Comparing Different Parameterizations of the z-expansion

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- Decay Rate Expression

Differential Decay Rate (Massless Lepton Limit)

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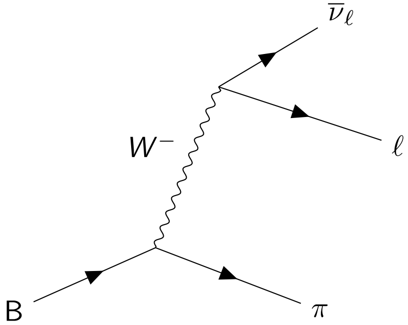

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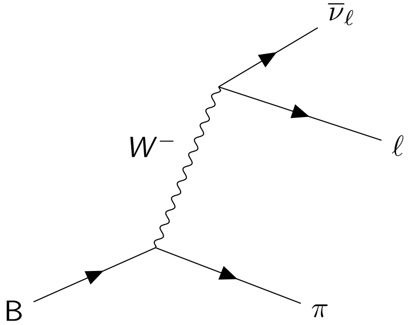

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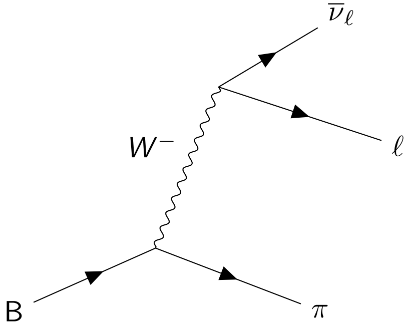
- $\lambda(q^2) = ((m_B^2 + m_\pi^2 - q^2)^2 - 4m_B^2 m_\pi^2)$
- Exclusive and inclusive decays have determinations of V_{ub} which differ by 2.4σ [1]



Conformal Mapping

- Transform $q^2 \to z(q^2, t_0) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}$ [5]

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- Visually what is happening:

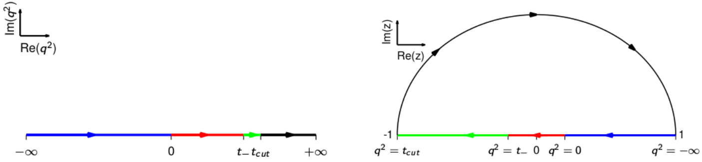

Figure: Image is borrowed from upcoming Fermilab $B \to K$ paper, Image Credit: Yuzhi Liu

BGL expansion

Parameterization of vector form factor

$$f_+(q^2; t_0) = \frac{1}{B(q^2)\phi(q^2)} \sum_{n=0}^{N} a_n z^n \text{ [4]}$$

- $B(q^2)$ is a function which characterizes the pole in the q^2 plane
- $\phi(q^2)$ is a function which arises from unitarity requirements and imposes a simple constraint on the coefficients

BCL Expansion

Parameterization of the vector form factor

$$f_+(q^2; t_0) = \frac{1}{1-q^2/m_{B^*}^2} \sum_{n=0}^{N-1} b_n \left(z^n - (-1)^{N-n} \frac{n}{N} z^N\right) \text{ [3]}$$

- The complicated function of z comes from the conservation of angular momentum requirement that: $\frac{df_+(q^2)}{dz}|_{z=-1} = 0$. $z = -1$ corresponds to the threshold for B^*
- Fixes issue with BGL parameterization by having the appropriate $1/q^2$ falloff behavior

Outline of methodology

- 1.) Fit the parameterization of the form factor over different regions of experimental data.

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- 4.) Test stability of fit coefficients
- 5.) We do not use any lattice data

Efficacy of predictions: BGL parameterization

$$X_p^2 = 1/N_{\text{data points}} \sum_i^{\text{unfitted region}} (\Delta B_{\text{exp}} - \Delta B_{\text{fit}})_i \,/(\sigma_i^2)$$

- X_p^2 is not minimized.

fit region	3 params	4 params.	5 params
$5 - 26.4$ GeV^2	1.02	0.88	1.00
$10 - 26.4$ GeV^2	2.12	3.23	5.15
$15 - 26.4$ GeV^2	3.42	1.90	7.79
$17 - 26.4$ GeV^2	17.56	897	809

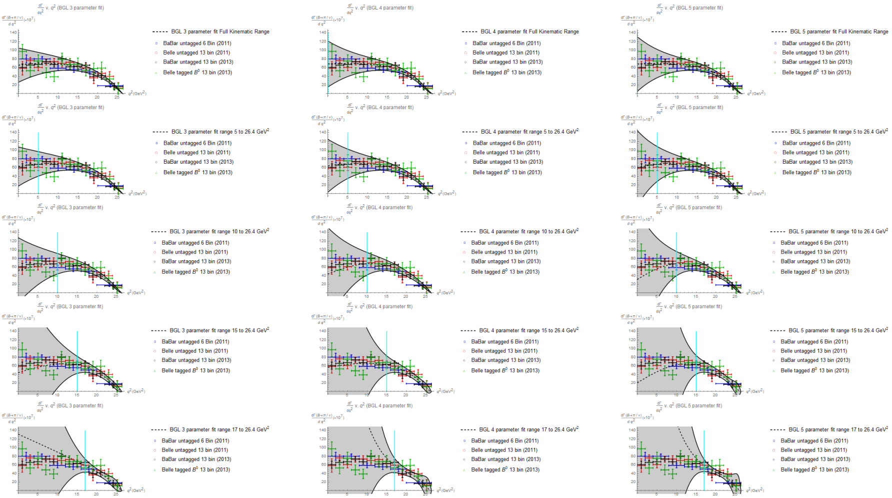
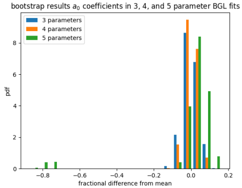
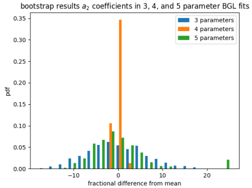
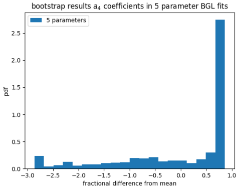


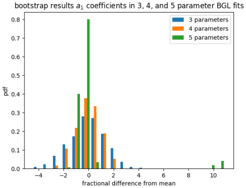
Figure: Traditional BGL fits with number of parameters ranging from 3 to 5 (left to right) and fit ranges decreasing (largest: top to smallest: bottom)

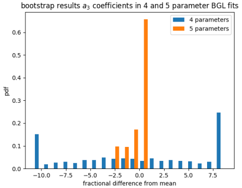
stability of fits: coefficients











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fit region	2 params.	3 params.	4 params.
$5 - 26.4$ GeV^2	1.04	1.05	0.95
$10 - 26.4$ GeV^2	1.793	2.073	3.77
$15 - 26.4$ GeV^2	2.62	3.34	4.33
$17 - 26.4$ GeV^2	7.97	48.5	156

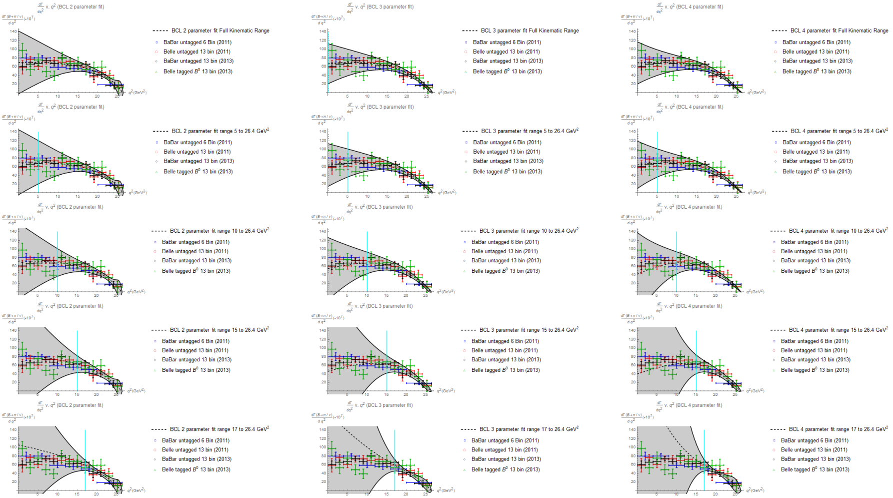
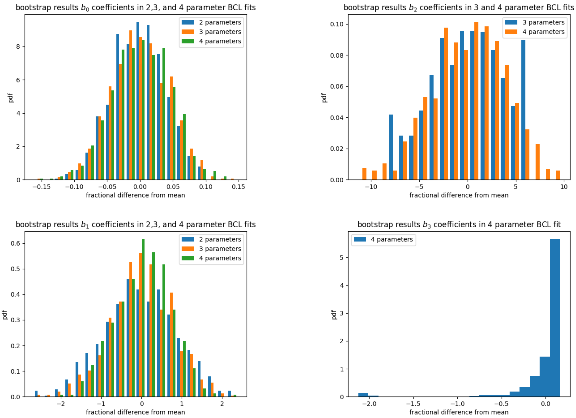


Figure: Traditional BCL fits with number of parameters ranging from 2 to 4 (left to right) and fit ranges decreasing (largest: top to smallest: bottom)

stability of fits: Coefficients b_i



- stable coefficients: b_0 , b_1 , and b_2
- coefficient b_3 is less well distributed.

BCL takeaway

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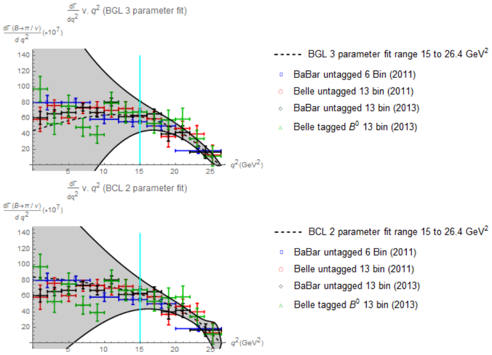
- The BCL parameterizations is stable up to order z^3 (3 parameters)
- The overestimation of the partial branching fractions is likely caused by overfitting due to the large statistical uncertainties in the large q^2 regime.
- Predictions become far more accurate when extended to the $15\ \mathrm{GeV}^2 < q^2 < 26.4\ \mathrm{GeV}^2$ region, slightly outside the region where we have lattice determinations of the form factors.

Comparison of BGL and BCL near lattice range $(15 - 26.4 \text{ GeV}^2)$ at maximal order z^2



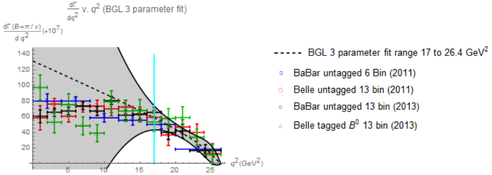
- BGL fit:

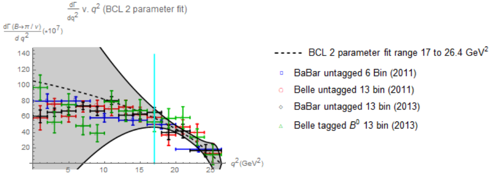
a_0	0.0245(21)
a_1	-0.013(20)
a_2	-0.13(19)
χ^2/d.o.f.	0.91
X_p^2	3.23

- BCL fit:

b_0	0.406(11)
b_1	-0.42(10)
b_2	[0.70(67)]
χ^2/d.o.f.	0.97
X_p^2	2.62

Comparison of BGL and BCL in lattice range $(17 - 26.4\ \mathrm{GeV}^2)$ at order z^2





- BGL Data

a_0	0.0240(20)
a_1	-0.009(32)
a_2	-0.03(41)
χ^2/d.o.f.	0.96
X_p^2	17.59

- BCL Data

b_0	0.405(11)
b_1	-0.30(16)
b_2	[-0.6(1.5)]
χ^2/d.o.f.	0.96
X_p^2	7.97

Examination

- for $15 - 26.4$ GeV^2 fit region predictions are nearly identical. BCL errorbands are smaller.

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- Considering only the lattice region ($17 - 26.4$ GeV^2) BCL parameterization overestimates partial branching fractions less than BGL parameterization.
- Comparing χ^2/d.o.f. are nearly equivalent.

What is the take away?

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- order z^2 and z^3 fits provide determinations determinations of the decay spectrum than z^4 parameter fits.
- Efficacy of this tool when examining $B \to \pi\ell\nu$ is limited by the statistical uncertainty associated with partial branching fractions measured in the high q^2 region due to phase space suppression.

Why should the lattice community care?

- this procedure can help us identify which parameterizations of the form factors provide better a better extrapolation of our lattice calculations.

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- this procedure can help us identify which parameterizations of the form factors provide better a better extrapolation of our lattice calculations.
- this procedure can identify possible energy regions of interest to examine using lattice calculations that have not been currently unexamined due to noise in signal extraction.

Where to go?

- Examine other semileptonic decay: e.g. $B_s \to K\ell\nu$, $B \to D\ell\nu$

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- Examine FCNC decays: e.g. $B \to \pi\ell\ell$, $\Lambda_b \to \Lambda\ell\ell$

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- Examine other semileptonic decay: e.g. $B_s \to K\ell\nu$, $B \to D\ell\nu$
- Examine FCNC decays: e.g. $B \to \pi\ell\ell$, $\Lambda_b \to \Lambda\ell\ell$
- Re-examine $B \to \pi\ell\nu$ when LHCb releases the results.

Acknowledgements

We would like to thank A. Schwartz for discussions regarding $B \to D$ decays. This research was supported in part by the Department of Energy under Award Numbers DOE grant DE-SC0010113

Further Reading I

J. A. Bailey *et al.* [Fermilab Lattice and MILC Collaborations], "$|V_{ub}|$ from $B \to \pi \ell \nu$ decays and (2+1)-flavor lattice QCD," Phys. Rev. D **92**, no. 1, 014024 (2015) doi:10.1103/PhysRevD.92.014024 [arXiv:1503.07839 [hep-lat]].

M. C. Arnesen, B. Grinstein, I. Z. Rothstein and I. W. Stewart, Phys. Rev. Lett. **95**, 071802 (2005) doi:10.1103/PhysRevLett.95.071802 [hep-ph/0504209].

C. Bourrely, I. Caprini and L. Lellouch, Phys. Rev. D **79**, 013008 (2009) Erratum: [Phys. Rev. D **82**, 099902 (2010)] doi:10.1103/PhysRevD.82.099902, 10.1103/PhysRevD.79.013008 [arXiv:0807.2722 [hep-ph]].

Further Reading II

- C.Glenn Boyd, Benjamn Grinstein, Richard F. Lebed, Model-independent extraction of —Vcb— using dispersion relations, Physics Letters B, Volume 353, Issues 23, 1995, Pages 306-312, ISSN 0370-2693, https://doi.org/10.1016/0370-2693(95)00480-9. (http://www.sciencedirect.com/science/article/pii/0370269395004809)
- Okubo, Susumu, "Exact Bounds for K_{l3} Decay Parameters", Phys Rev. D. 3, 2807-2813, 1971.

Appendix: BGL functions

- $B(q^2) = \frac{z(q^2,t_0)-z(m_{B*}^2,t_0)}{1-z(q^2,t_0)z(m_{B*}^2,t_0)}$
- $$\phi(q^2,t_0) = \sqrt{\frac{1}{32\pi\chi_{1^-}(0)}}\left(\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}\right) \times \frac{t_+ - q^2}{(t_+ - t_0)^{1/4}}\left(\sqrt{t_+ - q^2} + \sqrt{t_+}\right)^{-5} \times \left(\sqrt{t_+ - q^2} + \sqrt{t_+ - t_-}\right)^{3/2}$$