Scattering length from BS wave function inside the interaction range

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Contents

Main messages

- Lattice QCD can provide on-shell scattering amplitude using Bethe-Salpeter(BS) wave function not only "outside" the interaction range but also "inside" the interaction range
  - ◇ Our result of the scattering length agrees with the value of Lüscher's formula
- Lattice QCD can provide not only on-shell scattering amplitude but also half-off-shell scattering amplitude
  - ◇ Half-off-shell scattering amplitude can be an additional input for effective theories and models of hadrons, as a supplement to experiments

1 Introduction

Hadron interactions can be studied by lattice QCD

- Direct approach:
  - ◇ The standard method is Lüscher formula, which utilizes BS wave function outside the interaction range of two hadrons Lüscher(1986,1990),...
    cf. many talks in Lattice 2018
  - ◇ Related issue: a relation between on-shell scattering amplitude and BS wave function inside the interaction range was discussed in the infinite volume Lin et al.(2001),CP-PACS(2005),Yamazaki and Kuramashi(2017)
    cf. talk by Yamazaki-san
    → We explore this relation by a finite volume simulation
- Indirect approach: ex. HAL QCD method (indirect method through a potential from BS wave function) cf. talks by Doi-san, Iritani-san

2 Formulation(in brief)

Scattering amplitude $H(p;k)$ is obtained by BS wave function $\phi(\mathbf{x};k)$

- Integration range for $H(p;k)$ can be changed from ∞ to finite value R, called interaction range, if $(\Delta + k^2)\phi(\mathbf{x};k) = 0$ for $x > R$
  $\therefore$ Lattice simulation for $H(p;k)$ is possible, if $R < L/2$

◇ We consider $I = 2$ two-pion below inelastic threshold. A_1^+ projection is applied for S-wave in center of mass frame. Exp tails are assumed to be tiny and ignored.

$$\phi(\mathbf{x};k) := \langle 0|\Phi(\mathbf{x},t)|\pi^+\pi^+, E_k\rangle e^{E_k t},$$

$$\Phi(\mathbf{x},t) := \sum_{\mathbf{r}} \pi^+(R_{A_1^+}[\mathbf{x}] + \mathbf{r}, t)\pi^+(\mathbf{r},t),$$

$$R_{A_1^+}[\mathbf{x}] : \text{ projector onto } A_1^+ \text{ cubic group}, \quad E_k = 2\sqrt{m_\pi^2 + k^2}$$

$$\Delta\phi(\mathbf{x};k) := \sum_{i=1}^{3} (\phi(\mathbf{x}+\hat{i};k) + \phi(\mathbf{x}-\hat{i};k) - 2\phi(\mathbf{x};k)): \text{ Laplacian on lattices}$$

$$H(p;k) := -\int_{-\infty}^{\infty} d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}} \left(\Delta + k^2\right)\phi(\mathbf{x};k)$$

$$= -\sum_{|x|<R} e^{-i\mathbf{p}\cdot\mathbf{x}} \left(\Delta + k^2\right)\phi(\mathbf{x};k), \quad \text{if } (\Delta + k^2)\phi(\mathbf{x};k) = 0 \text{ for } x > R$$

[Formulation(continued)]

- Once $H(p;k)$ at on-shell $p=k$ is obtained, we can extract scattering phase shift $\delta(k)$ and scattering length a_0, as in Lüscher formula

  ◇ NB. $H(k;k)$ appears in Lüscher's formalism, though $H(k;k)$ is removed in the final form of Lüscher formula
  → Our claim is "$H(k;k)$ also keeps scattering info."

$$
\begin{aligned}
H(k;k) &= \frac{4\pi}{k} e^{i\delta(k)} \sin\delta(k) \\
a_0 &= \tan\delta(k)/k + O(k^2)
\end{aligned}
$$

[Quick derivation on Lüscher formula]

$$
\begin{aligned}
\phi(\mathbf{x};k) \quad \xrightarrow[x>R]{} & \quad v_{00} G(\mathbf{x};k), \quad G(\mathbf{x};k) : \text{solution of } (\Delta + k^2)\phi(\mathbf{x};k) = 0 \\
= & \quad C_{00} e^{i\delta(k)} \sin(kx+\delta(k))/kx + (l \geq 4 \text{ terms}), \quad v_{00}, C_{00} : \text{constants}
\end{aligned}
$$

Expanding $G(\mathbf{x};k)$ by $j_l(kx)$ and $n_0(kx)$ and comparing their coefficients leads to

$$
\begin{aligned}
C_{00} H(k;k) &= v_{00} \\
k \cot\delta(k) C_{00} H(k;k) &= 4\pi v_{00} g_{00}(k)
\end{aligned}
$$

Taking a ratio of the above two equations gives Lüscher formula,

$$
k \cot\delta(k) = 4\pi g_{00}(k)
$$

3 Set up of simulation

We use $I = 2$ $\pi\pi$ system in quenched lattice QCD as a test bed

- Iwasaki gauge action at $\beta = 2.334(a^{-1} = 1.207[\text{GeV}])$ CP-PACS(2001,2005)
- Valence Clover quark action with $C_{\text{SW}} = 1.398$

  ◇ Four random $Z(2)$ sources avoiding Fierz contamination and Wall sources for comparison + Coulomb gauge fixing

  ◇ Periodic boundary condition in space, Dirichlet boundary condition in time

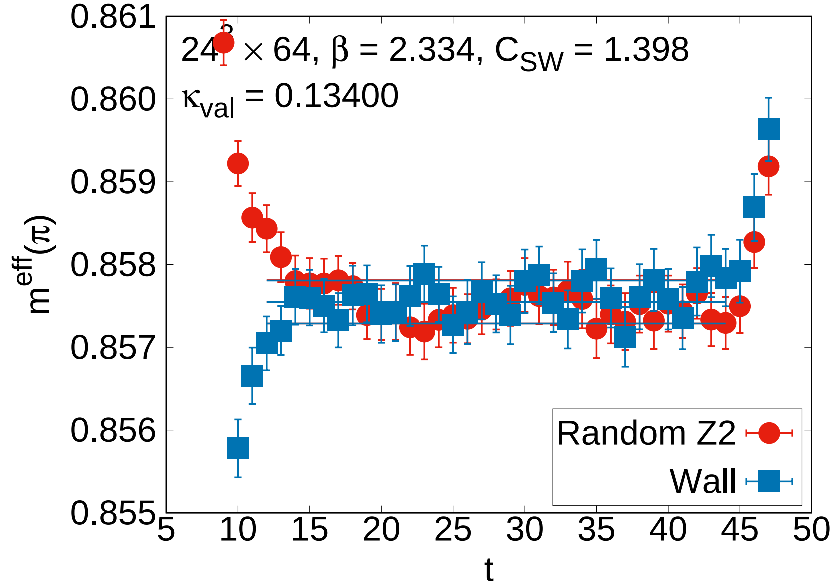
Lattice	κ_{val}	m_π [GeV]	N_{config}	N_{src}
$24^3 \times 64$	0.1340	0.86	200	32
$24^3 \times 96$	0.1340	0.86	200	24
	0.1358	0.67	200	24
	0.1369	0.52	200	24

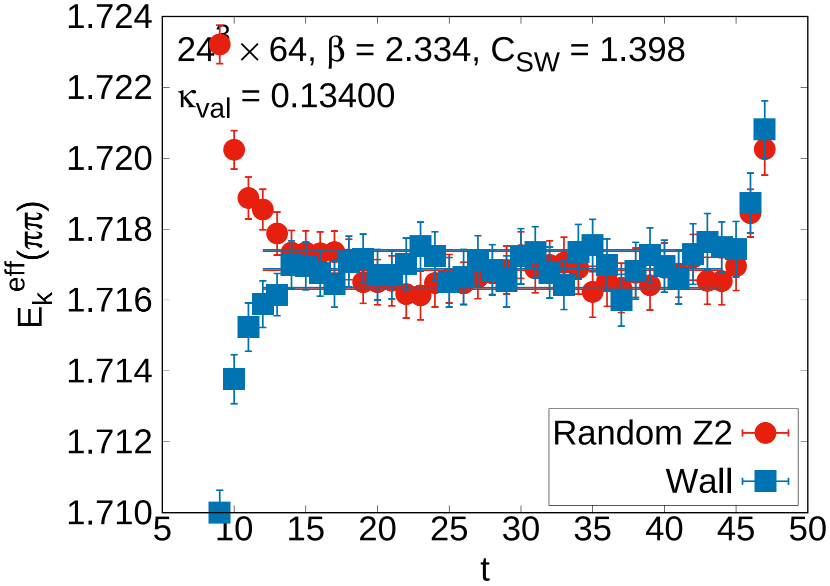
4 Result

[Check of plateau of temporal correlators]

- Effective masses of one-pion $m^{\mathrm{eff}}(\pi)$ and $I = 2$ two-pion $E_k^{\mathrm{eff}}(\pi\pi)$ have plateau in $t = [12, 44]$
  $\rightarrow$ No source dependence is observed for our case (fake plateau can appear for two baryons T.T.Takahashi and Y.K.Enyo(2009);HAL(Iritani et al.(2016));...)

  ♢ k_t^2 is determined from $E_k(\pi\pi) = 2\sqrt{m_\pi^2 + k_t^2}$, $k_t^2 \neq p^2 := (2\pi/L)^2 n^2$



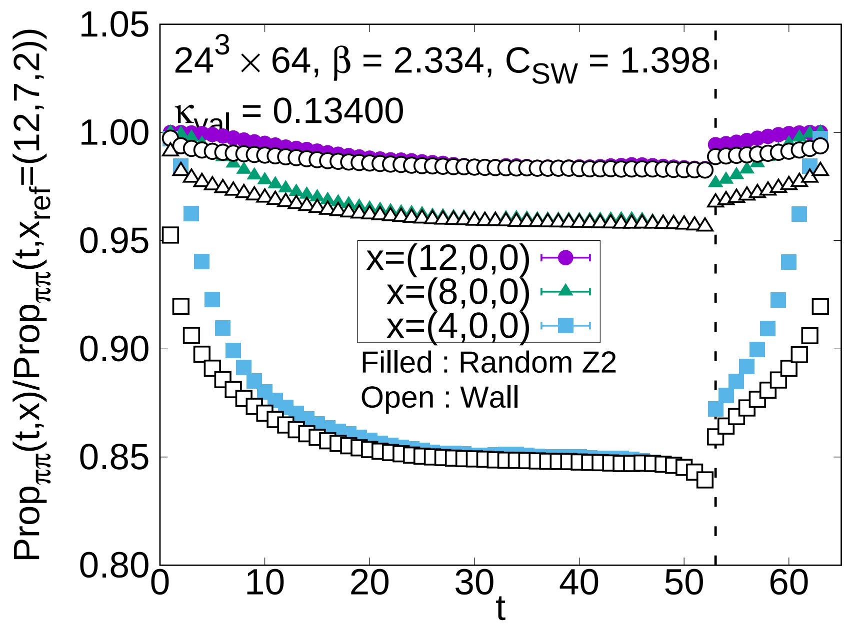


[Check of plateau of BS wave functions $\phi(\mathbf{x};k)$]

- Ratio of wave functions $\phi(\mathbf{x};k)/\phi(\mathbf{x}_{\rm ref};k)$ have plateau in $t=[32,44]$
  - ♢ Larger t is required for wave functions, but still under control cf. temporal correlators have plateau in $t=[12,44]$
  - ♢ No source dependence is observed

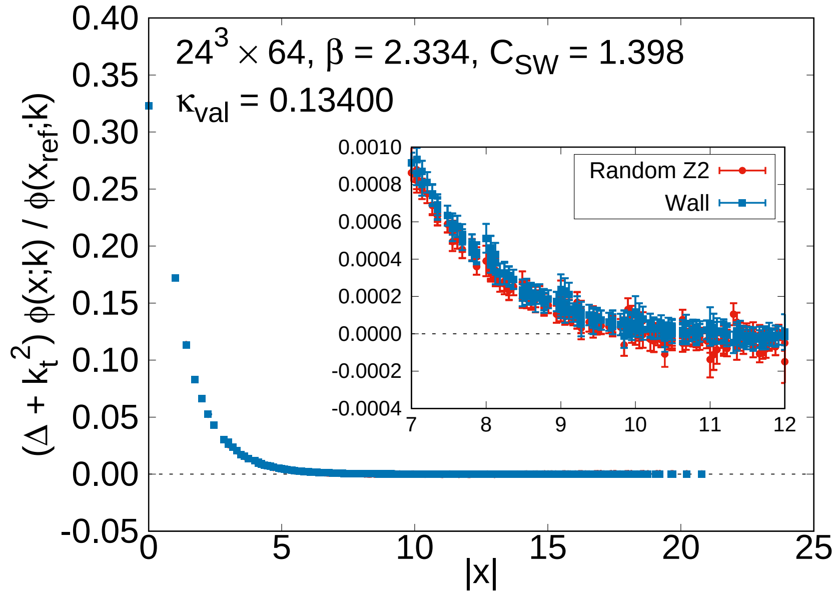
$$\phi(\mathbf{x};k)$$
$$= \text{const} \times prop_{4\rm pt}(\mathbf{x},t)e^{E_k t}$$

$$\phi(\mathbf{x};k)/\phi(\mathbf{x}_{\rm ref};k)$$
$$= prop_{4\rm pt}(\mathbf{x},t)/prop_{4\rm pt}(\mathbf{x}_{\rm ref},t)$$



[Check of sufficient condition: $(\Delta + k^2)\phi(\mathbf{x}; k) = 0,\ R < x < L/2$]

- We confirm $R \sim 10$, which is consistent with the result by CP-PACS(2005)
  $\rightarrow$ The sufficient condition is satisfied within our statistical errors
- k_s^2 can be inversely obtained by $k_s^2 = -\Delta\phi(\mathbf{x}; k)/\phi(\mathbf{x}; k)$,
  which is more precise than $k_t^2 = E_k^2/4 - m_\pi^2$ CP-PACS(2005)



◇ Reference point $\mathbf{x}_{\mathrm{ref}} = (12, 7, 2)$ is chosen to minimize $l = 4$ contribution:
$\phi(\mathbf{x}; k) = (l = 0 \text{ term}) + (l = 4 \text{ term}) + \ldots$

◇ Strictly speaking, there must be exp tail, which is below our statistical error

[Comparison of scattering length a_0]

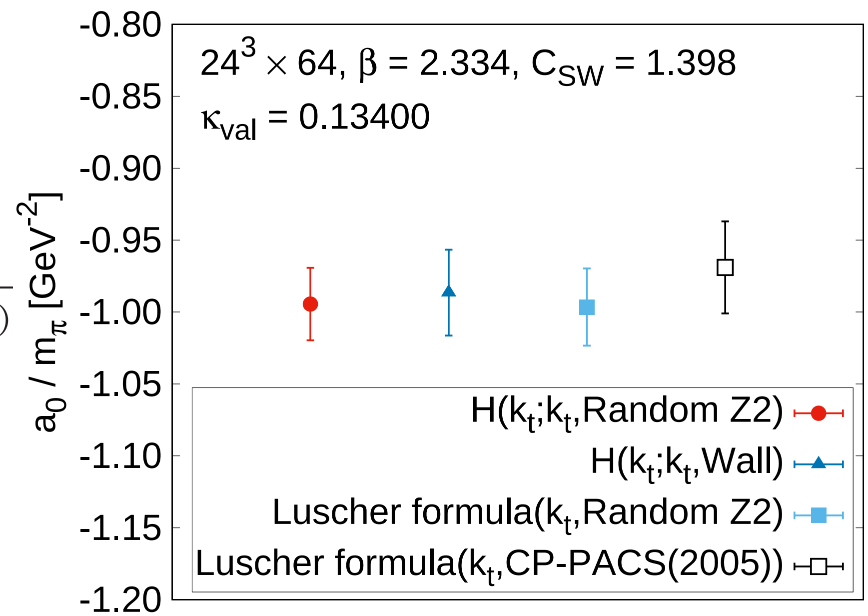
- a_0 is evaluated by $H(k_t; k_t)$ inside the interaction range or by Lüscher's formula outside the interaction range
  - ◇ Both results agree well
  - ◇ No source dependence is observed

$$a_0/m_\pi = \tan\delta(k)/(k m_\pi) + O(k^2)$$

$$\tan\delta(k) = \frac{\sin(k x_{\rm ref})}{4\pi x_{\rm ref}\,\phi(\mathbf{x}_{\rm ref}; k)/H_L(k;k) - \cos(k x_{\rm ref})}$$

or

$$= 1/((4\pi/k) g_{00}(k)) \ : \ \text{Lüscher's formula}$$

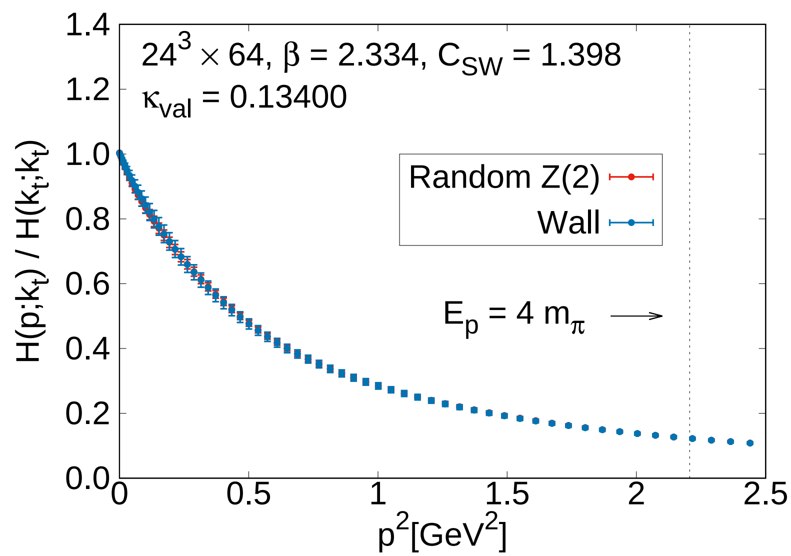


[Additional output: half-off-shell scattering amplitude $H(p;k)$]

- Not only on-shell amplitude $H(k;k)$ but also $H(p;k)$ can be estimated
  - ♢ $H(p;k)$ can be supplemental input to theoretical models of hadrons
  - ♢ Effective range $r_{\rm eff}$ can be extracted from $H(p;k)$
  - ♢ NB. $H(p;k)$ / $H(k;k)$ is available below 4π threshold, although there is no true inelastic threshold in quenched QCD (quenched artificial inelastic effects may appear)

$$
\begin{aligned}
H_L(p;k) &= -\sum_{\mathbf{x}\in L^3} j_0(px)(\Delta + k^2)\phi(\mathbf{x};k) \\
H(p;k)/H(k;k) &= H_L(p;k)/H_L(k;k)
\end{aligned}
$$

$$
\begin{aligned}
k\cot\delta(k) &= a_0^{-1} + r_{\rm eff}k^2 + O(k^4) \\
r_{\rm eff} &\sim -\frac{2k^2H' + \sin^2\delta(k)}{2k\sin\delta(k)\cos\delta(k)} \\
H' &:= \partial_{p^2} H(p;k)|_{p^2=k^2}/H(k;k)
\end{aligned}
$$



[Chiral extrapolation of a_0]

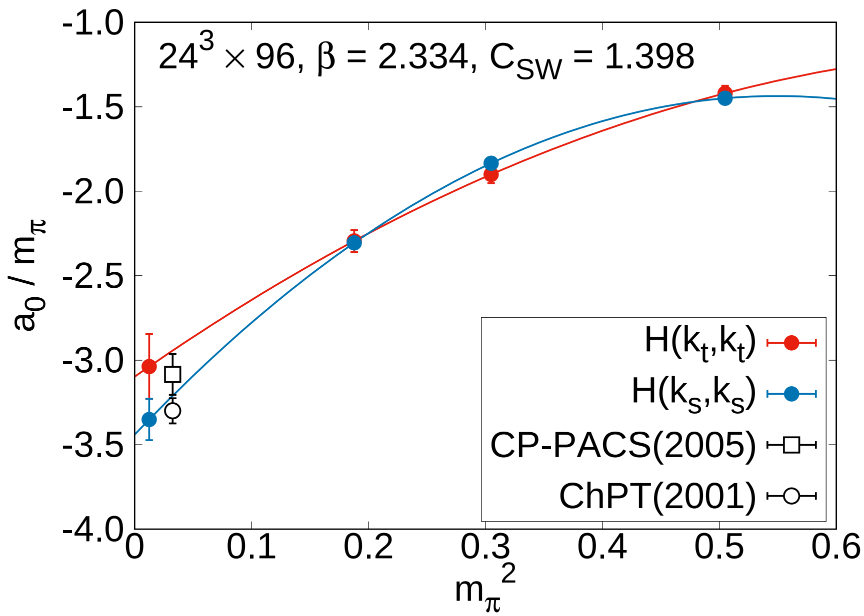
a_0 is extrapolated to the physical point using a ChPT motivated formula, ignoring chiral log due to our large pion masses

- Our result is consistent with ChPT(2001) and CP-PACS(2005)
  - ♢ For definite comparison, more realistic data ($N_f = 2+1$ full QCD on the physical point) is needed

$$a_0/m_\pi = A + Bm_\pi^2 + Cm_\pi^4 + O(m_\pi^2 \log m_\pi^2)$$

$$k_t^2 = E_k^2/4 - m_\pi^2$$

$$k_s^2 = -\Delta\phi(\mathbf{x};k)/\phi(\mathbf{x};k)$$

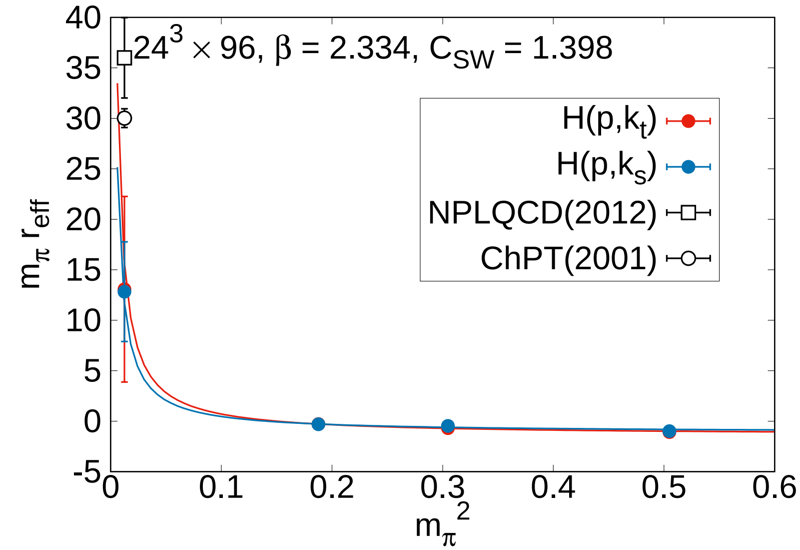


[Chiral extrapolation of $r_{\rm eff}$]

$r_{\rm eff}$ is extrapolated to the physical point using a ChPT motivated formula, ignoring chiral log due to our large pion masses

- Our result is consistent with CP-PACS(2005) at the simulation point, but underestimates ChPT(2001) and NPLQCD(2012) at the physical point
    - ◇ For definite comparison, more realistic data ($N_f = 2{+}1$ full QCD on the physical point) is needed

$$m_\pi r_{\rm eff} = A/m_\pi^2 + B + O(\log m_\pi^2)$$



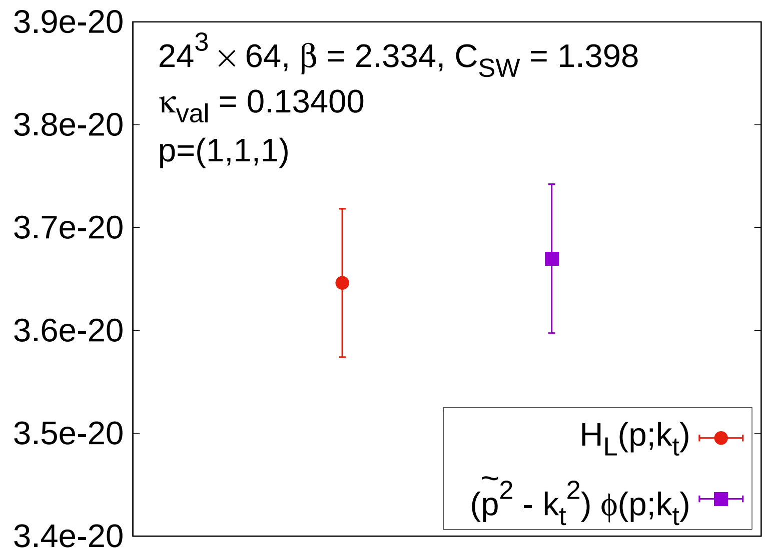
[Remark: scattering amplitude in momentum space $H(p;k)$]

- $H(p;k)$ can also be calculated using LSZ reduction formula in momentum space, instead of laplacian Δ. cf. J.Carbonell and V.A.Karmanov(2016)
  - ◇ Care is needed. If we change the integration range of $H(p;k)$ from ∞ to the interaction range R, a surface term appears.

$$
\begin{aligned}
H(p;k) &:= -\int_{-\infty}^{\infty} d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}(\Delta + k^2)\phi(\mathbf{x};k) \\
&= -\int_{-R}^{R} d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}(\Delta + k^2)\phi(\mathbf{x};k) \\
&\Downarrow \quad \text{partial integration} \\
&= (p^2 - k^2)\int_{-R}^{R} d^3x\, e^{-i\mathbf{p}\cdot\mathbf{x}}\phi(\mathbf{x};k) \\
&\quad +[\text{surface term}]_{-R}^{R}
\end{aligned}
$$

NB. on the lattice, $p^2 \to \tilde{p}^2$

$$\tilde{p}_i := \frac{2}{a}\sin\frac{ap_i}{2}, \quad p_i = (2\pi/L)n_i$$



5 Summary

We evaluate a scattering length a_0 of $I = 2$ $\pi\pi$ system in the quenched lattice QCD as a test bed. We utilize Bethe-Salpeter wave function not only "outside" the interaction range but also "inside" the interaction range

- Consistency is checked
  Our result using the scattering amplitude "inside" the interaction range agrees with the value of standard Lüscher's finite volume method using data "outside" the interaction range
- Additional output is obtained
  A half-off-shell scattering amplitude $H(p; k)$ is estimated by lattice QCD, which can be an additional input to theoretical models of hadrons, as a supplement to experiments

[Future work]
Apply our strategy to

- More realistic case ($N_f = 2 + 1$ full QCD on the physical point)
- More complicated system (other 2-body system with not only light quarks but also heavy quarks, and hopefully 3-body system)