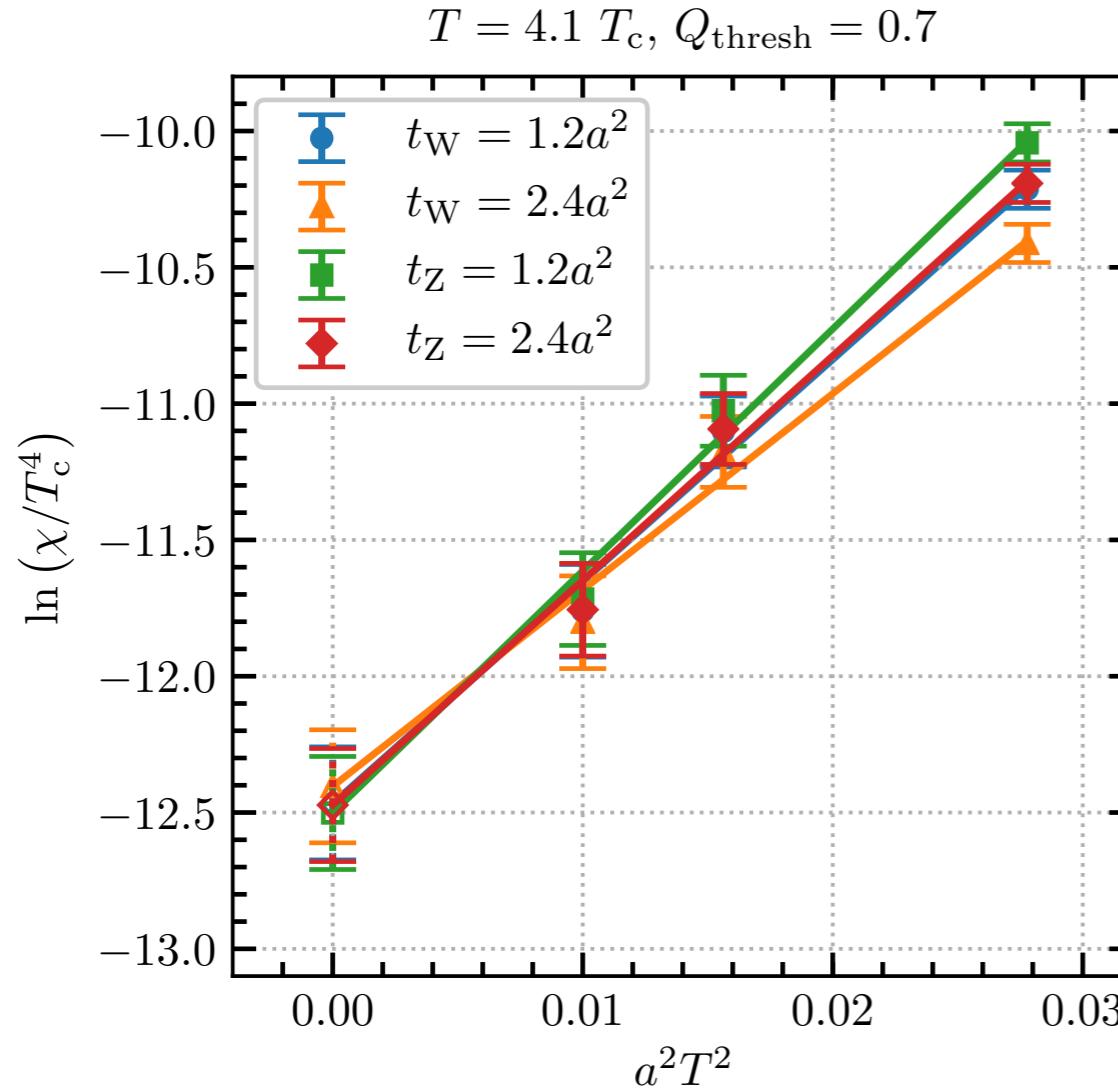
$$\chi_{\text{top}} = \frac{T}{L^3} \frac{\sum_{\text{configs}} e^{-W(Q')} Q_{\text{flowed}}^2}{\sum_{\text{configs}} e^{-W(Q')}}$$

we use the topological charge after gradient flow, thresholded to be an integer and compare 3 thresholds, 2 flow depth, and 2 flow types



Results

Continuum Extrapolation



- ▶ continuum extrapolation should be performed in $\ln \chi$:

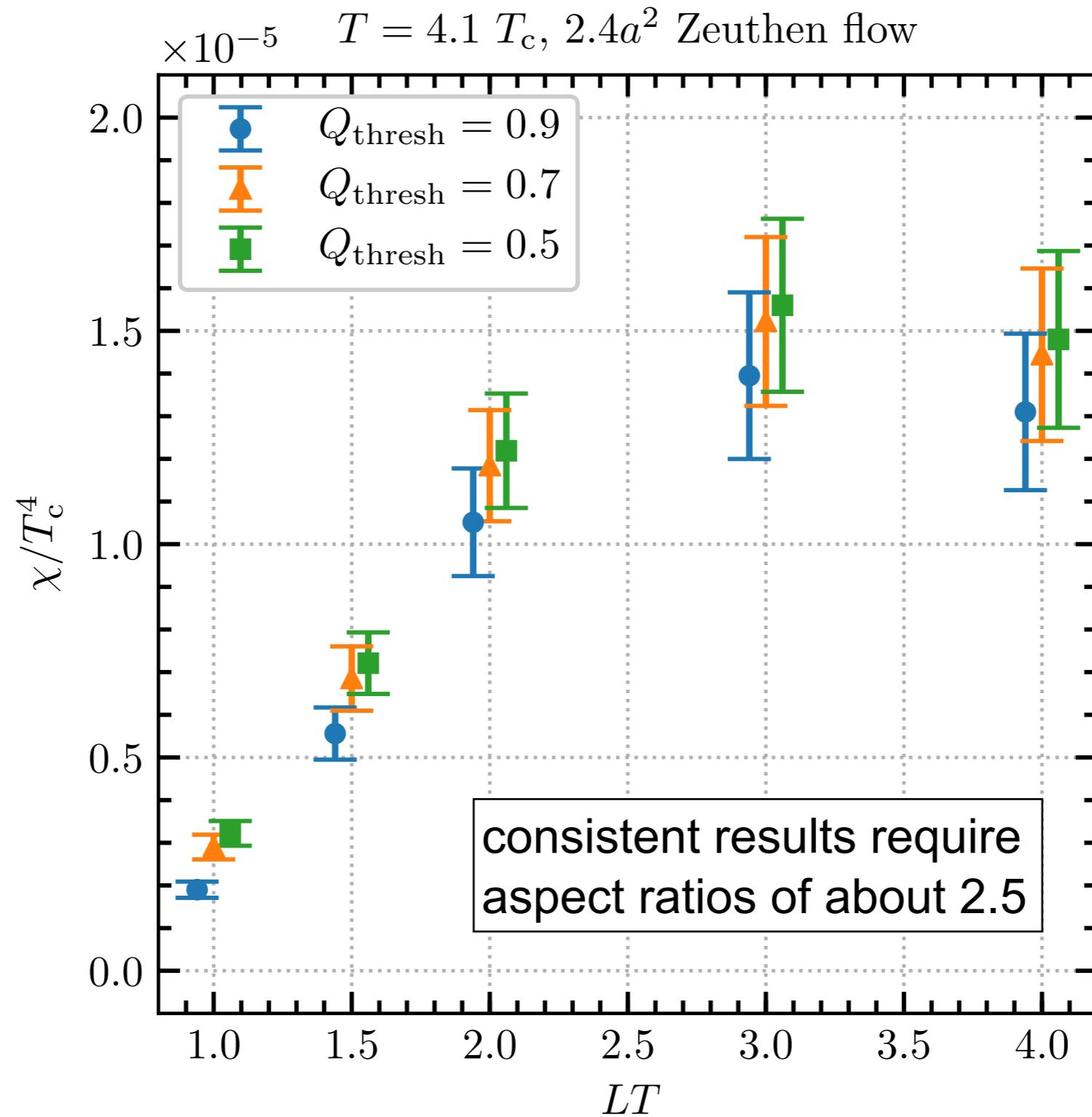
$$\chi \propto \exp \left[-S_{\text{caloron}} \left(1 + \mathcal{O} (aT)^2 \right) \right]$$

PTJ, G.D. Moore, and D. Robaina,
arXiv:1806.01162

- ▶ $N_\tau = 6$ seems to be too coarse

Results

Finite Volume Study



Results

Final Numbers and Comparison

our continuum extrapolated results:

$$\chi_{\text{top}}(2.5 T_c) = 2.22 \times 10^{-4} e^{\pm 0.18} T_c^4, \quad \chi_{\text{top}}(4.1 T_c) = 3.83 \times 10^{-6} e^{\pm 0.21} T_c^4$$

S. Borsanyi et al., PLB **752**, 175 (2016):

$$\chi_{\text{top}}(2.5 T_c) = 1.9 \times 10^{-4} T_c^4, \quad \chi_{\text{top}}(4.1 T_c) = 5.6 \times 10^{-6} T_c^4$$

conventional calculation with heat bath/overrelaxation algorithm up to $4 T_c$,
much more statistics, grand fit to all continuum extrapolated data

E. Berkowitz et al., PRD **92**, 034507 (2015):

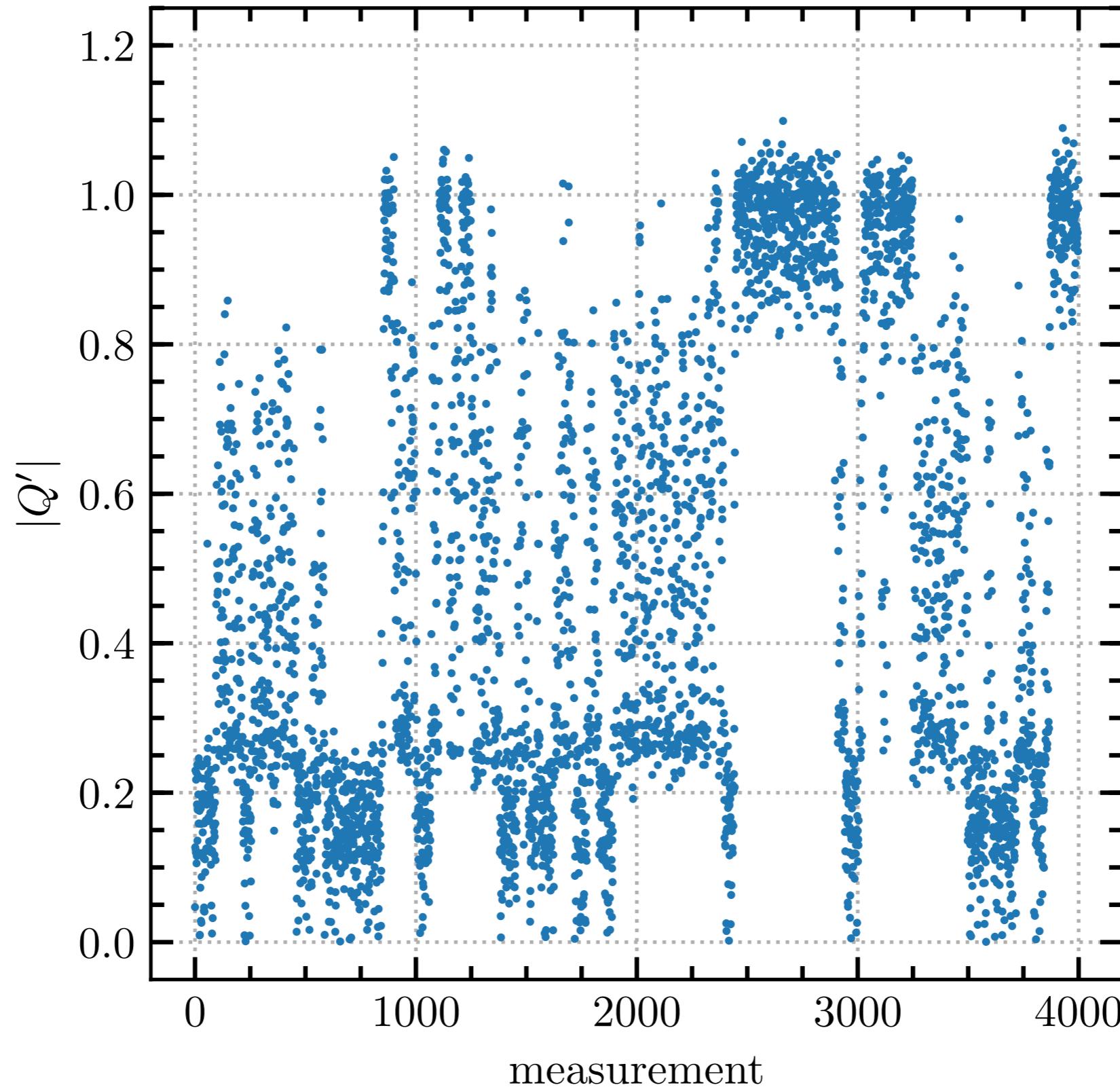
$$\chi_{\text{top}}(2.5 T_c) = 5.3(1) \times 10^{-4} T_c^4$$

conventional calculation up to $2.5 T_c$, no continuum extrapolation

(our result at finite a : $\chi_{\text{top}}(2.5 T_c) = 5.4 \times 10^{-4} e^{\pm 0.07} T_c^4$)

Still Some Problems...

8×24^3 lattice, $4.1 T_c$



Conclusions

- ▶ QCD topological susceptibility at high temperatures relevant for axion cosmology
- ▶ however, very hard to measure at high temperatures due to bad statistics and topological freezing
- ▶ reweighting overcomes both problems and allows for direct measurements of χ_{top} at high temperatures
- ▶ results in quenched approximation agree well with literature

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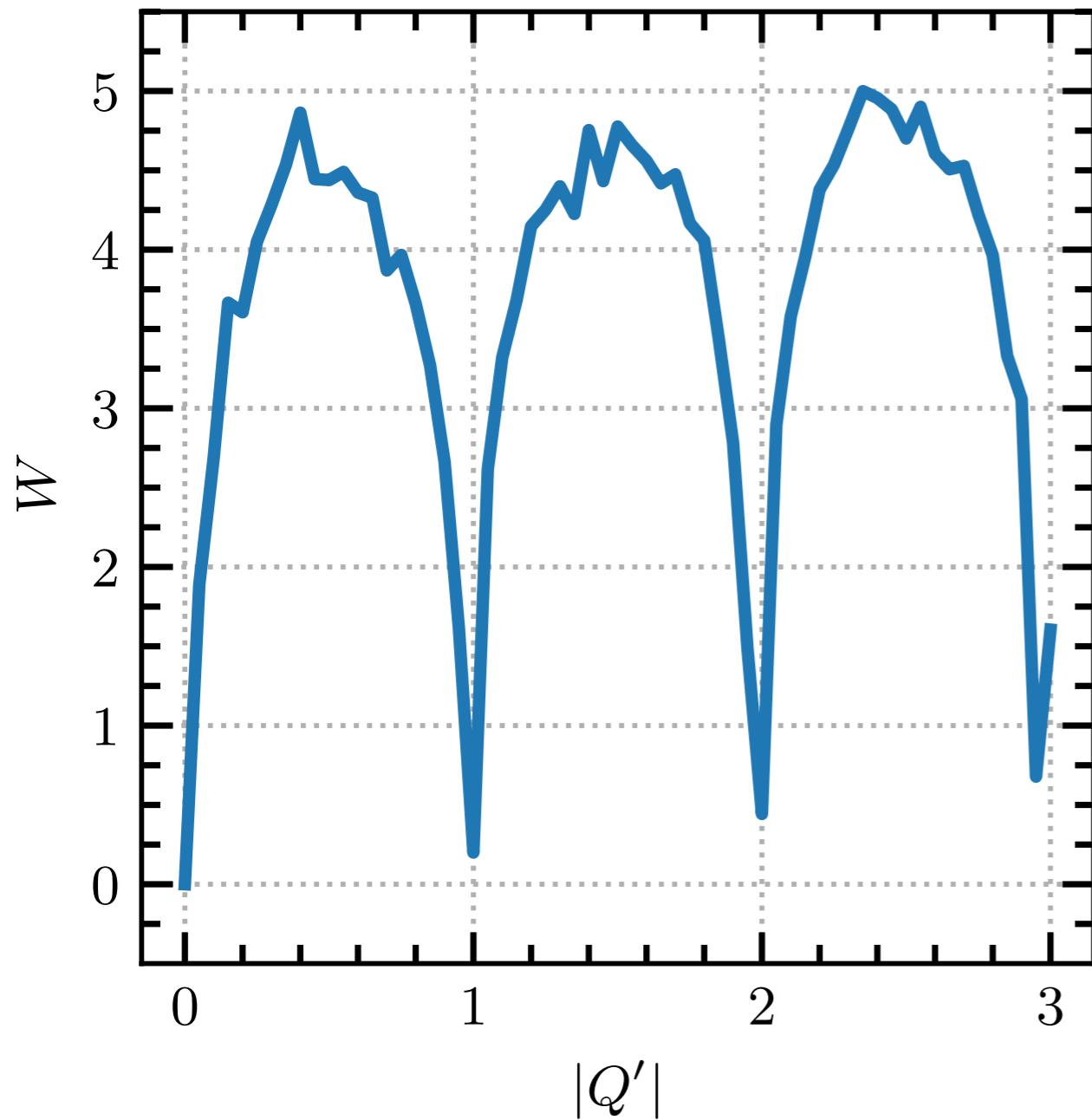
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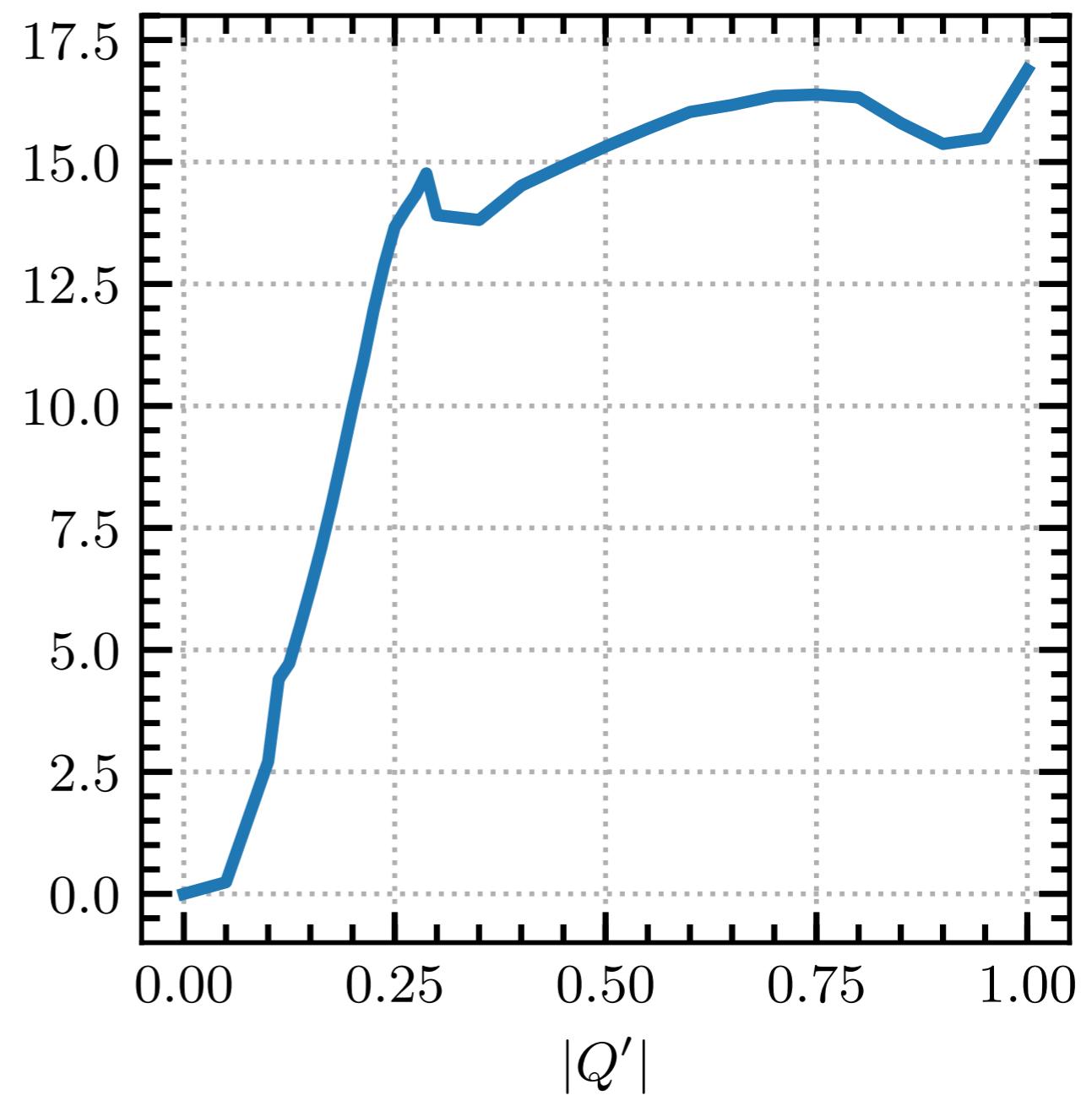
Thank You!

Reweighting Functions

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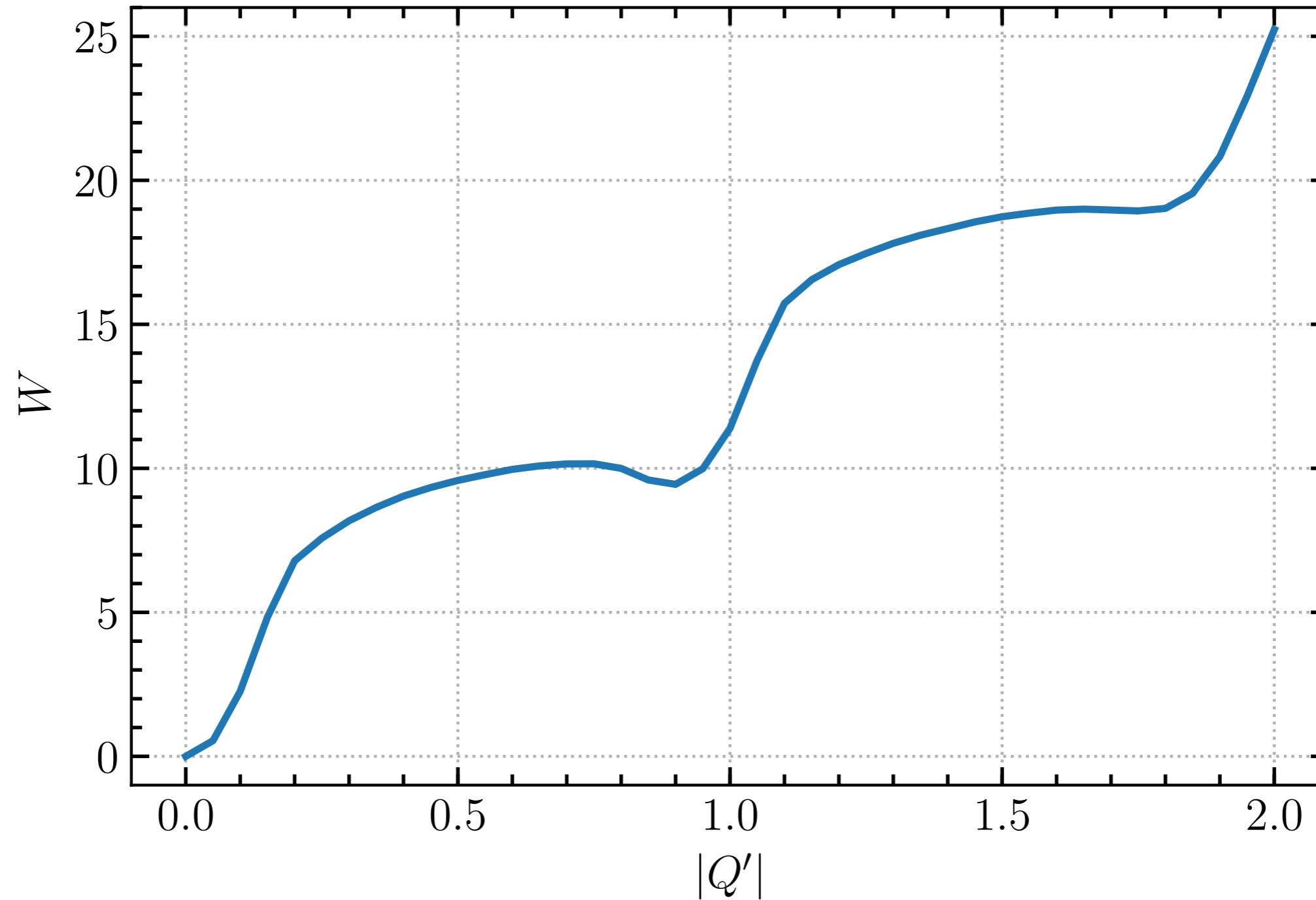


$T = 4.1 T_c$



Need of Higher Topological Sectors

$T = 2.5 T_c$, 6×16^3 lattice

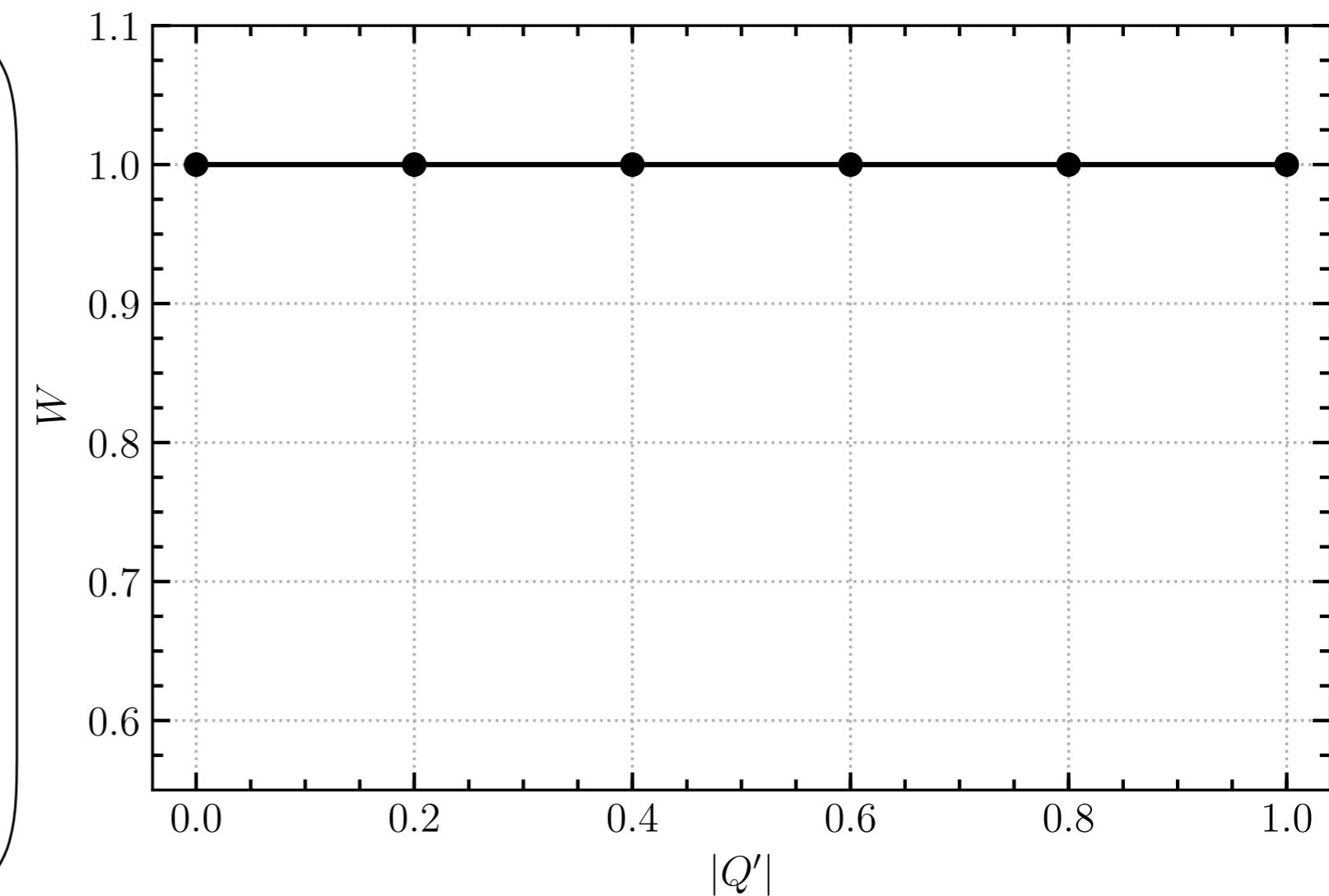


Building the Reweighting Function Step by Step

constrain to $|Q'| \in [0, 1]$ and perform extra Markov chain for building W

similar to M. Laine and K. Rummukainen, Nucl. Phys. **B535**, 423 (1998); F. Wang and D.P. Landau, PRL **86**, 2050 (2001)

discretize W on this interval and make it piecewise linear



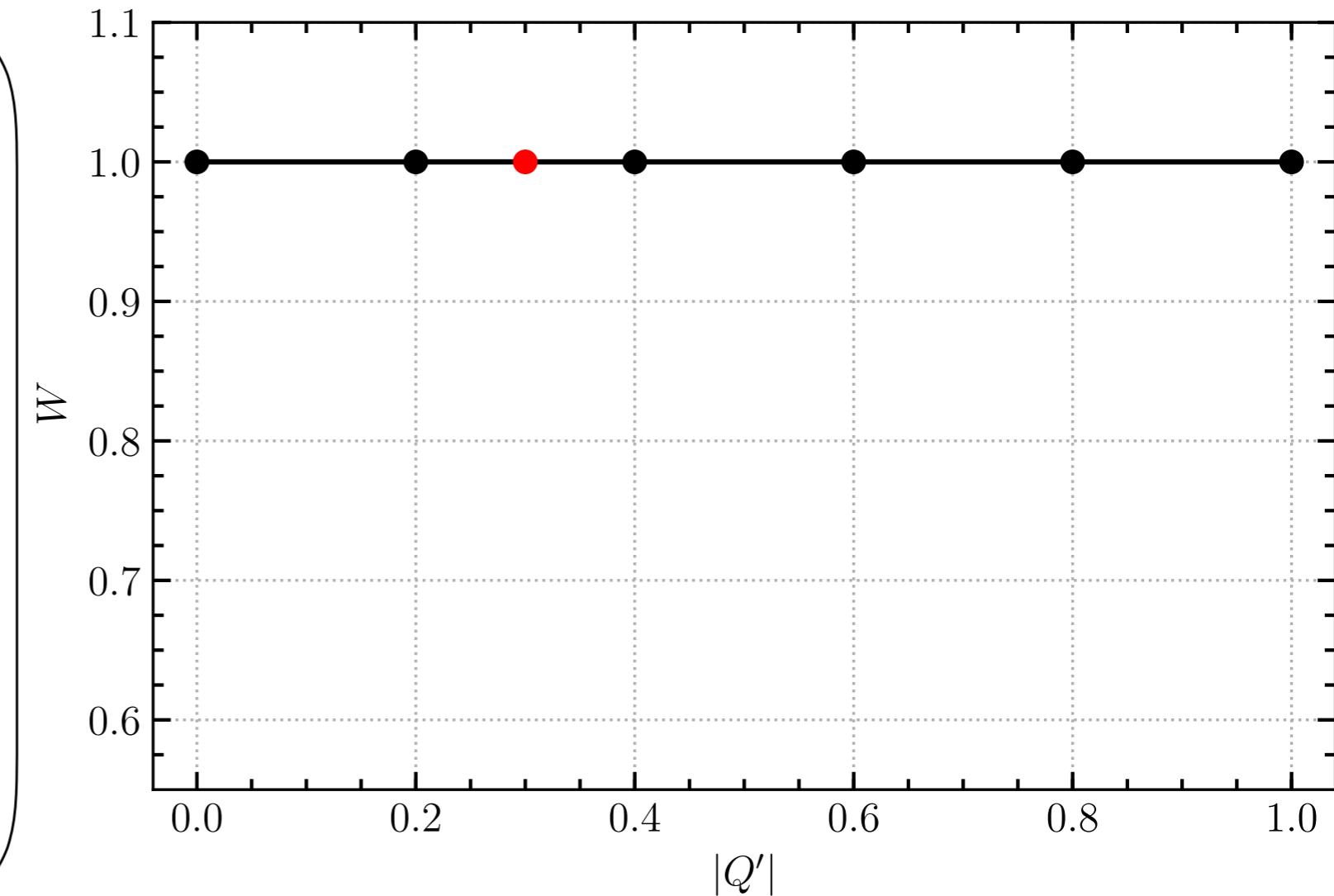
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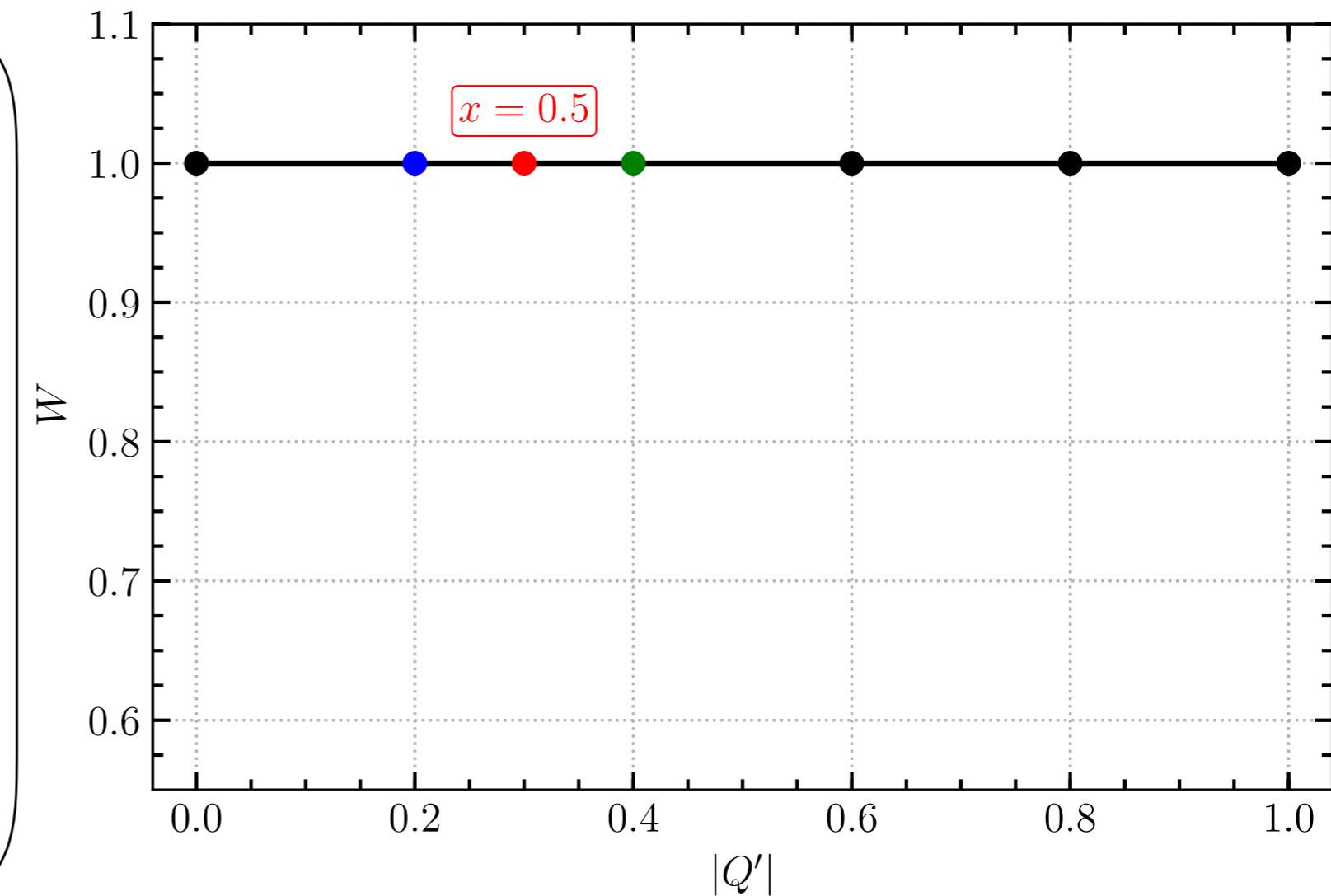
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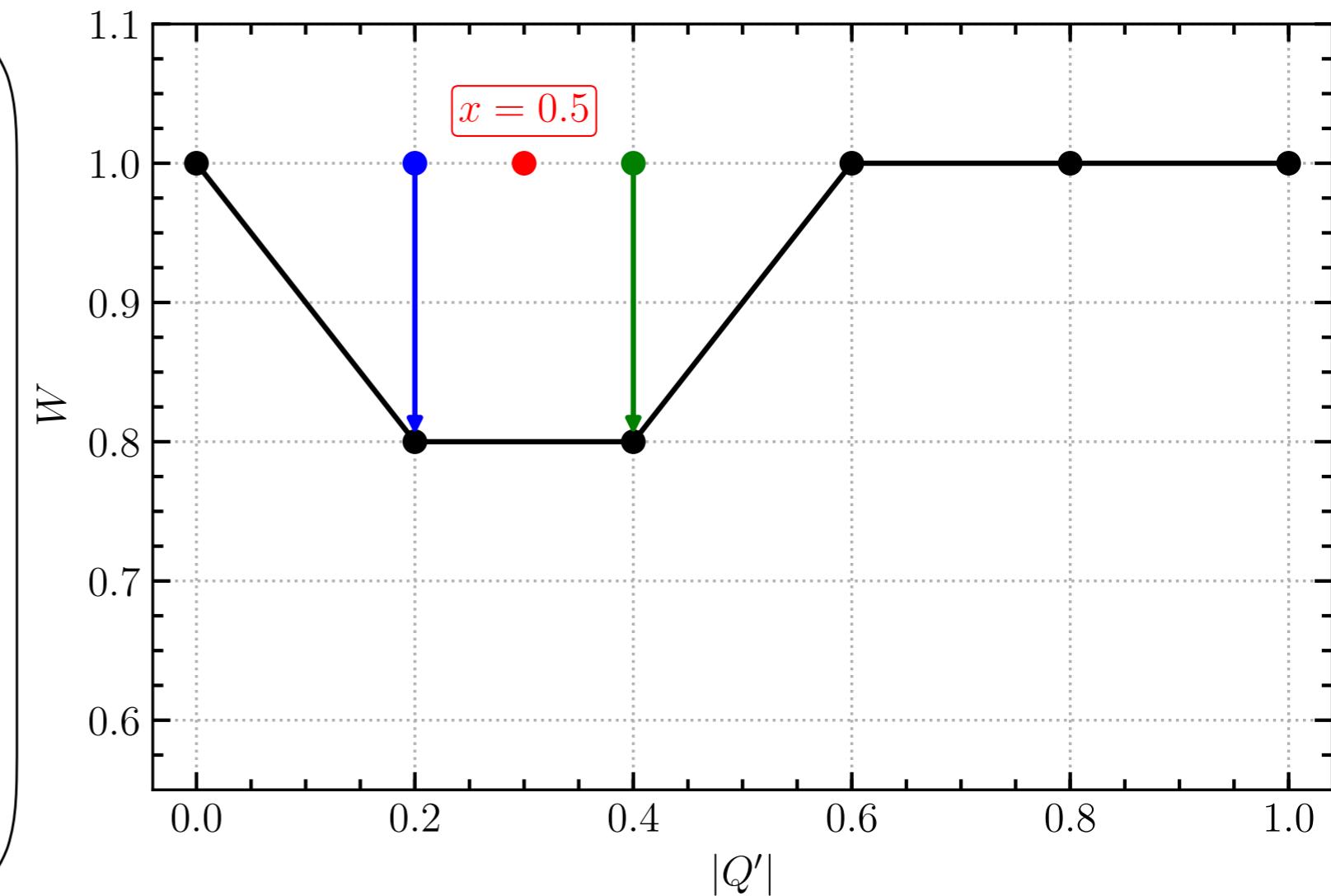
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 $W(Q'_i) = s(1 - x)$
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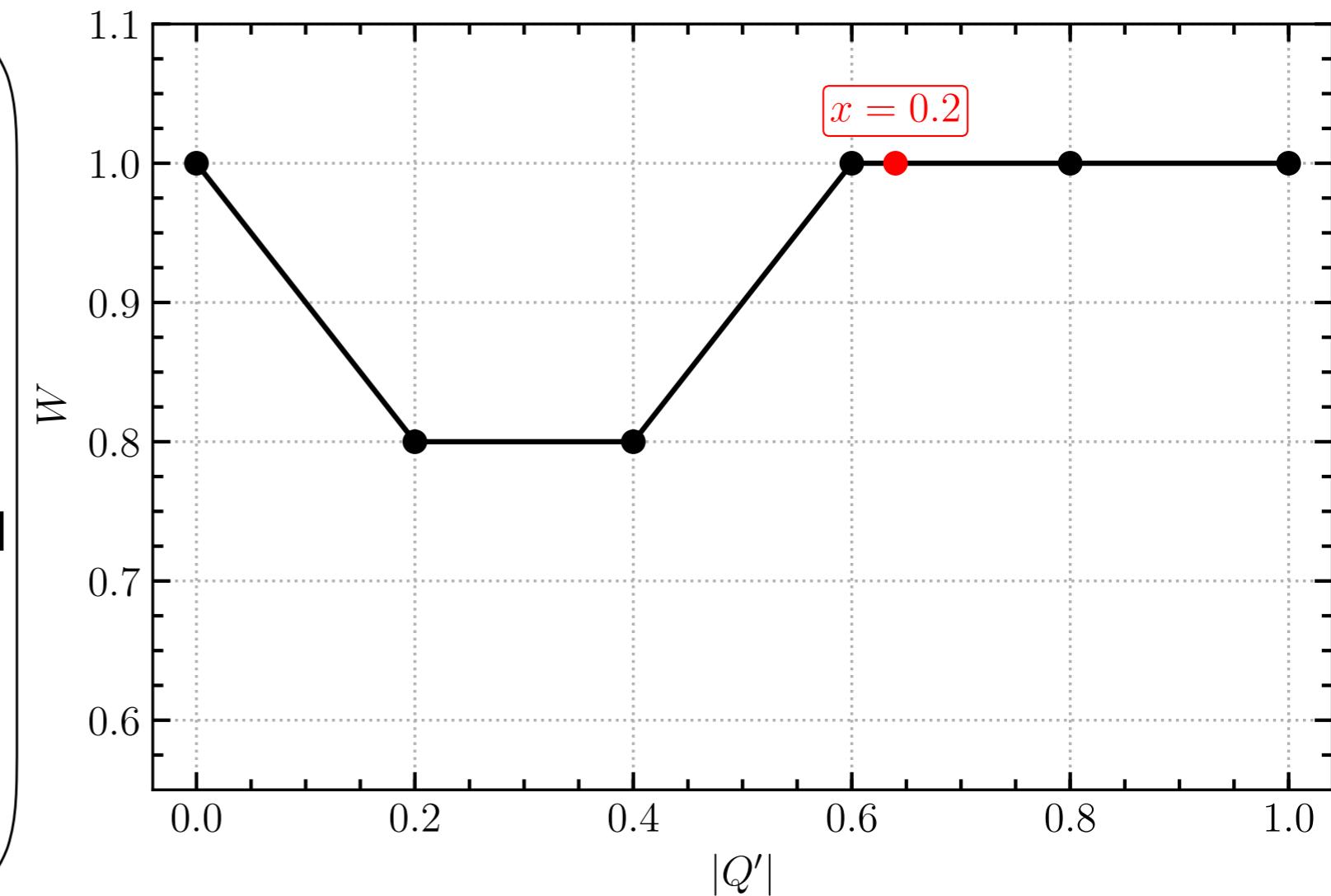
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- 4a. lower s if a sweep is completed
4b. repeat previous steps



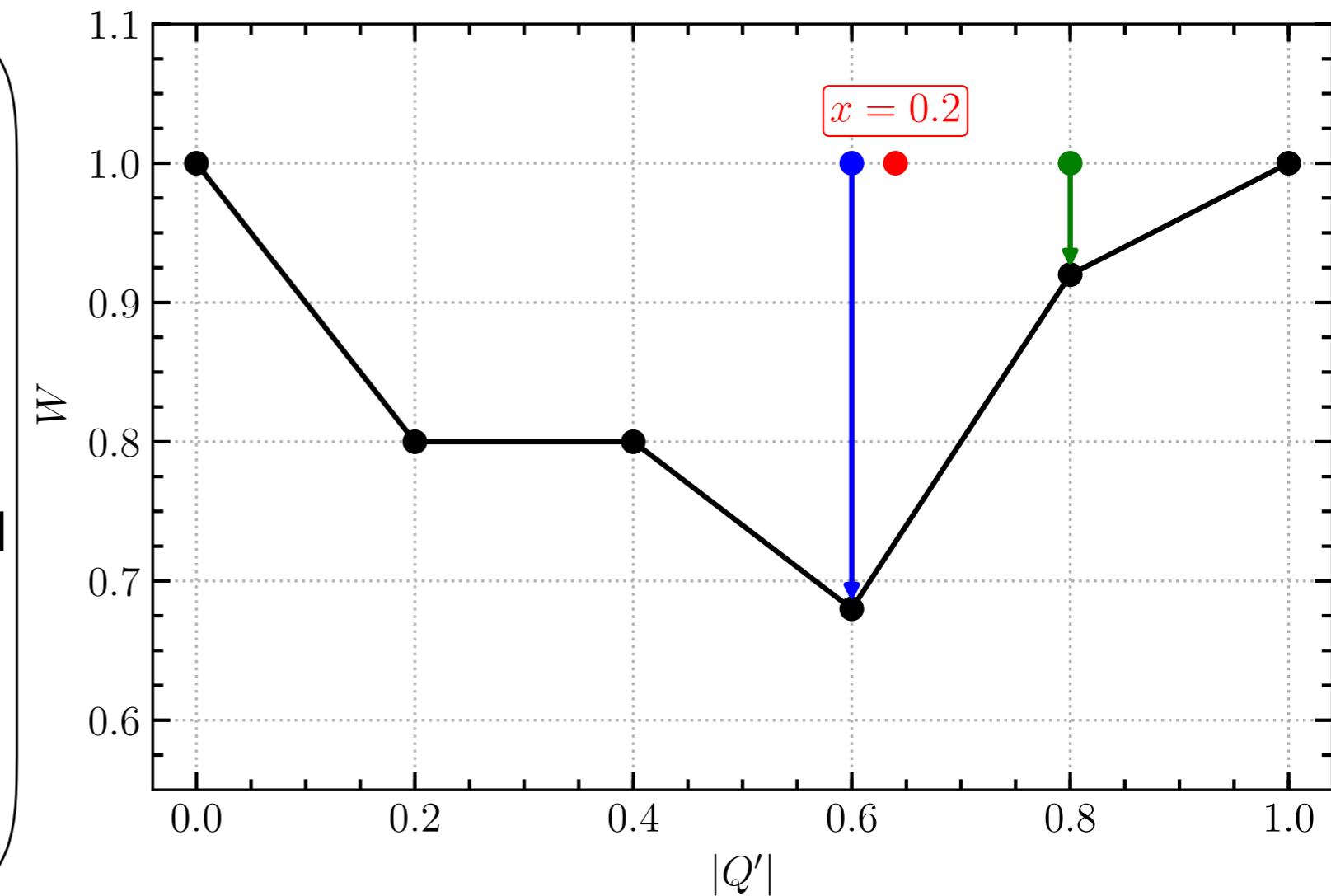
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4b. repeat previous steps
5. if W does not change much any more, stop building

