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Numerical study of QED finite-volume effects using lattice scalar QED

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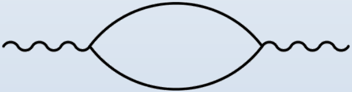
Introduction

Two projects, to calculate QED finite-volume effects (FVEs) for

- Hadron self-energy in moving frame
- Hadronic vacuum polarisation (HVP)

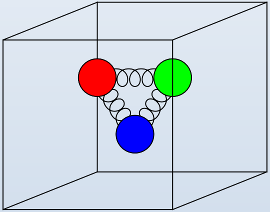
using scalar QED as an effective theory.

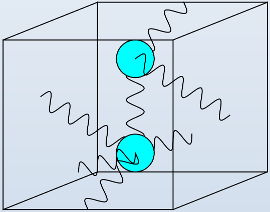
I will focus on the numerical aspect of these projects (lattice scalar QED simulations).



QED FVEs

- As lattice QCD errors approach 1%, QED corrections become important.
- QED FVEs are larger than QCD FVEs, due to long-range interaction.
- Low-energy effects - can be calculated in effective theory (e.g. scalar QED).



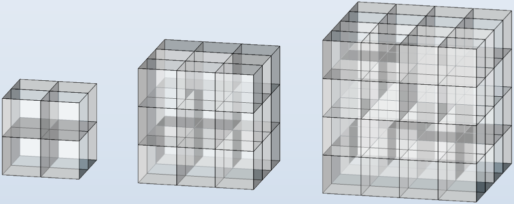


QED FVEs

- Using the QED_L framework, FVEs have been calculated for hadron masses [Davoudi & Savage, arXiv:1402.6741; Borsanyi et al., arXiv:1406.4088] and leptonic decay amplitudes [Lubicz et al., arXiv:1611.08497].
- We extend the calculation for hadron masses to moving frames, for both on-shell and off-shell momentum, and calculate QED FVEs for the HVP.

Numerical strategy

- Simulate lattice scalar QED on several volumes.
- Cross-check analytic calculations.
- Cheap simulations - high precision, low computational cost.
- Understand features of QED on the lattice which may otherwise be overlooked.



Scalar QED on the lattice

Discretised scalar QED action:

$$S[\phi, A] = S_\phi[\phi, A] + S_{\text{Feyn.}}[A]$$

Feynman-gauge QED action:

$$\begin{aligned} S_{\text{Feyn.}}[A] &= a^4 \sum_x \left\{ \frac{1}{4} \sum_{\mu,\nu} \left(\delta_\mu A_\nu(x) - \delta_\nu A_\mu(x)\right)^2 + \frac{1}{2} \sum_\mu \left(\delta_\mu A_\mu(x)\right)^2 \right\} \\ &= -\frac{a^4}{2} \sum_x A_\mu(x) \delta^2 A_\mu(x) \end{aligned}$$

- In momentum space, $\tilde{A}_\mu(k)$ is Gaussian - cheap to sample gauge configurations [Duncan, Eichten & Thacker, arXiv:hep-lat/9602005].
- Subtract zero mode using QED_L scheme [Uno & Hayakawa, arXiv:0804.2044].

Scalar QED on the lattice

Scalar action:

$$
\begin{aligned}
S_\phi[\phi, A] &= \frac{a^4}{2} \sum_x \phi^*(x) \Delta \phi(x) \\
\Delta &= -\sum_\mu D_\mu^* D_\mu + m^2 \\
D_\mu f(x) &= a^{-1} \left[e^{iqaA_\mu(x)} f(x + a\hat{\mu}) - f(x) \right]
\end{aligned}
$$

Quenched theory: set scalar determinant = 1 in path integral.

Calculating the scalar propagator

Alternative to CG, making use of FFT.

Expand scalar propagator around $q = 0$:

$$\Delta = -\sum_{\mu} D_{\mu}^{*} D_{\mu} + m^2 = \Delta_0 + q\Delta_1 + q^2\Delta_2 + \mathcal{O}\left(q^3\right)$$

$$\begin{aligned} \Delta^{-1} &= \Delta_0^{-1} - q\Delta_0^{-1}\Delta_1\Delta_0^{-1} \\ &\quad + q^2\left[\Delta_0^{-1}\Delta_1\Delta_0^{-1}\Delta_1\Delta_0^{-1} - \Delta_0^{-1}\Delta_2\Delta_0^{-1}\right] + \mathcal{O}\left(q^3\right) \end{aligned}$$

Calculating the scalar propagator

Free scalar propagator:

$$\Delta_0^{-1} = \mathcal{F}^{-1} \frac{1}{\hat{k}^2 + m^2} \mathcal{F}$$

Photon vertex operators:

$$\begin{aligned}
\Delta_1 &= -ia^{-1}\mathcal{F}^{-1} \sum_\mu \left[\mathcal{F} A_\mu \mathcal{F}^{-1} e^{iak_\mu} - e^{-iak_\mu} \mathcal{F} A_\mu \mathcal{F}^{-1} \right] \mathcal{F} \\
\Delta_2 &= \frac{1}{2}\mathcal{F}^{-1} \sum_\mu \left[\mathcal{F} A_\mu^2 \mathcal{F}^{-1} e^{iak_\mu} + e^{-iak_\mu} \mathcal{F} A_\mu^2 \mathcal{F}^{-1} \right] \mathcal{F}
\end{aligned}$$

where $\mathcal{F}$ represents the Fourier transform.

Scalar self-energy

- Self-energy $\Sigma(p)$ is defined by the amputated $\mathcal{O}\left(q^2\right)$ corrections to the scalar propagator.
- When p is a lattice mode (off-shell), calculate $\Sigma(p)$ directly from momentum-space scalar propagator.
- When p is on-shell ($p_0 = i\sqrt{\mathbf{p}^2 + m^2}$), obtain $\Sigma(p)$ from large-time behaviour of correlators.

Scalar self-energy - on-shell

Time-momentum scalar propagator:

$$C(t,\mathbf{p}) = C_0(t,\mathbf{p}) + q^2 C_1(t,\mathbf{p}) + \mathcal{O}\left(q^4\right)$$

In continuous space-time (for simplicity):

$$\begin{aligned} C_0(t,\mathbf{p}) &= \int \frac{dp_0}{2\pi} \frac{e^{ip_0 t}}{p^2+m^2} = \frac{e^{-\omega(\mathbf{p})|t|}}{2\omega(\mathbf{p})} \\ C_1(t,\mathbf{p}) &= \int \frac{dp_0}{2\pi} \frac{\Sigma(p)}{\left(p^2+m^2\right)^2} e^{ip_0 t} \end{aligned}$$

where $\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2+m^2}$

Scalar self-energy - on-shell

Ground state contribution is from pole at $p_0 = i\omega(\mathbf{p})$:

$$C_1^{(\Sigma)}(t, \mathbf{p}) = \frac{e^{-\omega(\mathbf{p})|t|}}{4\omega(\mathbf{p})^3} \left\{ [1 + |t|\,\omega(\mathbf{p})]\,\Sigma(p_{\text{o.s.}}) - i\omega(\mathbf{p}) \frac{\partial \Sigma}{\partial p_0}(p_{\text{o.s.}}) \right\}$$

so we can construct an effective self-energy:

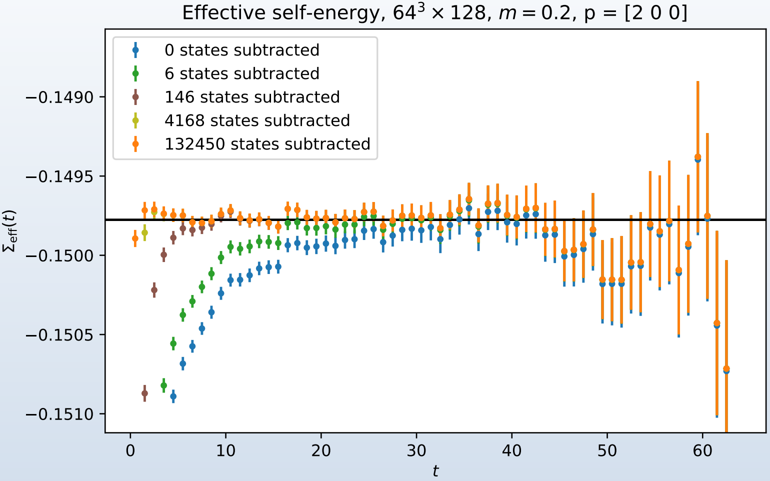
$$\Sigma_{\text{eff.}}(t, \mathbf{p}) = 2\omega(\mathbf{p})\,\text{sign}(t)\,\partial_t \left[\frac{C_1(t, \mathbf{p})}{C_0(t, \mathbf{p})} \right]$$

[de Divitiis et al., arXiv:1303.4896; Giusti et al., arXiv:1704.06561]

Scalar self-energy - excited states

- $C_1(t, \mathbf{p})$ also gets excited state contributions from poles of $\Sigma(p)$ at $p_0 = i\omega_\gamma(\mathbf{p}, \mathbf{k})$, where $\omega_\gamma(\mathbf{p}, \mathbf{k}) = |\mathbf{k}| + \omega(\mathbf{p} - \mathbf{k})$.
- Ground state dominance relies on energy gap $\omega_\gamma(\mathbf{p}, \mathbf{k}) - \omega(\mathbf{p})$, which vanishes in the infinite-volume limit
- In scalar QED we can calculate excited state contributions analytically and subtract them.
- In QCD calculations this is not possible - severe contamination from excited states at large volumes.

Effective self-energy



Results: scalar self-energy

On-shell:

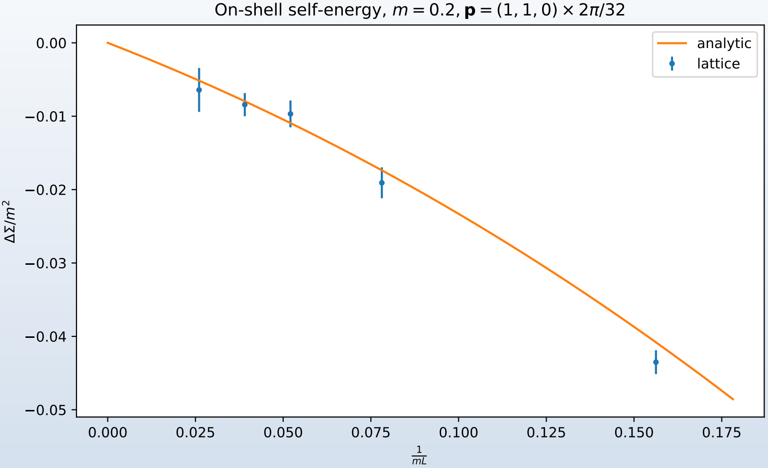
$$\Delta\Sigma\left(p_{\mathrm{o.s.}}\right) = m^2 \left\{ \frac{1}{\gamma\left(|\mathbf{v}|\right)} \frac{c_{2,1}\left(\mathbf{v}\right)}{4\pi^2 mL} + \frac{c_1}{2\pi\left(mL\right)^2} + \left[\frac{1}{\gamma\left(|\mathbf{v}|\right)^3} - \frac{1}{\gamma\left(|\mathbf{v}|\right)} \right] \frac{1}{2\left(mL\right)^3} + \mathcal{O}\left(e^{-mL}\right) \right\}$$

Off-shell:

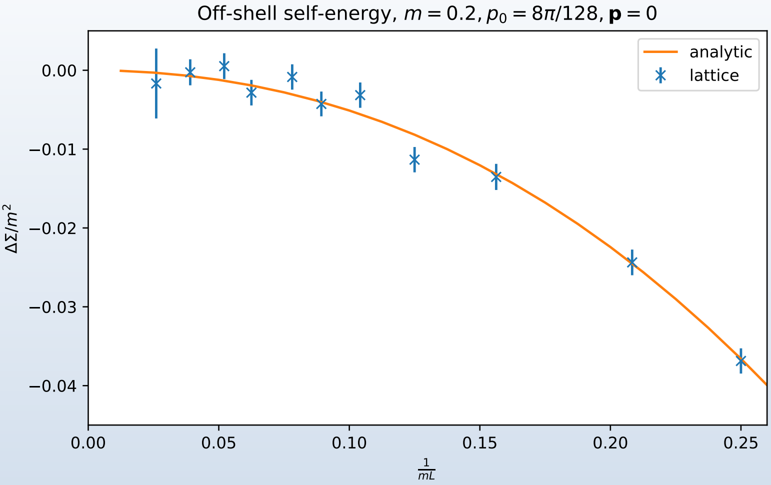
$$\Delta\Sigma\left(p\right) = m^2 \left\{ \frac{1}{\gamma\left(|\mathbf{v}|\right)^2} \frac{c_1}{\pi\sigma\left(mL\right)^2} + \left[\left(\frac{4}{\sigma^2} - \frac{2}{\sigma}\right) \frac{1}{\gamma\left(|\mathbf{v}|\right)^3} + \left(\frac{1}{2} - \frac{2}{\sigma}\right) \frac{1}{\gamma\left(|\mathbf{v}|\right)} \right] \frac{1}{\left(mL\right)^3} + \mathcal{O}\left(\frac{1}{\left(mL\right)^4}\right) \right\}$$

where $\sigma = p_0^2/\omega\left(\mathbf{p}\right)^2 + 1$, $\mathbf{v} = \mathbf{p}/\omega\left(\mathbf{p}\right)$ and $\gamma\left(\beta\right) = 1/\sqrt{1-\beta^2}$.

Results: scalar self-energy

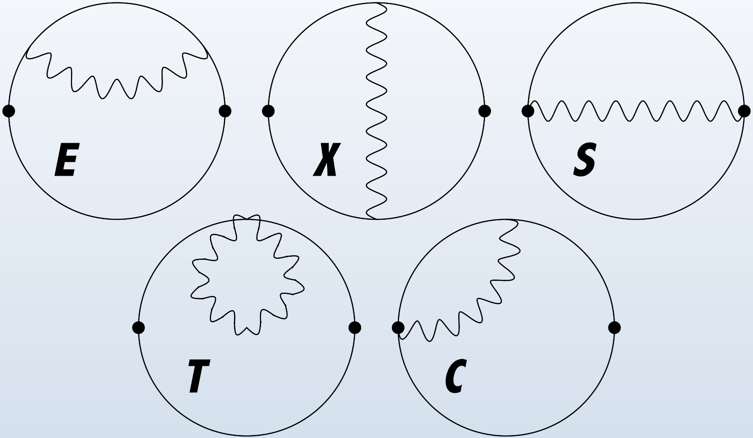


Results: scalar self-energy



HVP

5 NLO connected diagrams:



$$\Pi\left(q^2\right) = 2 \cdot E + 2 \cdot T + X + S + 4 \cdot C$$

HVP - renormalisation

- FVEs are IR effects, so naïvely expect unrenormalised theory to have same FVEs as renormalised theory.
- Not true at 2 loops - exponential FVE multiplied by quadratic divergence.
- Need to calculate counter-terms.

HVP - renormalisation

Three counter-terms relevant here:

- $\delta_m \phi^\dagger \phi$ (mass renormalisation)
- $\delta_Z \hat{p}^2 \phi^\dagger \phi$ (wavefunction renormalisation)
- $\delta_V \left(iq\right) \left(\widehat{p_1 + p_2}\right)^\mu \phi^\dagger A_\mu \phi$ (vertex correction)

Renormalisation conditions:

- $\Sigma(0) = 0$ fixes δ_m
- $\Sigma(q = 2\pi/128) = 0$ fixes δ_Z
- Ward identity fixes $\delta_V = \delta_Z$

Results: HVP

- For each diagram (E, T, S, C, X), leading FV correction is $\mathcal{O}\left(1/L^2\right)$.
- For full HVP, $\mathcal{O}\left(1/L^2\right)$ terms cancel and leading correction is $\mathcal{O}\left(1/L^3\right)$.
- Absence of $\mathcal{O}\left(1/L\right)$ and $\mathcal{O}\left(1/L^2\right)$ corrections is universal.

Results: HVP

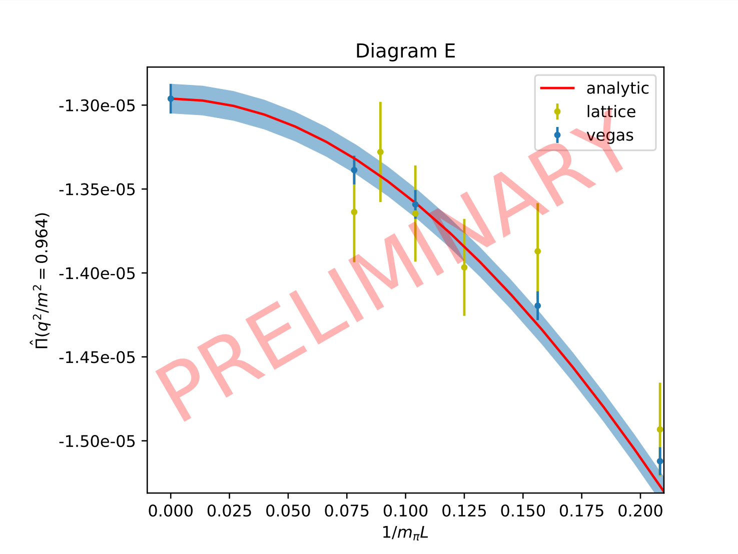
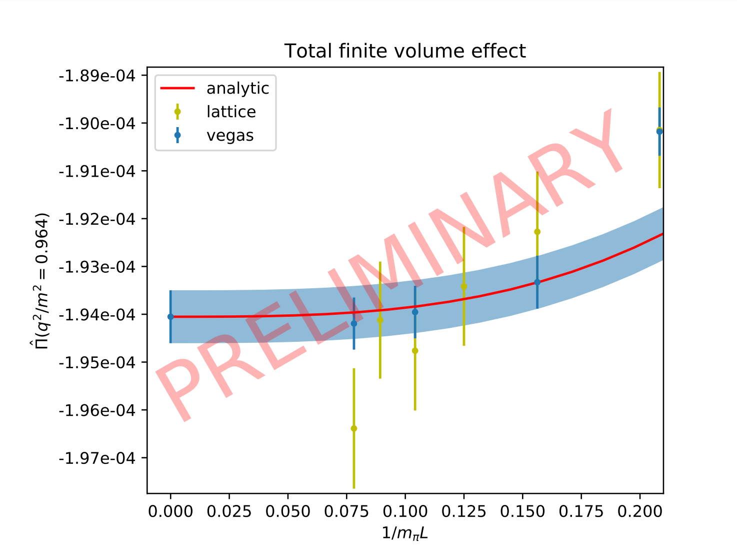


Diagram E
analytic
lattice
vegas
PRELIMINARY
-1.30e-05
-1.35e-05
-1.40e-05
-1.45e-05
-1.50e-05
$\hat{\Pi}(q^2/m^2 = 0.964)$
0.000
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200
$1/m_\pi L$

Results: HVP

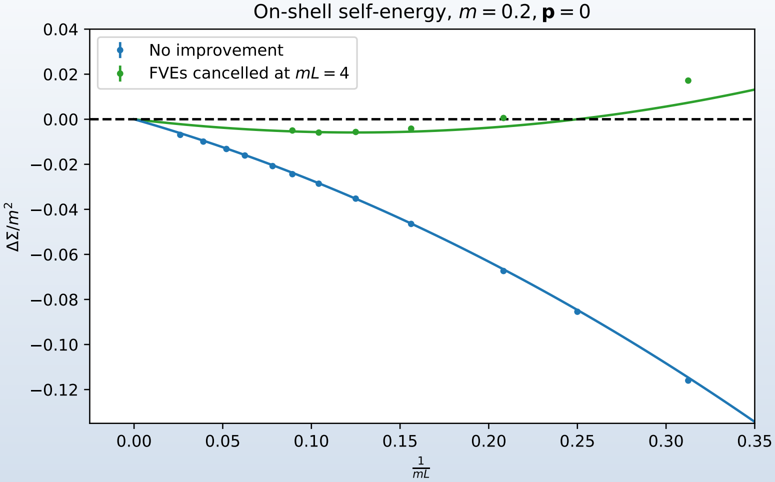


Total finite volume effect
analytic
lattice
vegas
PRELIMINARY
-1.89e-04
-1.90e-04
-1.91e-04
-1.92e-04
-1.93e-04
-1.94e-04
-1.95e-04
-1.96e-04
-1.97e-04
$\hat{\Pi}(q^2/m^2 = 0.964)$
0.000
0.025
0.050
0.075
0.100
0.125
0.150
0.175
0.200
$1/m_\pi L$

Infrared improvement

- We are free to modify a small subset of photon modes in addition to the zero-mode.
- We can tune these modes to suppress FVEs for a given quantity.

Infrared improvement



Conclusions

- Extended hadron self-energy QED_L FVE calculations to moving frame.
- Calculated QED_L FVEs for the HVP.
- Verified results using numerical simulations of scalar QED.
- Through verification process, understood features which need to be handled in numerical simulations:
  - excited states
  - renormalisation
- Outlook:
  - Use these results in the development of QCD + QED calculations,
  - Further investigation of infrared improvement.