Can the complex Langevin method see the deconfinement phase transition in QCD at finite density?

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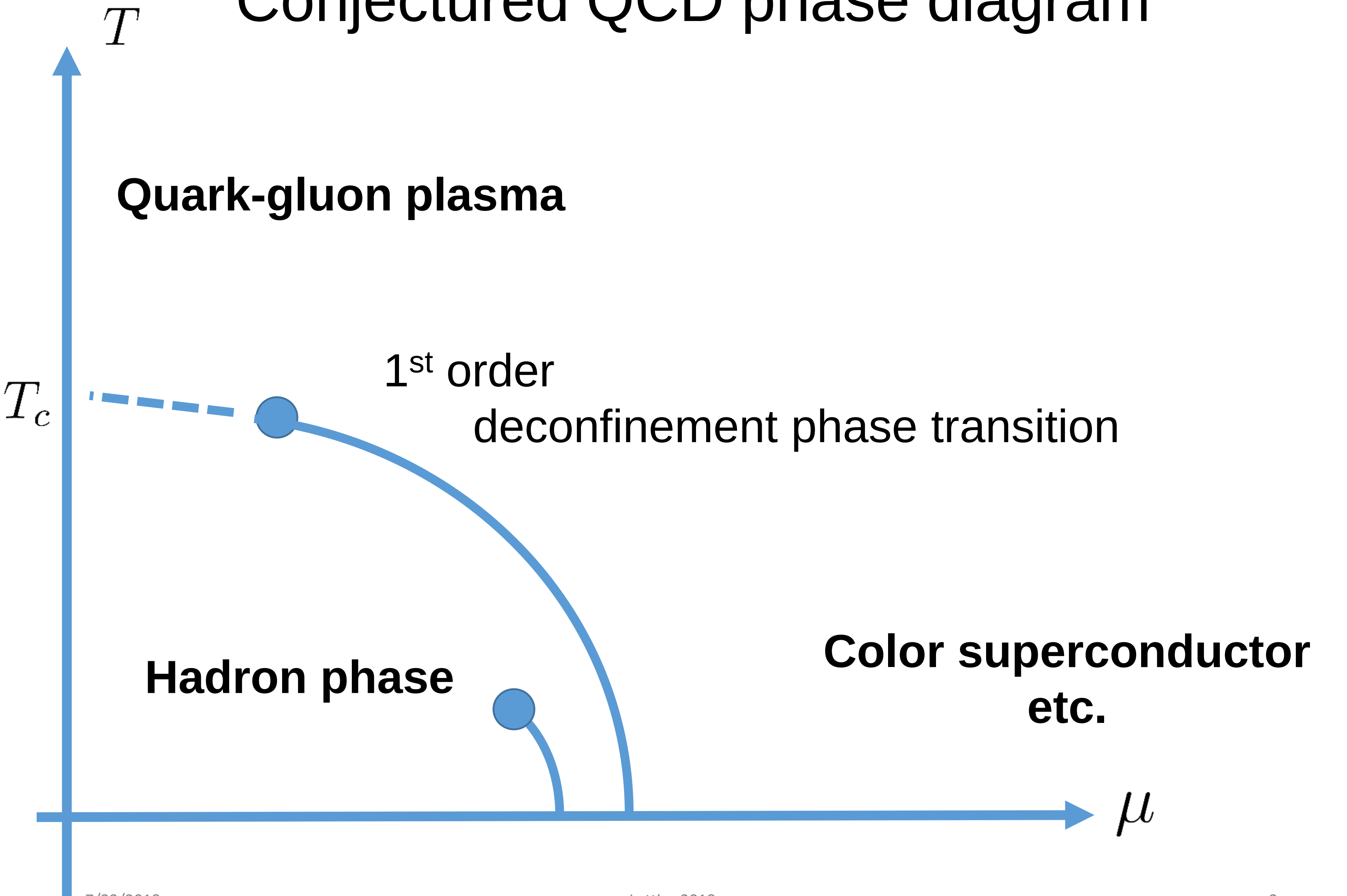
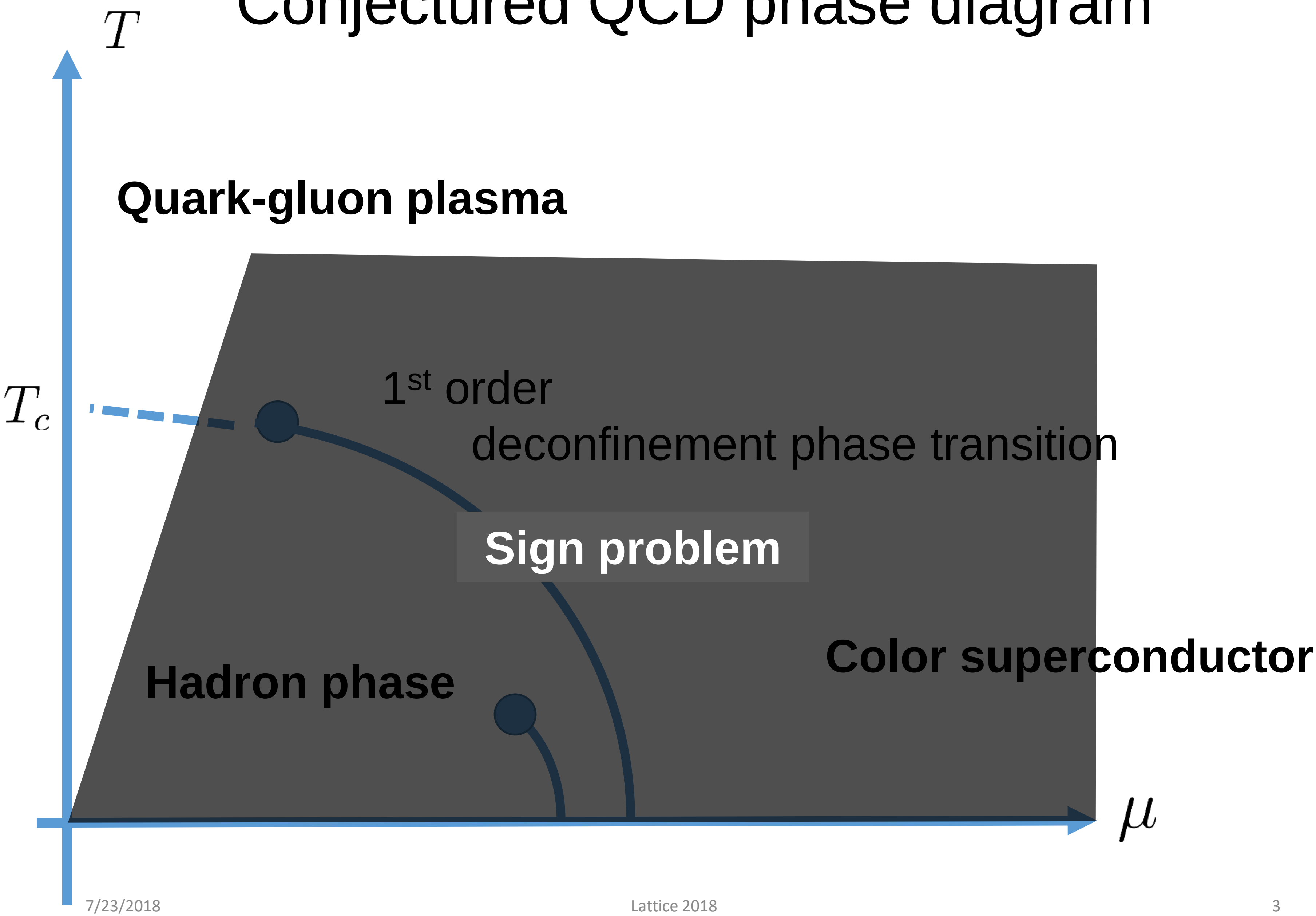
Conjectured QCD phase diagram



Conjectured QCD phase diagram



Finite density QCD

QCD partition function

$$Z = \int dU \det M[U,\mu] \mathrm{e}^{-S_g[U]}$$

$$M = D + m$$

The origin of the sign problem

$\det M$ is complex when $\mu \neq 0$

A promising way to solve the sign problem:
complex Langevin method

Complex Langevin method for QCD

$$Z = \int dU \det M[U,\mu] \mathrm{e}^{-S_g[U]}$$

[Parisi ‘83], [Klauder ‘84]
[Aarts, Seiler, Stamatescu ‘09]
[Aarts, James, Seiler, Stamatescu ‘11]
[Seiler, Sexty, Stamatescu ‘13]
[Sexty ‘14] [Fodor, Katz, Sexty, Torok ‘15]
[Sinclair, Kogut ‘16]
[Nishimura, Shimasaki ‘15]
[Nagata, Nishimura, Shimasaki ‘15]

Complexification

$$U_{x\mu} \in SU(3) \to \mathcal{U}_{x\mu} \in SL(3,\mathbb{C}) \qquad S(U) \to S(\mathcal{U})$$

The complex Langevin eq. of QCD

$$\mathcal{U}_{x\mu}(t+\epsilon) = \exp\left[i\left(-\epsilon \boxed{\mathcal{D}_{x\mu} S[\mathcal{U}]} + \sqrt{\epsilon}\eta_{x\mu}(t)\right)\right]\mathcal{U}_{x\mu}(t)$$

Drift term

Criterion of correctness

Exponential falloff of the drift distribution

Complex Langevin is reliable

Power-law falloff of the drift distribution

Complex Langevin gives incorrect answer

[Nagata, Nishimura, Shimasaki ‘15]

The main causes of the power-law falloff:

Excursion problem: large deviation of the link variables from SU(3)

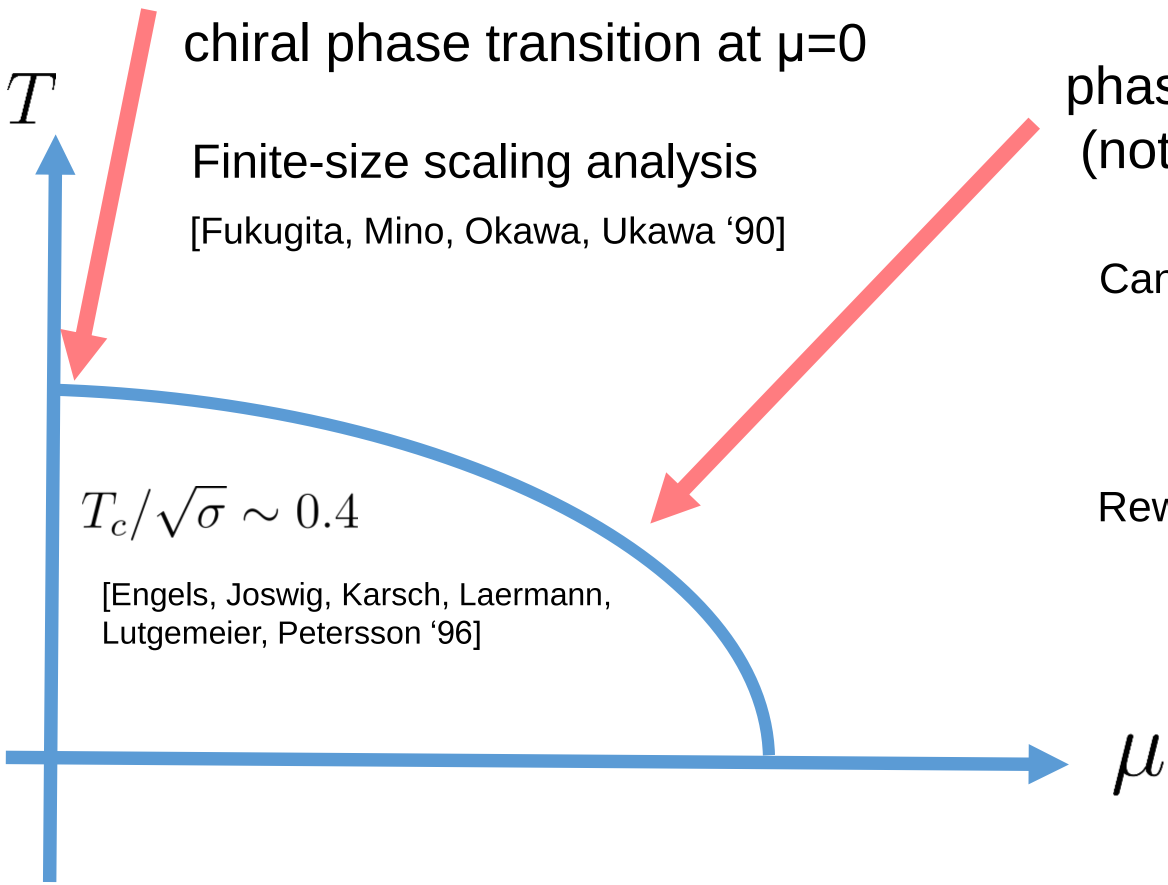
Singular drift problem: small eigenvalues of the fermion matrix

Phase diagram of QCD with 4-flavor staggered fermion

1st order
chiral phase transition at μ=0

Finite-size scaling analysis

[Fukugita, Mino, Okawa, Ukawa '90]



$T_c/\sqrt{\sigma} \sim 0.4$

[Engels, Joswig, Karsch, Laermann, Lutgemeier, Petersson '96]

phase transition at finite μ
(not completely established)

Canonical method

[de Forcrand, Kratochvila '06]
[Li, Alexandru, Liu, Meng '10]

Reweighting and complex Langevin

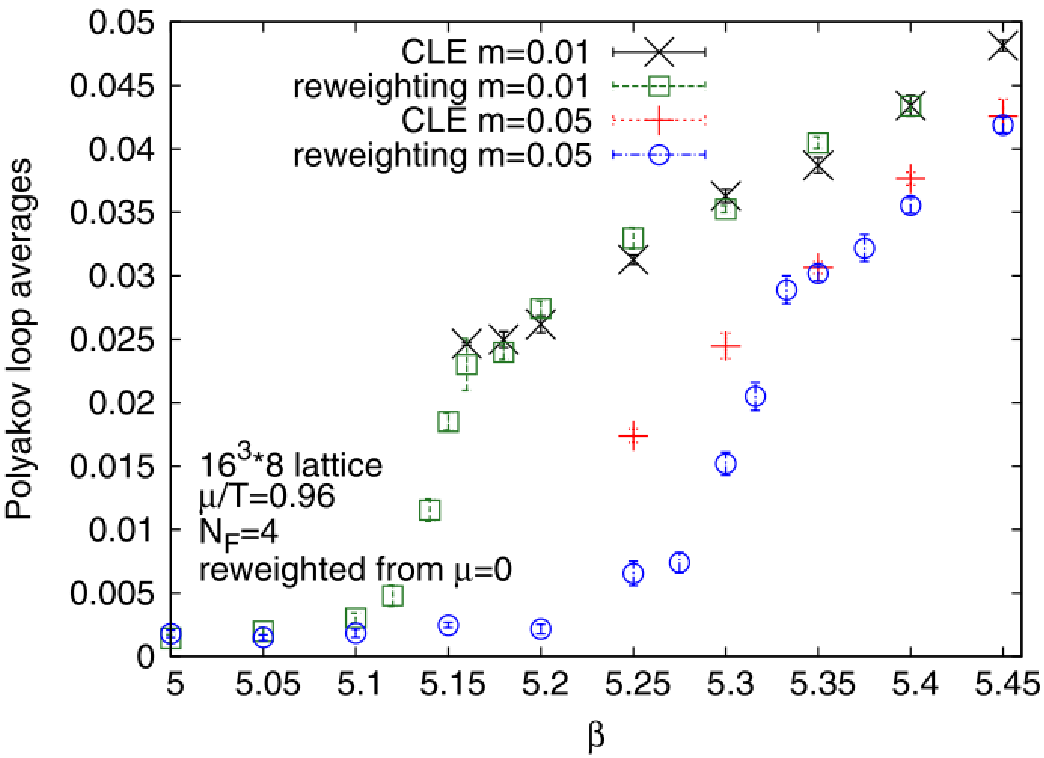
[Fodor, Katz, Sexty, Torok '15]

Previous study

[Fodor, Katz, Sexty, Torok '15]

Previous studies of N_f = 4 high density QCD:

Lattice size: $16^3 \times 8$



For m=0.01,

Reweighting method implies phase transition at $\beta \sim 5.15$

However, complex Langevin breaks down at $\beta < 5.15$

Motivation of our study

If the temporal lattice size is large enough,
complex Langevin may be able to detect the phase transition.

For instance, when $\beta = 5.2$, $m_q a = 0.01$,
the temperature becomes…

$N_T = 6$	$T \sim 300$ MeV	[Fodor, Katz, Sexty, Torok '15]
$N_T = 8$	$T \sim 220$ MeV	
$N_T = 12$	$T \sim 150$ MeV	Our study

If the phase transition is first order, we should be also careful of **hysteresis**.

Setup

- N_f = 4, staggered fermion
- Lattice size: $20^3 \times 12$, $24^3 \times 12$
- β = 5.2 - 5.6
- μ/T = 1.2
- Quark mass: $m_q a$ = 0.01
- Number of Langevin steps = 10^4 – 10^5
- Computer resources: K computer

Physical scales:

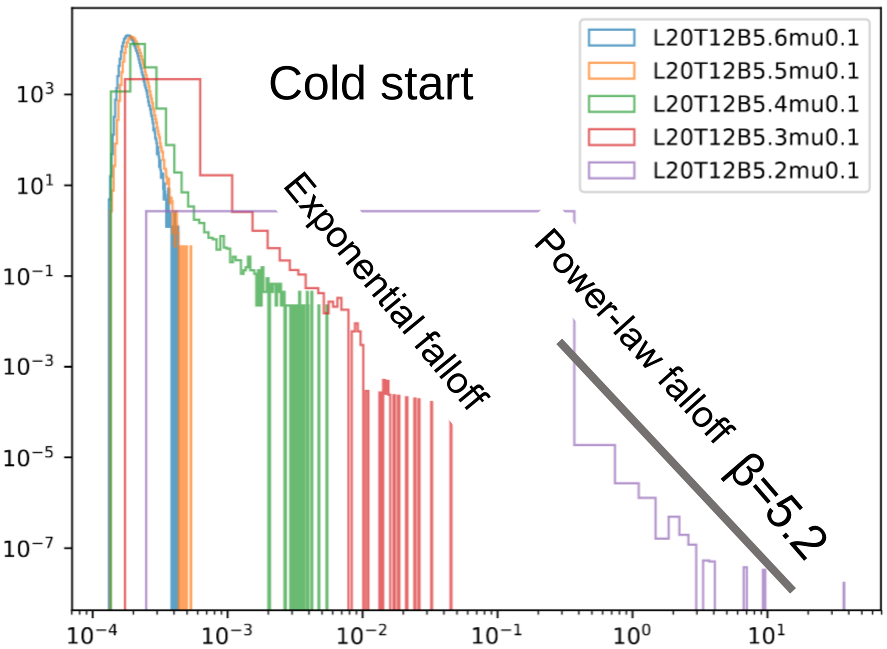
(β=5.2) $a \simeq 0.11\mathrm{fm} \quad m_\pi \simeq 530\mathrm{MeV}$

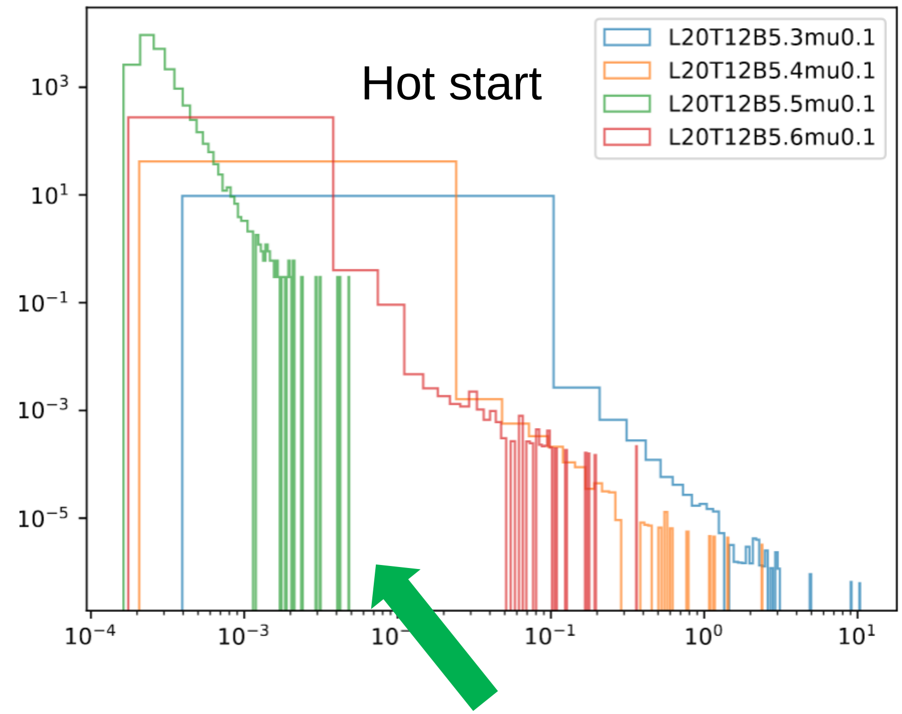
(β=5.4) $a \simeq 0.07\mathrm{fm} \quad m_\pi \simeq 740\mathrm{MeV}$

[Fodor, Katz, Sexty, Torok '15]

Reliability of the simulation (L=20)

Histograms of the drift term (only the fermionic contribution is shown)





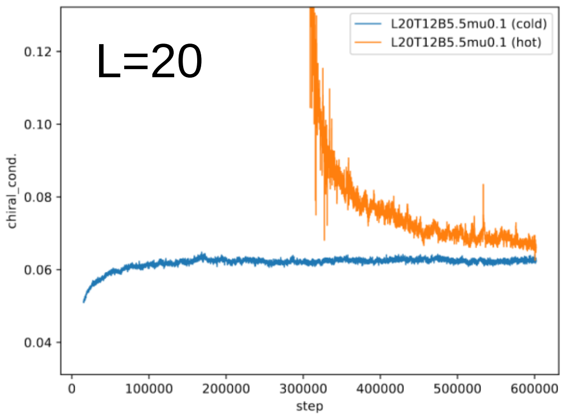
Reliable: β=5.3-5.6

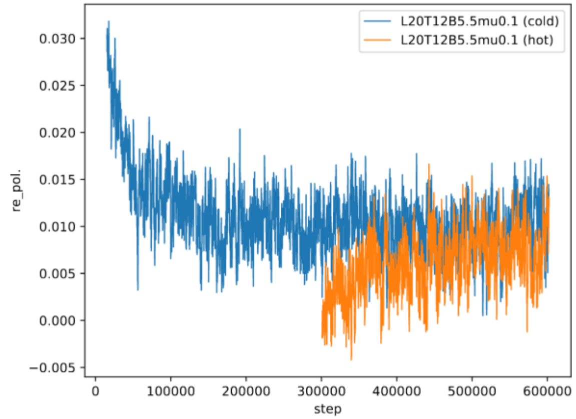
Reliable: β=5.5

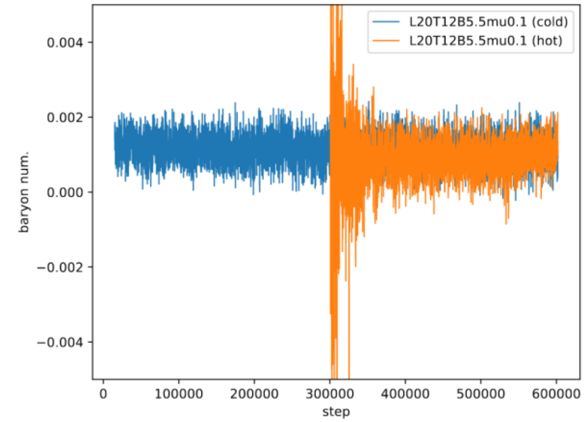
β=5.3, 5.4, 5.6 are not thermalized yet, and sample sizes are relatively small.

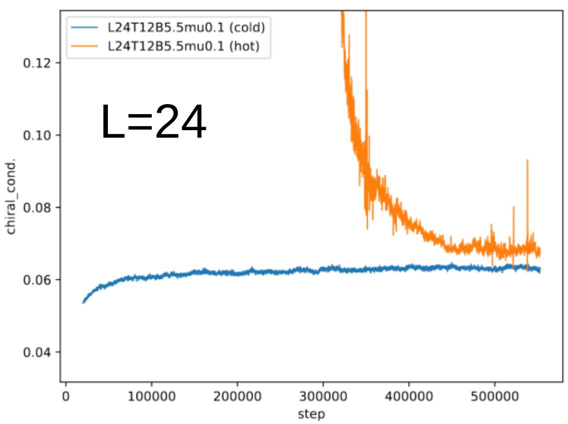
History of observables (β=5.5)

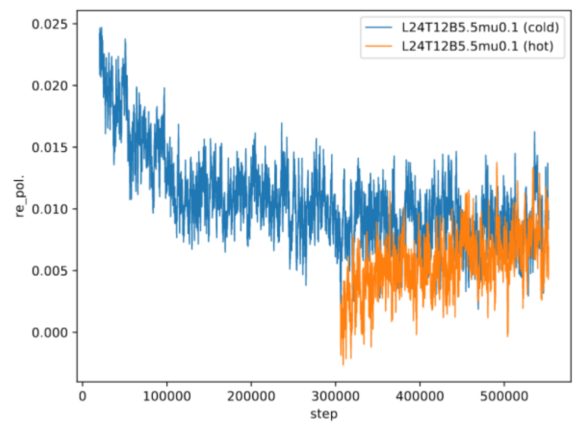
Chiral condensate | Polyakov loop | Baryon number density

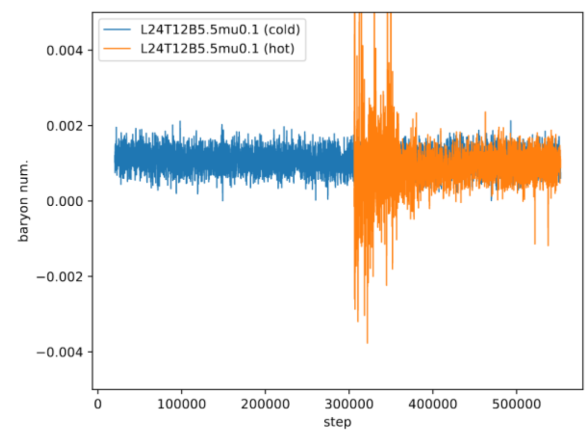












Current data suggest that observables at β=5.5 shows hysteresis.

Summary and outlook

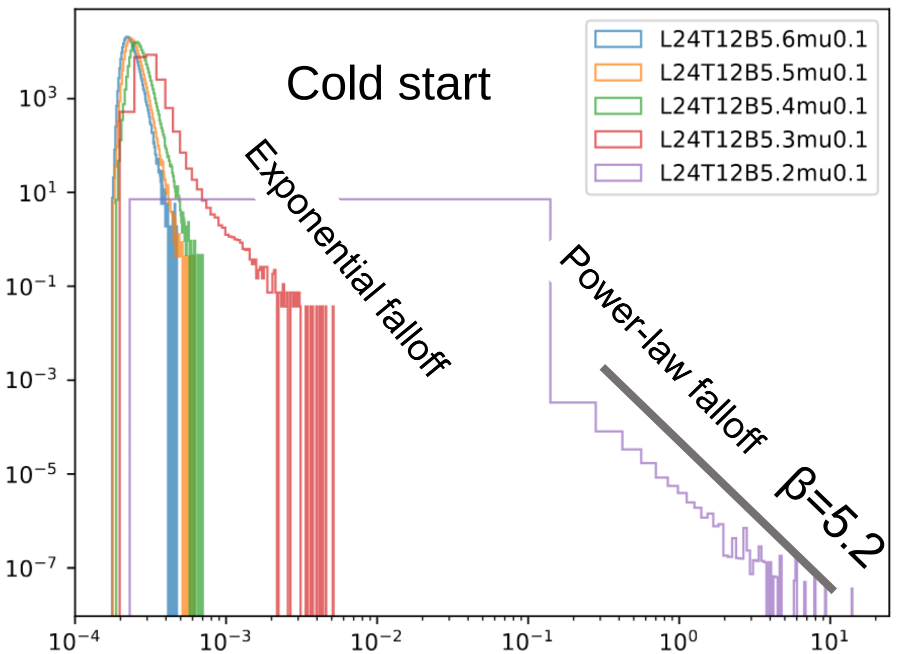
- ➢ Complex Langevin method is applied to explore the (possibly first order) phase transition of 4-flavor QCD in finite density region.
- ➢ We compare histories of the chiral condensate with different initial conditions.
- ➢ Simulation result at β=5.5 is reliable.
- ➢ Current data suggest that observables at β=5.5 shows hysteresis.

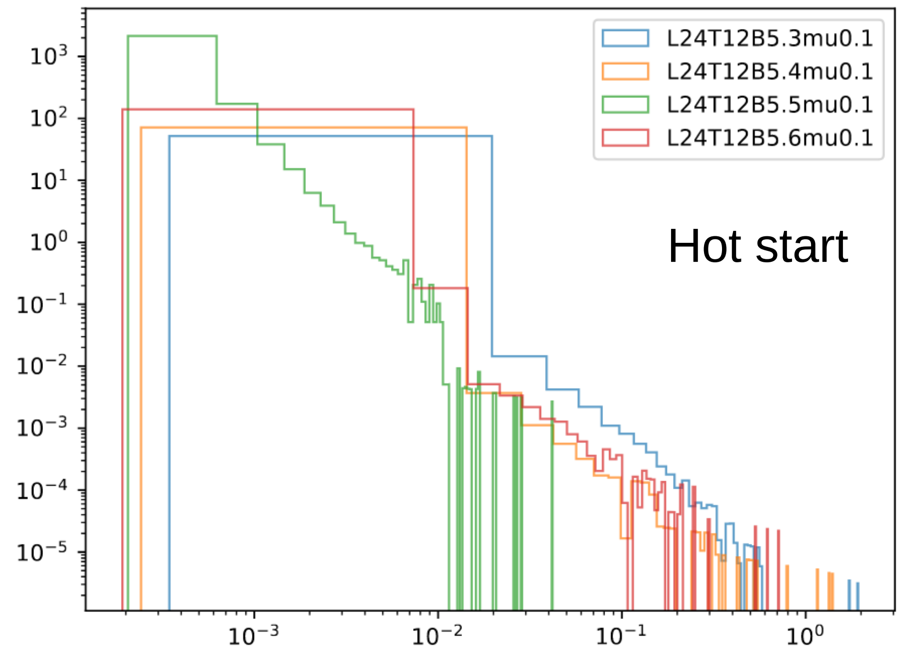
- ◆ For data at β=5.3, 5.4, 5.6, we need more Langevin steps to check their reliability.
- ◆ It is important to determine the critical β where the hysteresis vanishes.

Appendix

Reliability of the simulation (L=24)

Histograms of the drift term (only the fermionic contribution is shown)





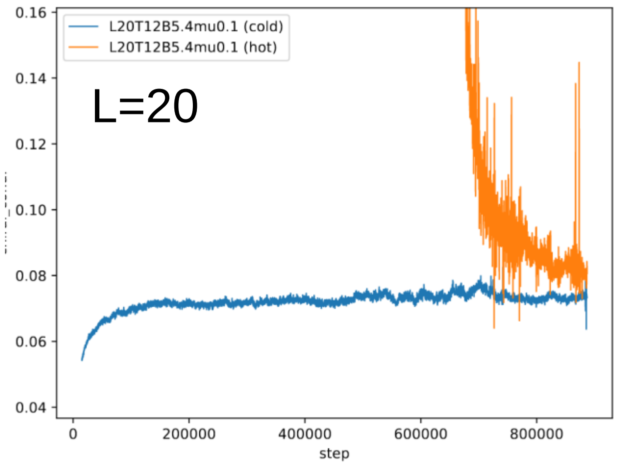
Reliable: β=5.3-5.6

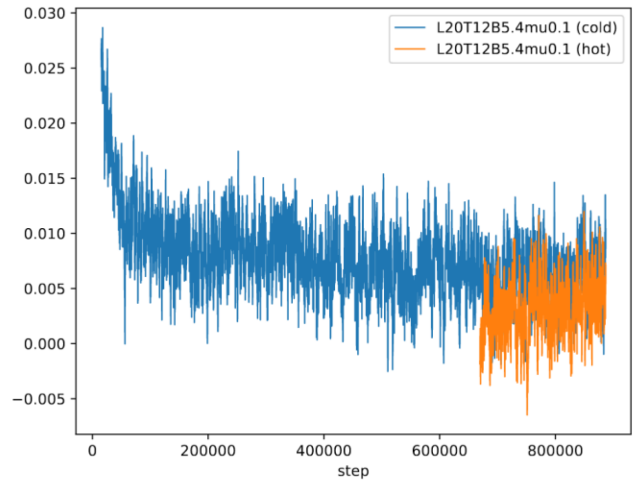
Reliable: β=5.5

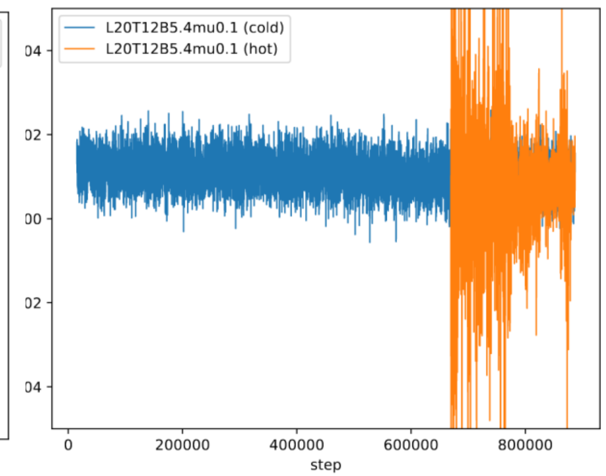
β=5.3, 5.4, 5.6 are not thermalized yet, and sample sizes are relatively small.

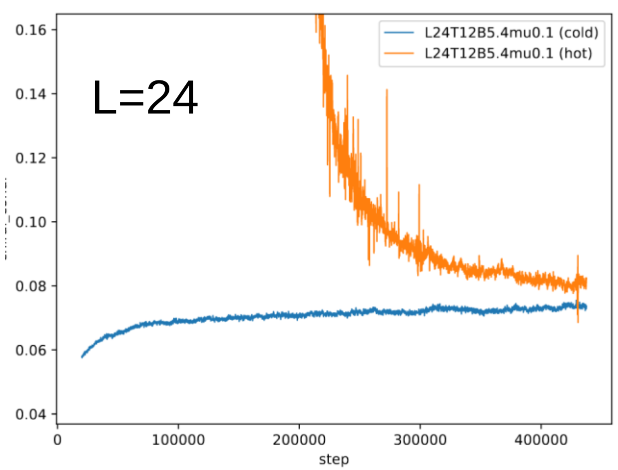
History of observables (β=5.4)

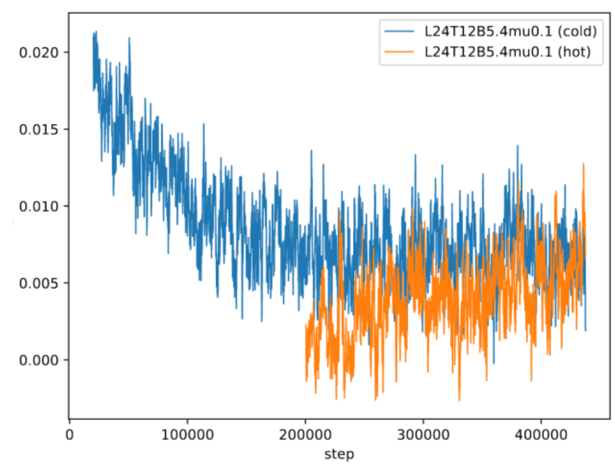
Chiral condensate | Polyakov loop | Baryon number density

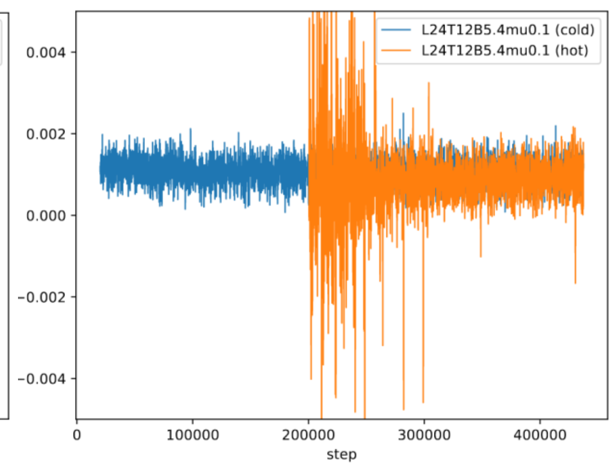






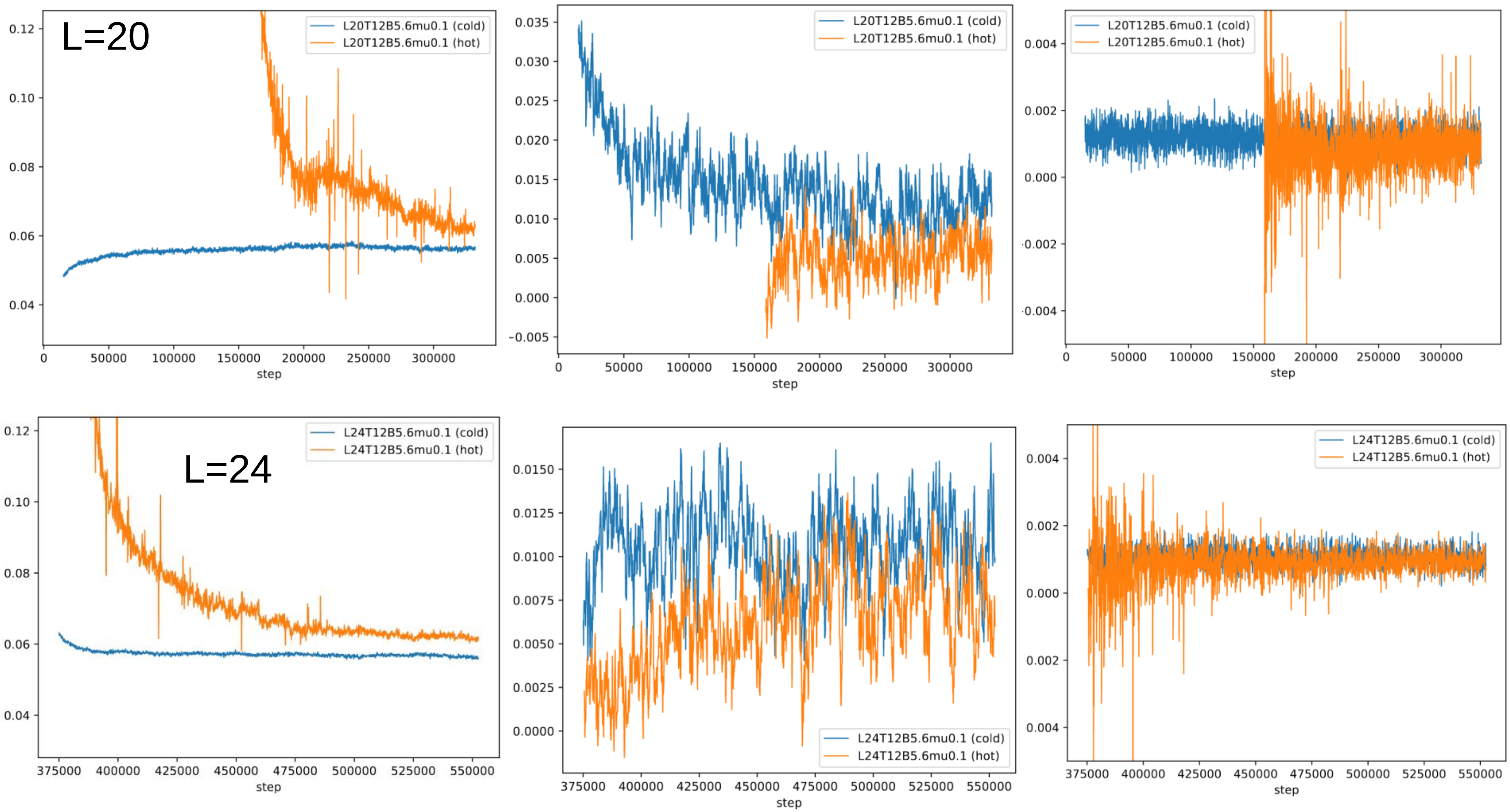


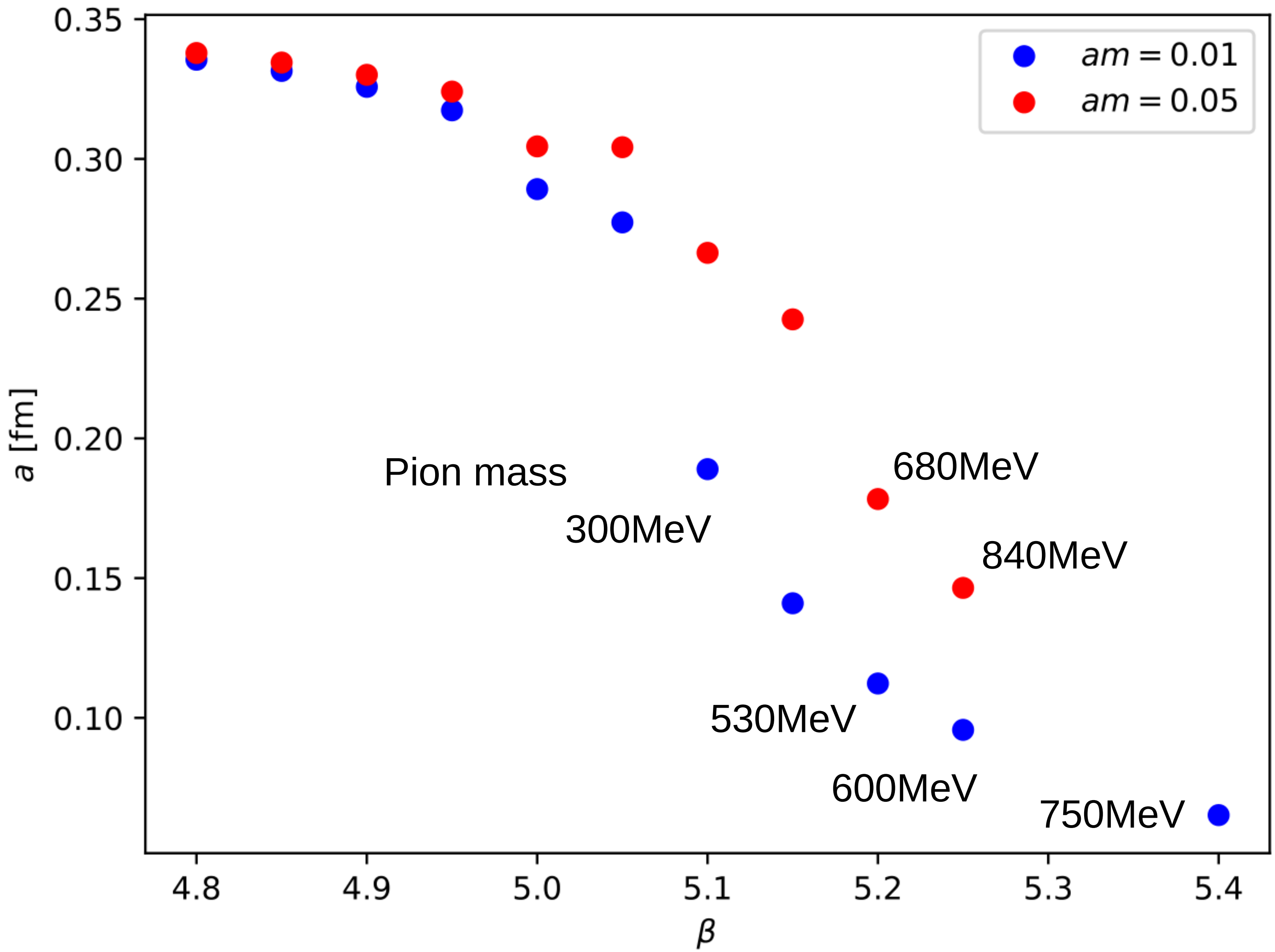




History of observables (β=5.6)

Chiral condensate | Polyakov loop | Baryon number density




0.35
0.30
0.25
0.20
0.15
0.10
a [fm]
4.8
4.9
5.0
5.1
5.2
5.3
5.4
β
am = 0.01
am = 0.05
Pion mass
300MeV
680MeV
840MeV
530MeV
600MeV
750MeV

Basic idea of complex Langevin method

$$Z = \int_{-\infty}^{\infty} dx e^{-S(x)}, \quad S(x) \in \mathbb{C}$$

[Parisi 83], [Klauder 84]
[Aarts, Seiler, Stamatescu 09]
[Aarts, James, Seiler, Stamatescu 11]
[Seiler, Sexty, Stamatescu 13]
[Sexty 14] [Fodor, Katz, Sexty, Torok 15]
[Nishimura, Shimasaki 15]
[Nagata, Nishimura, Shimasaki 15]

Complexification

$$x \in \mathbb{R} \to z \in \mathbb{C} \quad S(x) \to S(z)$$

Complex Langevin equation

$$\frac{dz}{dt} = -\frac{\partial S(z)}{\partial z} + \eta(t)$$

$$\langle \eta(t) \rangle = 0$$

$$\langle \eta(t)\eta(t') \rangle = 2\delta(t-t')$$

$\langle \dots \rangle$:noise average

We identify the noise effect as a *quantum fluctuation.*

Justification of complex Langevin method

Associated Fokker-Planck-like equation becomes,

$$\frac{\partial}{\partial t}\Phi(x,y,t) = \left[\frac{\partial}{\partial x_i}\left\{\mathrm{Re}\left(\frac{\partial S}{\partial z_i}\right) + N_R\frac{\partial}{\partial x_i}\right\} + \frac{\partial}{\partial y_i}\left\{\mathrm{Im}\left(\frac{\partial S}{\partial z_i}\right) + N_I\frac{\partial}{\partial y_i}\right\}\right]\Phi(x,y,t)$$

Under *certain conditions*,

$$\int dx dy O(x+iy)\Phi(x,y) = \int dx O(x) P(x)$$

$$\frac{\partial}{\partial t}P(x,t) = \frac{\partial}{\partial x}\left(\frac{\partial S}{\partial x} + \frac{\partial}{\partial x}\right)P(x,t)$$

The stationary solution reads

$$P_{\mathrm{eq}}(x) \propto e^{-S(x)} \qquad \langle O(z(t))\rangle \to \frac{1}{Z}\int dx O(x) e^{-S(x)}, \quad t\to\infty$$

Criterion of correctness

A criterion for the correctness of the complex Langevin method

K. Nagata, J. Nishimura, S. Shimasaki [1508.02377, 1606.07627]

$$\mathcal{U}_{x\mu}(t+\epsilon) = \exp\left[i\left(-\epsilon\mathcal{D}_{x\mu}S[\mathcal{U}] + \sqrt{\epsilon}\eta_{x\mu}\right)\right]\mathcal{U}_{x\mu}(t)$$

Drift term $v_{x\mu}(\mathcal{U}) = -\mathcal{D}_{x\mu}S[\mathcal{U}]$

Probability distribution of the magnitude of the drift term plays a key role.

$$p(u) = \frac{1}{4N_V}\left\langle \sum_{x\mu} \delta(u - u_{x\mu}) \right\rangle \qquad u_{x\mu} = \sqrt{\frac{1}{N_c^2 - 1}\,\mathrm{tr}(v_{x\nu}v_{x\nu}^{\dagger})}$$