Fermions on Simplicial Lattices and their Dual Lattices

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What Am I Talking About?

Background

Naive and Staggered Fermions on an A_4 lattice

Naive and Staggere Fermions on an A_4^* lattice

Final Remarks and Sales Pitch

The isotropic lattices in every dimension

The notation comes from the book by Conway and Sloane.

- Z^n; The hypercubic lattices. Automorphism group has $2^n\, n!$ elements (=384 in 4-d).
- A_n; Also called "simplicial." Group order $= 2 \cdot n!$ (=240 in 4-d). In 2-d, triangular lattice. FCC in 3-d. Pure gauge models were simulated on an A_4 lattice.
- A_n^*; The lattice dual to A_n. In 3-d A_3^* is the BCC lattice.
- D_n; Also known as the "checkerboard" lattice. $D_3 = A_3$ is FCC. $D_4 = F_4$ is self-dual. Automorphism group of D_4 has 1152 elements. D_3, D_4, and D_5 are the densest possible lattice packings in 3, 4 and 5 dimensions.
- Hyperdiamond lattice is not a Bravais lattice. Union of 2 A_n lattices.

Extremely Abridged History

Noticed a long time ago [Celmaster and Krausz, (1983)] that fermions on non-cubic lattices are problematic:

$$\sum \bar{\psi}_{\mathbf{n}}\, \mathbf{e}_i \cdot \gamma \left(\psi_{\mathbf{n}+\mathbf{e}_i} - \psi_{\mathbf{n}-\mathbf{e}_i}\right)$$

Equations for doublers break rotational symmetry. There must be a symmetry connecting doublers to have rotational invariance and a reduction to staggered fermions.

Could add Wilson term. On D_4 you have rotational symmetry broken only at $O(a^4)$.

In 4-d, staggered fermions have only been satisfactorily formulated on hypercubic lattices.

Drouffe and Moriarty (1983) did simulations of pure SU(2) and SU(3) gauge theories on the A_4 lattice.

A Lattice Fermion Popularity Contest

Counting papers on hep-lat since 2017 using lattice fermions:

- 155 Wilson/clover,
- 86 domain wall
- 62 staggered
- 57 overlap
- 0 on non-cubic lattices

The A_4 lattice

Coordinate vector of A_d lattice:
$(n_1,\ n_2, \ldots, n_{d+1})$ where $\sum n_i = 0$ Surface in Z_{d+1} lattice.

Nearest neighbor vectors:

$$\boldsymbol{\epsilon}_{12} = (1, -1, 0, 0, 0),\ \boldsymbol{\epsilon}_{13} = (1, 0, -1, 0, 0), \ldots, \boldsymbol{\epsilon}_{45} = (0, 0, 0, 1, -1)$$

and negatives of these.
So 20 neighbors in 4-d, compared to 8 for hc.
Take primitive lattice vectors $\boldsymbol{\tau}_\mu = \boldsymbol{\epsilon}_{\mu 5}$:

$$\boldsymbol{\tau}_1 = (1, 0, 0, 0, -1), \ldots, \boldsymbol{\tau}_4 = (0, 0, 0, 1, -1)$$

Reciprocal lattice vectors, $\mathbf{b}_\mu$, defined by $\mathbf{b}_\mu \cdot \boldsymbol{\tau}_\nu = 2\pi\delta_{\mu\nu}$ are

$$\mathbf{b}_1 = \kappa(4, -1, -1, -1, -1), \ldots, \mathbf{b}_4 = \kappa(-1, -1, -1, 4, -1)$$

with $\kappa = 2\pi/5$, generate the lattice A_4^*.

Also need a set of orthonormal vectors on A_4:
$\mathbf{e}_1 = (1, -1, 0, 0, 0)/\sqrt{2}$, $\mathbf{e}_2 = (1, 1, -2, 0, 0)/\sqrt{6}$,
$\mathbf{e}_3 = (1, 1, 1, -3, 0)/\sqrt{12}$, $\mathbf{e}_4 = (1, 1, 1, 1, -4)/\sqrt{20}$.

The action:

$$S_A = \frac{\sqrt{2}}{8} i \sum_{\mathbf{n}} \sum_{j>i}^{5} \bar{\psi}_{\mathbf{n}} \gamma_i \gamma_j \left(\psi_{\mathbf{n}+\boldsymbol{\epsilon}_{ij}} - \psi_{\mathbf{n}-\boldsymbol{\epsilon}_{ij}} \right)$$

$$\{\gamma_i, \gamma_j\} = 2\delta_{\mu\nu}$$

The inverse free propagator in momentum space:

$$D(k) \propto \sum_{j>i}^{5} \gamma_i \gamma_j \sin(\mathbf{k} \cdot \boldsymbol{\epsilon}_{ij})$$

which leads to the propagator

$$S(k) \propto \sum_{j>i} \gamma_i \gamma_j \sin(\mathbf{k} \cdot \boldsymbol{\epsilon}_{ij}) / \sum_{j>i} \sin^2(\mathbf{k} \cdot \boldsymbol{\epsilon}_{ij})$$

The modes

Poles at $\mathbf{k} = 0$ and at

$$\mathbf{k} = \mathbf{b}_\mu/2$$

and sums of 2, 3 and all 4 of these, 16 in total.

5 modes at $|\mathbf{k}| = \sqrt{\frac{4}{5}}\pi \Leftrightarrow \frac{\pi}{5}(-4, 1, 1, 1, 1), \ldots \frac{\pi}{5}(1, 1, 1, 1, -4)$
10 modes at $|\mathbf{k}| = \sqrt{\frac{6}{5}}\pi \Leftrightarrow \frac{\pi}{5}(3, 3, -2, -2, -2), \ldots$

Symmetries connecting modes

The action is invariant under

$$\psi_{\mathbf{n}} \to T(n)\,\psi_{\mathbf{n}}, \quad \bar{\psi}_{\mathbf{n}} \to \bar{\psi}_{\mathbf{n}}\,T(n)$$

where

$$T(n) = (-1)^{n_\mu}\gamma_\mu$$

and products of these.

Since all modes are equivalent need only examine the one at $\mathbf{k} \approx 0$

For $k \approx 0$

$$D(k) \approx -\frac{1}{\sqrt{5}} \sum_{j>i} \gamma_i \gamma_j \, \mathbf{k} \cdot \boldsymbol{\epsilon}_{ij} \equiv i \sum_{\mu=1}^{4} \Gamma_\mu \mathbf{k} \cdot \mathbf{e}_\mu$$

Solving for Γ_μ:

$$\Gamma_\mu = i \sum_{i=1}^{5} e^i_\mu \gamma_i A$$

where

$$A = \frac{1}{\sqrt{5}} \sum_{i=1}^{5} \gamma^i$$

The Γ_μ comprise a set of Euclidean Dirac matrices:

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$$

Thus the action describes 16 Dirac fermions. We also have

$$\Gamma_5 = A = \frac{1}{\sqrt{5}} \sum_{i=1}^{5} \gamma^i$$

Short paws

Symmetry group of the A_4 lattice

Permuations of $(n_1, n_2, n_3, n_4, n_5)$, the "symmetric" group S_5.

Negation of all the coordinates is also a symmetry.

So 2 X 5! $=$ 240 elements.

S_5 is generated by single exchanges: e.g. (21345)

The action is invariant provided

$$\psi_{\mathbf{n}} \to \frac{1}{\sqrt{2}}(\gamma_1 - \gamma_2)\psi_{\mathbf{n}'}$$

$$\bar{\psi}_{\mathbf{n}} \to \bar{\psi}_{\mathbf{n}'}\frac{1}{\sqrt{2}}(\gamma_1 - \gamma_2).$$

Representations of some lattice objects

ϵ_{ij}, $\gamma_i\gamma_j$, $U_{ij} = e^{i\,A_{ij}}$ transform as 10-d rep. of S_5.

Orthogonality of characters $\rightarrow \mathbf{10} = \mathbf{4} \oplus \mathbf{6}$

$$i\gamma_i\gamma_j = \sqrt{\tfrac{2}{5}}\,\epsilon^{\mu}_{ij}\,\Gamma_\mu + i\sum_{\nu>\mu}(e^i_\mu e^j_\nu - e^j_\mu e^i_\nu)\Gamma_\mu\Gamma_\nu$$

showing reduction to vector and antisymmetric tensor.

Likewise:

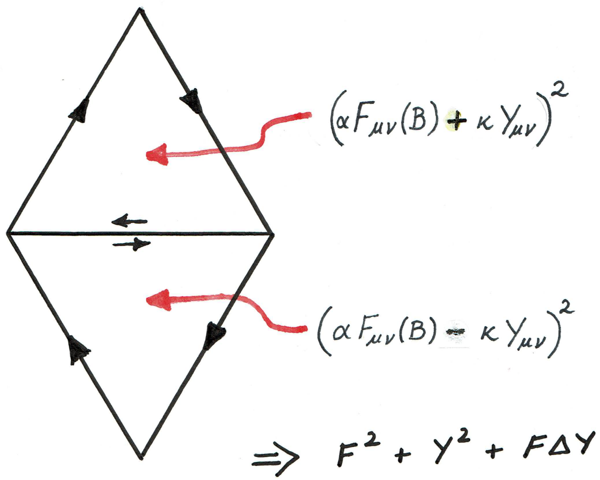
$$A_{ij} = \epsilon_{ij}^{\mu} B_\mu + \sum_{\nu>\mu} (e_\mu^i e_\nu^j - e_\mu^j e_\nu^i) Y_{\mu\nu}$$

the naive continuum limit:

$$\int d^4x \bar{\psi} \{ \Gamma_\mu (\partial_\mu - igB_\mu) + g\sigma_{\mu\nu} Y_{\mu\nu} \} \psi + m\bar{\psi}\psi$$

$Y_{\mu\nu}$ is short range $\rightarrow$ four-fermion interaction with coupling of order $a^2 g^2$.

The Action for the Link Variables



Absence of additive mass renormalization

Additive mass renormalization is forbidden, even though there is no exact axial symmetry. The action

$$S_A = \frac{\sqrt{2}}{8} i \sum_{\mathbf{n}} \sum_{j>i}^{5} \bar{\psi}_{\mathbf{n}} \, \gamma_i \gamma_j \, U_{\mathbf{n},ij} \, \psi_{\mathbf{n}+\epsilon_{ij}} + h.c.$$

is invariant under negation of all the coordinates provided

$$U_{ij} \to U_{ij}^{\dagger}; \quad \psi_{\mathbf{n}} \to \psi_{-\mathbf{n}}; \quad \bar{\psi}_{\mathbf{n}} \to -\bar{\psi}_{-\mathbf{n}}$$

This implies for the full propagator:

$$S(-p) = -S(p)$$

which forbids a mass term.

Mass or Wilson terms are not invariant.

No exact chiral symmetry $\rightarrow$ fermion determinant is not real (except for free fermions).

- In a simulation, the pseudo-fermion action

$$\phi(D^\dagger D + m^2)^{-1}\phi$$

  is real and $\approx det(D + m)$.

- Or to get to reality you can double the fermions $\psi \rightarrow (\psi_1,\ \psi_2)$ with a mass term $m\ \psi\sigma_3\psi$.

- Or go to a hyperdiamond lattice $(A_4 \cup A_4)$ with ψ_1 on one A_4 with mass m and ψ_2 on the other with mass $-m$. The coupling $\rightarrow$ axial-vector interaction mixing 1 and 2.

Axial Vector Interaction

Using

$$\gamma_i = -i \sum_{\mu} e^i_\mu \Gamma_\mu \Gamma_5 + \frac{1}{\sqrt{5}} \Gamma_5$$

a rotationally invariant, axial vector interaction is

$$\sum_{\mathbf{n}} \sum_{i}^{5} (\bar{\psi}_{\mathbf{n}}\, \gamma_i\, \psi_{\mathbf{n}+\mathbf{r}_i} + \bar{\psi}_{\mathbf{n}+\mathbf{r}_i}\, \gamma_i\, \psi_{\mathbf{n}}) Z_i(\mathbf{n})$$

the same for all doublers, where

$$\mathbf{r}_1 = (4, -1, -1, -1, -1), \ldots, \mathbf{r}_5 = (-1, -1, -1, -1, 4)$$

generate an A_4^* sublattice. So axial currents live on a dual sublattice.

Naive continuum limit $\Rightarrow \bar{\psi}\, \Gamma_\mu \Gamma_5 \psi\, A^5_\mu + \bar{\psi} \Gamma_5 \psi\, \phi$

Reduction to Staggered Fermions

Naive action is diagonalized by:

$$\psi_{\mathbf{n}} \to \gamma_1^{n_1}\gamma_2^{n_2}\gamma_3^{n_3}\gamma_4^{n_4}\gamma_5^{(n_1+n_2+n_3+n_4)}\psi_{\mathbf{n}}$$

leading to the staggered fermion action

$$S_{st} = \sum \bar{\chi}_{\mathbf{n}}\ \eta_i(n)\eta_j(n)\ (\chi_{\mathbf{n}+\epsilon_{ij}} - \chi_{\mathbf{n}-\epsilon_{ij}}) + m\bar{\chi}_{\mathbf{n}}\chi_{\mathbf{n}}$$

where χ_n is a single anticommuting variable and the phases are

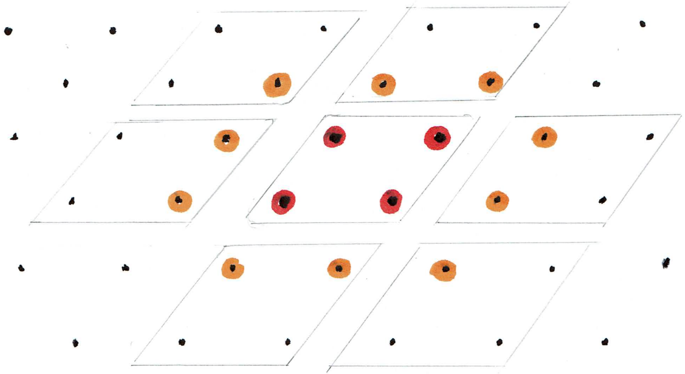
$$\eta_1 = 1,\ \eta_2 = (-1)^{n_1},\ \eta_3 = (-1)^{n_1+n_2},\ \eta_4 = (-1)^{n_1+n_2+n_3},$$
$$\eta_5 = (-1)^{n_1+n_2+n_3+n_4}$$

Can make blocks of 16 points as on hypercubic lattice.

Degrees of freedom in a block couple to degrees of freedom in 20 neighboring blocks.

All the symmetries of the naive fermions carry through to the staggered case. There is no additive mass renormalization.

Staggered Blocks on Triangular Lattice



Fermions on an A_4^* lattice

The action:

$$S = \frac{5}{16} \sum_{\mathbf{n}} \sum_{j}^{5} \bar{\psi}_{\mathbf{n}} \gamma_i \left(\psi_{\mathbf{n}+\mathbf{f}_j} - \psi_{\mathbf{n}-\mathbf{f}_j} \right)$$

where

$$\mathbf{f}_1 = \kappa(4, -1, -1, -1, -1), \ldots, \mathbf{f}_5 = \kappa(-1, -1, -1, -1, 4)$$

with $\kappa = 1/\sqrt{20}$.
Take the first 4 to be primitive vectors. The doubling symmetry is then

$$\psi_{\mathbf{n}} \to (-1)^{n_\mu} \gamma_\mu \psi_{\mathbf{n}}$$

The propagator

$$S(k) \propto \sum_i \gamma_i \sin(\mathbf{k} \cdot \mathbf{f}_i) / \sum_i \sin^2(\mathbf{k} \cdot \mathbf{f}_i)$$

has a mode at $\mathbf{k} = 0$, and 10 modes at

$$\alpha(1, -1, 0, 0, 0), \ldots, \alpha(0, 0, 0, 1, -1); \quad \alpha = 2\pi/\sqrt{5}$$

and 5 modes at

$$\alpha(0, 1, 1, -1, -1), \ldots, \alpha(1, 1, -1, -1, 0)$$

For $\mathbf{k} \approx 0$ the inverse propagator

$$\Rightarrow \frac{2}{\sqrt{5}} \sum_i \gamma_i \, \mathbf{k} \cdot \mathbf{f}_i \equiv \sum_{\mu=1}^{4} \Gamma_\mu \mathbf{k} \cdot \mathbf{e}_\mu$$

$$\Rightarrow \Gamma_\mu = \frac{2}{\sqrt{5}} \sum_{i=1}^{5} \mathbf{f}_i \cdot \mathbf{e}_\mu \gamma_i$$

which obey

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu}$$

and as for A_4

$$\Gamma_5 = \frac{1}{\sqrt{5}} \sum_{i=1}^{5} \gamma^i$$

The naive continuum limit is

$$\int d^4x\bar{\psi}\{\Gamma_\mu(\partial_\mu - igB_\mu) + g\Gamma_5\ \phi\}\psi + m\bar{\psi}\psi$$

Absence of additive mass renormalization works the same.

The staggered action is

$$S_{st} = \sum \bar{\chi}_{\mathbf{n}}\ \eta_i(n)\ (\chi_{\mathbf{n}+\mathbf{f}_i} - \chi_{\mathbf{n}-\mathbf{f}_i}) + m\bar{\chi}_{\mathbf{n}}\chi_{\mathbf{n}}$$

where

$$\eta_1 = 1,\ \ \eta_2 = (-1)^{n_1},\ \ \eta_3 = (-1)^{n_1+n_2},\ \ \eta_4 = (-1)^{n_1+n_2+n_3},$$
$$\eta_5 = (-1)^{n_1+n_3}$$

Axial Interactions on the A_4^* lattice

An axial interaction with the same charge for all the doublers is

$$\sum_{\mathbf{n}} \sum_{j>i}^{5} (\bar{\psi}_{\mathbf{n}} \, \gamma_i \gamma_j \, \psi_{\mathbf{n}+\mathbf{f}_i-\mathbf{f}_j} + \bar{\psi}_{\mathbf{n}+\mathbf{f}_i-\mathbf{f}_j} \, \gamma_i \gamma_j \, \psi_{\mathbf{n}}) A_{ij}$$

The vectors $\mathbf{f}_i - \mathbf{f}_j$ generate an A_4 sublattice.

So, again, axial interactions live on a dual sublattice.

The Last Slide

Fermions on A_4 and A_4^* lattices are interesting (at least to one person), and might be useful in simulations. Drouffe and Moriarty claimed that (quenched) simulations on A_4 are faster than on hypercubic.

Mean field calculations, including $1/d$ corrections, are better. The corrections are smaller because you're really expanding in $1/(\textit{kissing number})$.

The duality between vector and axial vector currents paralleling the duality between A_4 and A_4^* lattices is interesting.

Would be interesting to find a fermion formulation on D_n ($D_4 = F_4$) lattices, as they have more rotational symmetry (broken at $O(a^4)$). At least someone could try Wilson fermions.

all.jpg 

Odd numbers of exchanges, e.g. (23145) or (21435) are rotations. Subgroup of S_5 called A_5, the alternating group.

In even dimensions, negation of all the coordinates has det $= 1$, a 180 deg rotation.

S_5 has representations of dimensions 1, 1, 4, 4, 5, 5 and 6.

Chiral Symmetry

Recall

$$\Gamma_5 = \frac{1}{\sqrt{5}} \sum_{i=1}^{5} \gamma^i$$

Can't do:

$$\psi_{\mathbf{n}} \to e^{i\phi\,\Gamma_5}\,\psi_{\mathbf{n}}$$

No doubling symmetry.
Chiral transformation same for all modes:

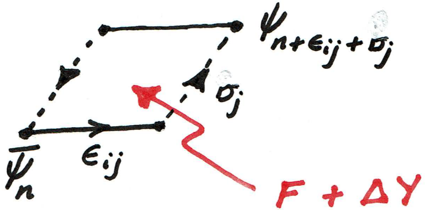
$$\psi_{\mathbf{n}} \to \psi_{\mathbf{n}} + \frac{i}{\sqrt{5}}\phi \sum_j \gamma_j \sum_{\boldsymbol{\sigma}_j} \psi_{\mathbf{n}+\boldsymbol{\sigma}_j}$$

e.g.

$$\boldsymbol{\sigma}_1 = (0,1,1,-1,-1),\ (0,1,-1,1,-1),\ldots(0,-1,-1,1,1)$$

The Anomaly

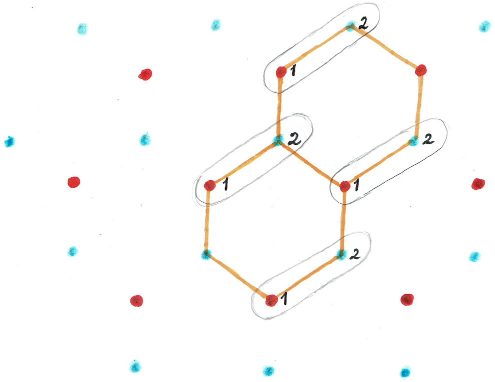
$$\delta S \Rightarrow i\varphi$$



$$\langle \bar\psi_n \gamma_i \psi_{n+\epsilon_{ij}+\sigma_j} - \ldots \rangle = c\tilde{F}$$

$$\Rightarrow \quad \delta S = c\varphi F\tilde{F}$$

Hexagonal Lattice



$$\left(\sigma_1 \Delta_1 + \sigma_2 \Delta_2 + m\right)\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0$$

Short Sales Pitch

More nearest-neighbors ⇒

- longer correlation length for given bare coupling constant.
- Faster thermalization times ⇒ Shorter auto-correlation times? At least in a disordered phase.
- More rotational symmetry.
-

The Bad: more nearest-neighbors ⇒

- More computation per simulation step.
- More link degrees of freedom per site.