

The $K_{\ell 3}$ form factor from four-flavor lattice QCD and $|V_{us}|$



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Outline

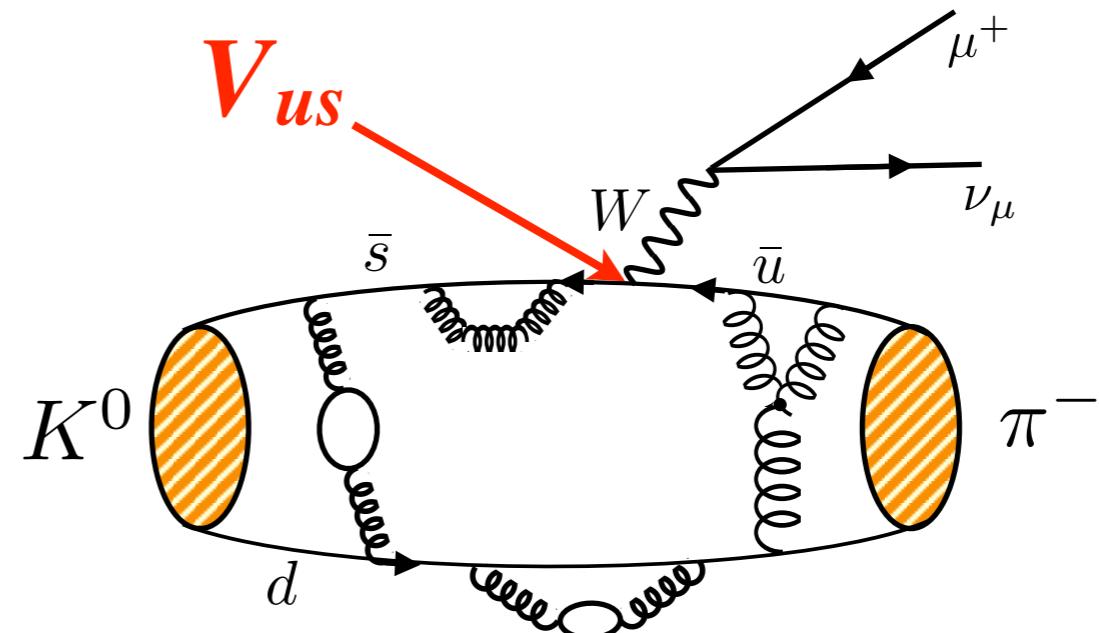
A Fermilab Lattice and MILC collaborations project

- Introduction
- Set-up
- Analysis
- Chiral-continuum fit
- Systematic error analysis
- Result in comparison
- Implications
- Summary and Outlook



Introduction

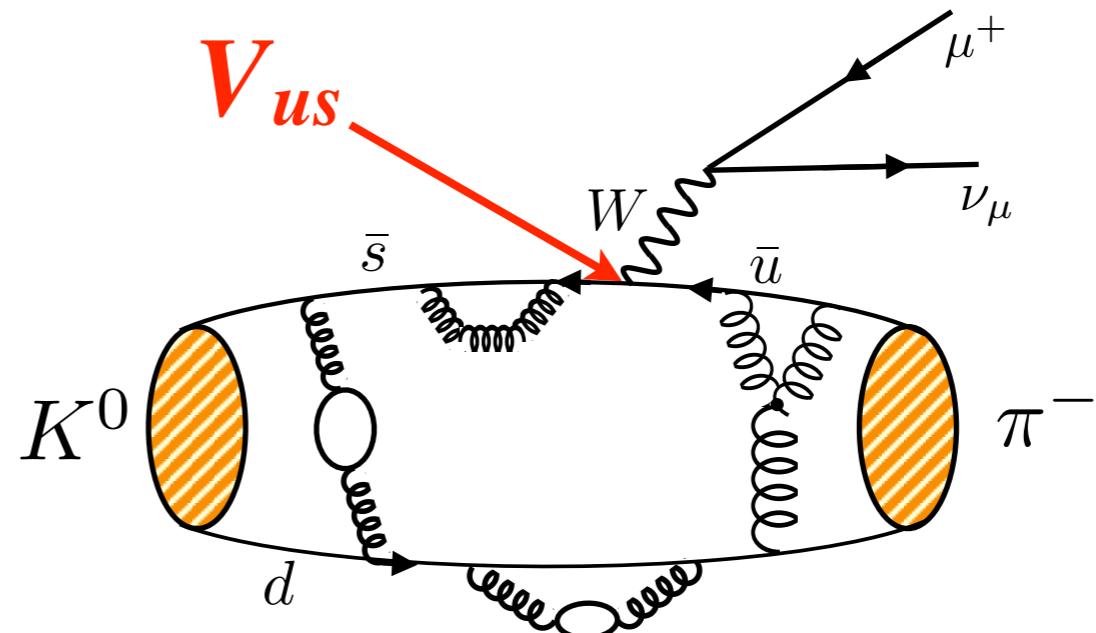
example: $K^0 \rightarrow \pi^- \ell^+ \nu_\ell$



$$\Gamma_{K\ell 3} = (\text{known}) \times \begin{pmatrix} \text{phase} \\ \text{space} \end{pmatrix} \times (1 + \delta_{\text{EM}}^{K\ell} + \delta_{\text{SU}(2)}^{K\pi}) \times |V_{us}|^2 \times |f_+^{K^0\pi^-}(0)|^2$$

Introduction

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$$\Gamma_{K\ell 3} = (\text{known}) \times \begin{pmatrix} \text{phase} \\ \text{space} \end{pmatrix} \times (1 + \delta_{\text{EM}}^{K\ell} + \delta_{\text{SU}(2)}^{K\pi}) \times |V_{us}|^2 \times |f_+^{K^0\pi^-}(0)|^2$$

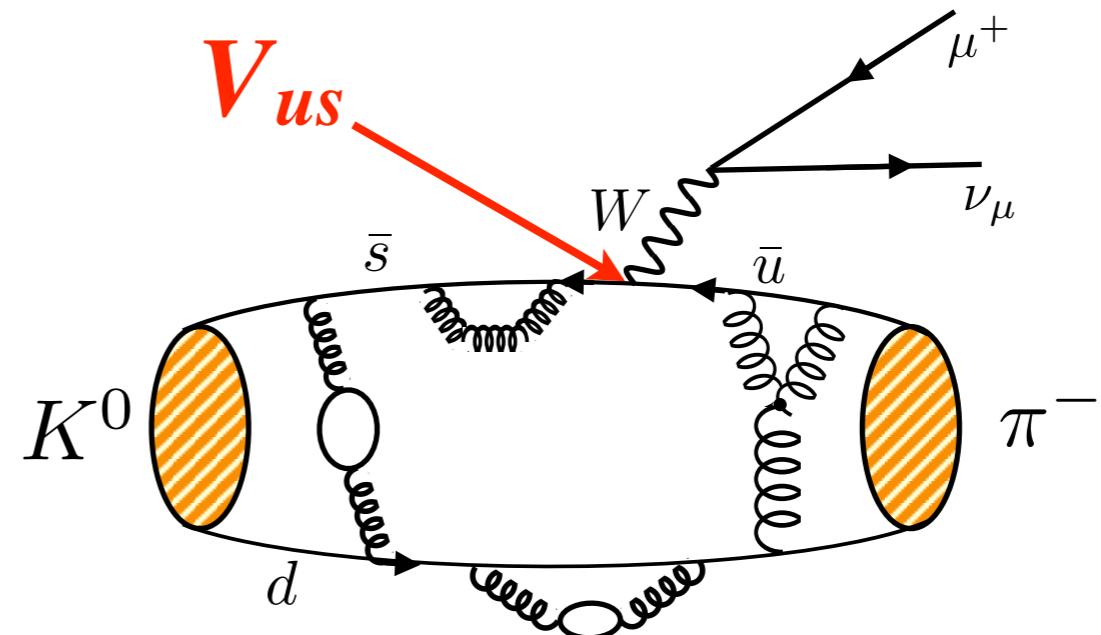
Needed to relate pure QCD form factor to experiment. Mode dependent.

Needed to include charged kaon decay in the experimental average.

Both are currently estimated phenomenologically. [Cirigliano et al, arXiv:1107.6001, RMP 2012].

Introduction

example: $K^0 \rightarrow \pi^- \ell^+ \nu_\ell$



experimental averages:

(M. Moulson @ CKM 2016, [arXiv:1704.04104](https://arxiv.org/abs/1704.04104))

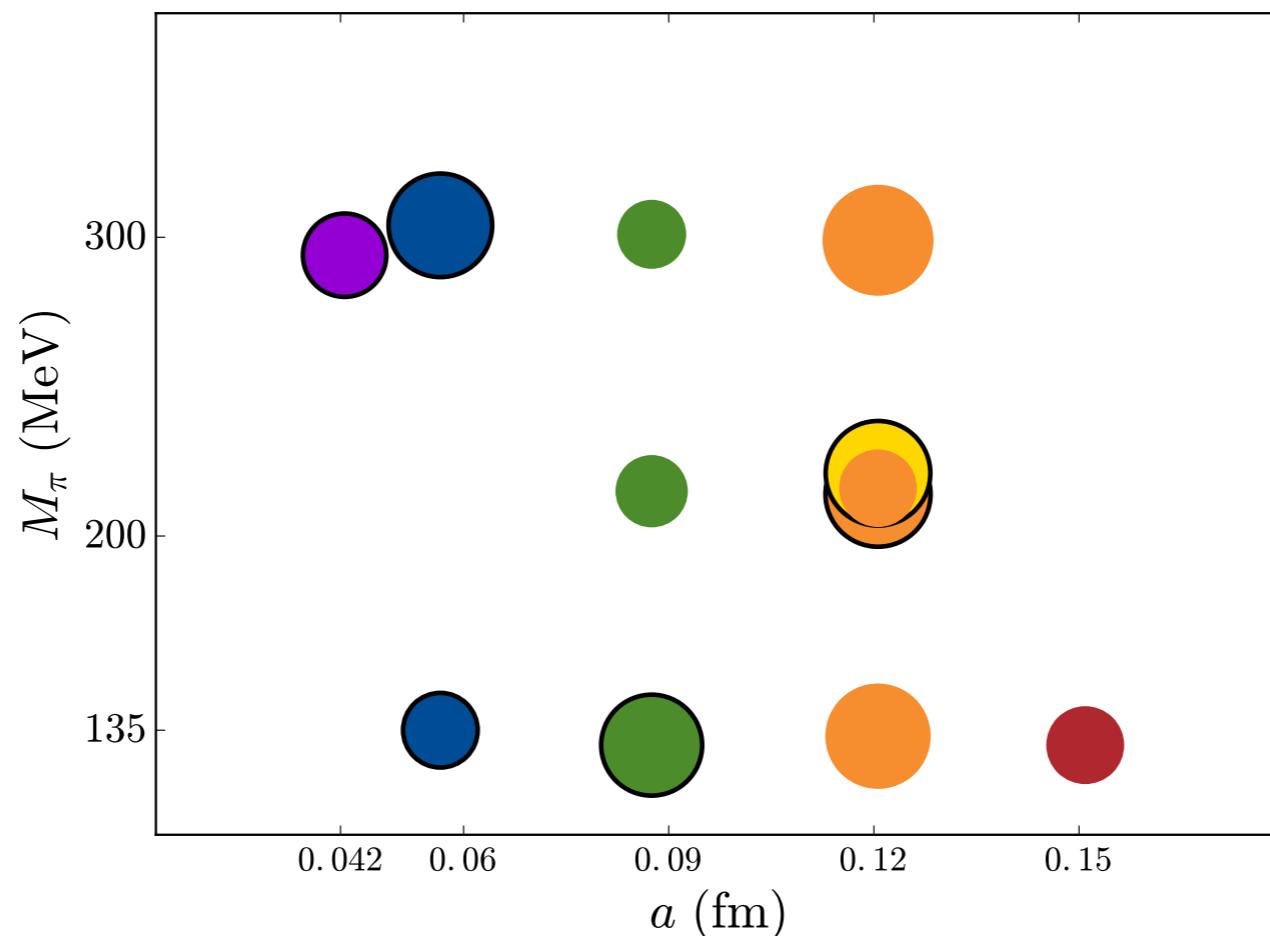
All modes: $|V_{us}| f_+^{K^0\pi^-}(0) = 0.21654(41)$ 0.19% uncertainty

K^0 only: $|V_{us}| f_+^{K^0\pi^-}(0) = 0.21633(44)_{\text{exp}}(24)_{\delta_{\text{EM}}^{K^0\ell}}$

K^\pm only: $|V_{us}| f_+^{K^0\pi^-}(0) = 0.21710(51)_{\text{exp}}(24)_{\delta_{\text{EM}}^{K^\pm\ell}}(42)_{\delta_{\text{SU}(2)}^{K^\pm\pi^0}}$

Set-up

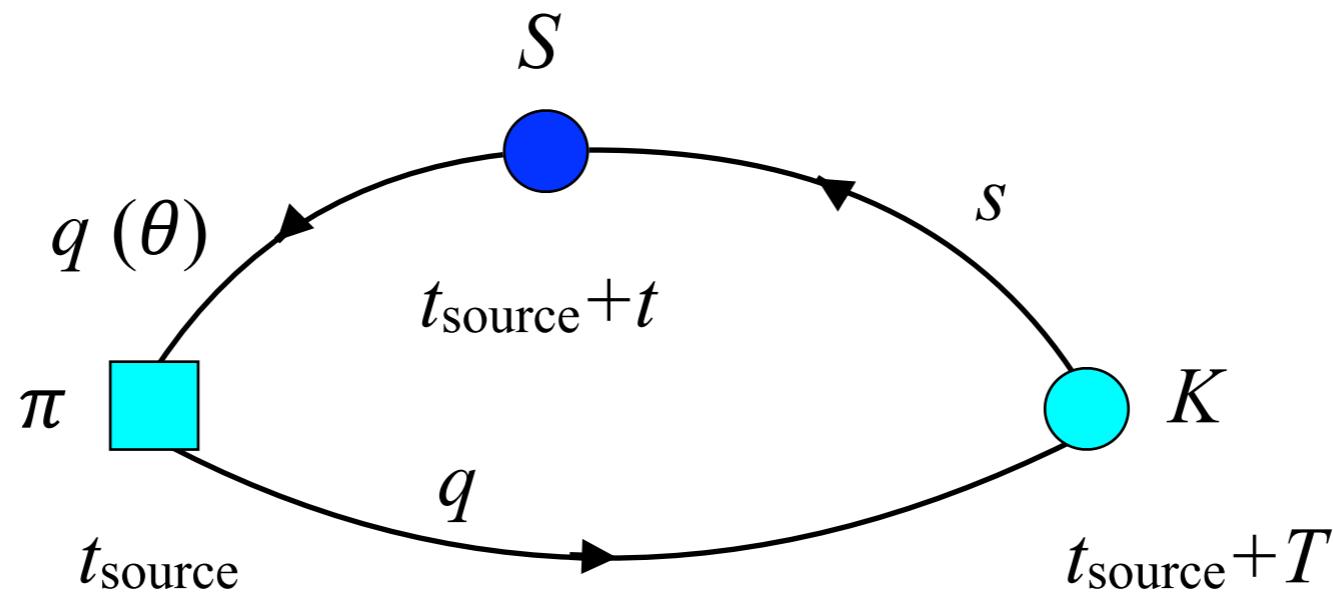
- MILC HISQ ensembles with $N_f = 2+1+1$ sea



- ♦ Size of disk: # of configurations
- ♦ Black circles: new ensembles or increased # confs since 2014
(see table in appendix for more details)

Set-up

- Use HISQ action also for valence strange and light quarks



- ♦ Kaon at rest, pion recoil momentum with twisted boundary conditions so that $q^2 = 0$.
- ♦ use Ward-Takahashi identity to calculate scalar form factor

$$f_0(q^2) = \frac{m_s - m_u}{m_K^2 - m_\pi^2} \langle \pi | S | K \rangle$$

together with kinematic constraint $f_+(0) = f_0(0)$

Analysis

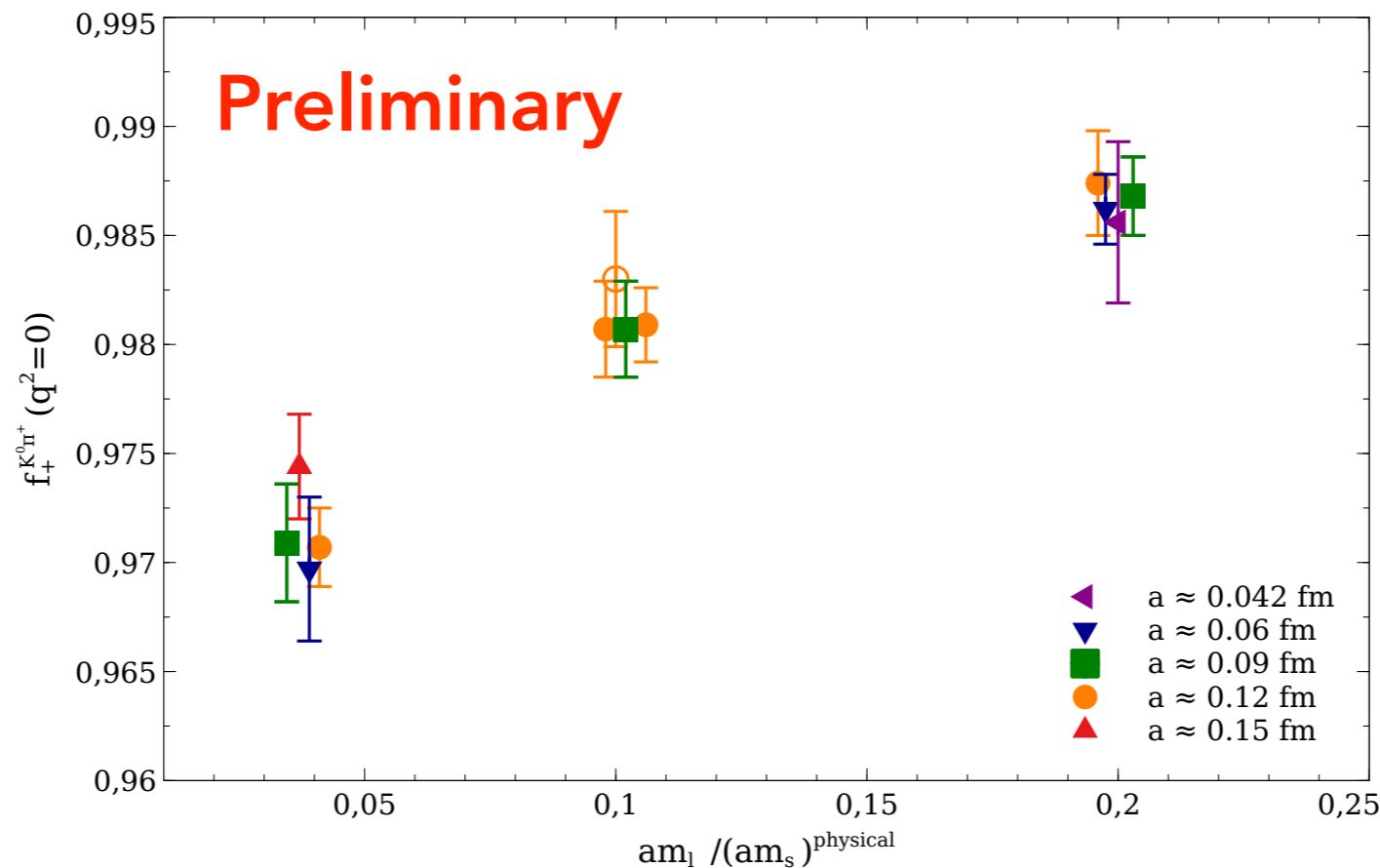
- Combined two- and three-point function fits

$$C_{2\text{pt}}^P(\vec{p}_P; t) = \sum_{m=0}^{N_{exp}} (-1)^{m(t+1)} (Z_m^P)^2 (e^{-E_P^m t} + e^{-E_P^m (L_t - t)})$$
$$C_{3\text{pt}}^{K \rightarrow \pi}(\vec{p}_\pi, \vec{p}_K; t, T) = \sum_{m,n=0}^{N_{exp}^{3pt}} (-1)^{m(t+1)} (-1)^{n(T-t+1)} A^{mn}(q^2) Z_m^\pi Z_n^K \\ \times (e^{-E_\pi^m t} + e^{-E_\pi^m (L_t - t)}) (e^{-E_K^n (T-t)} + e^{-E_K^n (L_t - T+t)})$$

- ♦ 3+3 states, $t_{\min} \sim 0.6\text{-}0.7$ fm with $t \in [t_{\min}, T - t_{\min}]$
- ♦ For more details see 2014 paper (A. Bazavov et al, [arXiv:1312.1224](#), [2014 PRL](#))

Analysis

- results for $f_+(0)$ at each lattice spacing and sea quark mass



- Correct the above form factors **before** the chiral-continuum fit for
 - ♦ finite volume effects
 - ♦ effects due to poorly sampled topology
(see appendix)

Finite volume corrections

- use ChPT to calculate the leading-order FV corrections for each twisting angle and light-quark mass [Bernard et al, arXiv: [1702.03416](#), 2017 JHEP]

$$\begin{aligned}\Delta^V f_+(0) &\equiv f_+^V(0) - f_+^\infty(0) \\ &= \frac{(m_s - m_d)\Delta^V \langle\pi|S|K\rangle}{(m_K^V)^2 - (m_\pi^V)^2} - \frac{(m_s - m_d)\langle\pi|S|K\rangle^V(\Delta^V m_K^2 - \Delta^V m_\pi^2)}{[(m_K^V)^2 - (m_\pi^V)^2]^2}\end{aligned}$$

- the resulting corrections are $\leq 0.1\%$ on all ensembles

Chiral-continuum fit function

- for chiral interpolation and continuum extrapolation
- use ChPT for light-quark mass dependence, discretization effects, finite volume, and isospin breaking effects.
- in isospin limit

$$f_+^{K\pi}(0) = 1 + f_2 + f_4 + f_6 + \dots$$

f_i : chiral corrections of $O(p^i)$

- f_2 : NLO PQSChPT
 f_4 : NNLO continuum ChPT + $O(\alpha_s a^2, \alpha_s^2 a^2, a^4)$
+ N³LO and N⁴LO analytic terms

$$f_+^{K\pi}(0) = 1 + f_2^{\text{PQSChPT}}(a) + f_4^{\text{cont.}} + g_{1,a} + r_1^4(m_\pi^2 - m_K^2)^2 \left[\tilde{C}_4 + g_{2,a} + h_{m_\pi} \right]$$

Chiral-continuum fit function

$$f_+^{K\pi}(0) = 1 + f_2^{\text{PQS}\chi\text{PT}}(a) + f_4^{\text{cont}} + g_{1,a} + r_1^4(m_\pi^2 - m_K^2)^2 \left[\tilde{C}_4 + g_{2,a} + h_{m_\pi} \right]$$

- * $f_2^{\text{PQS}\chi\text{PT}}(a)$: one-loop (NLO) partially quenched SChPT **Bernard, Bijnens, E.G., 1311.7511**
- * f_4^{cont} : Two-loop (NNLO) continuum ChPT **Bijnens & Talavera, 0303103**
- * $\tilde{C}_4 \propto (C_{12} + C_{34} - L_5^2)(\mu)$
- * $g_{1,a}$ and $g_{2,a}$ account for higher order discretization effects:

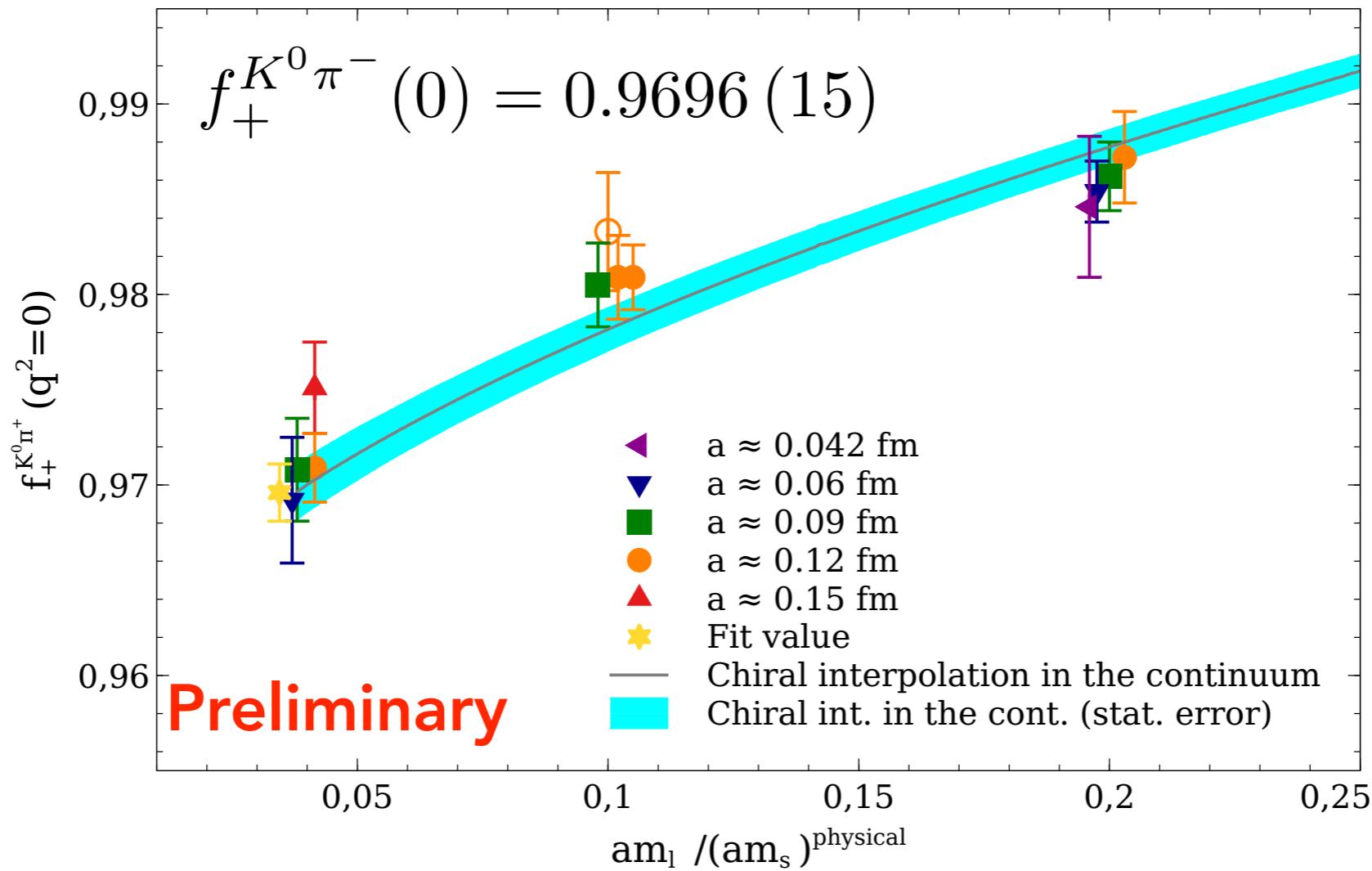
$$g_{1,a} = K_1 \sqrt{r_1^2 a^2 \bar{\Delta} \left(\frac{a}{r_1} \right)^2} + K_3 \left(\frac{a}{r_1} \right)^4, \quad g_{2,a} = K_2 \sqrt{r_1^2 a^2 \bar{\Delta} \left(\frac{a}{r_1} \right)^2} + K'_2 r_1^2 a^2 \bar{\Delta}$$

with $r_1^2 a^2 \bar{\Delta}$ used as a proxy of $\alpha_s^2 a^2$

- * h_{m_π} includes analytical terms that parametrize higher order chiral effects

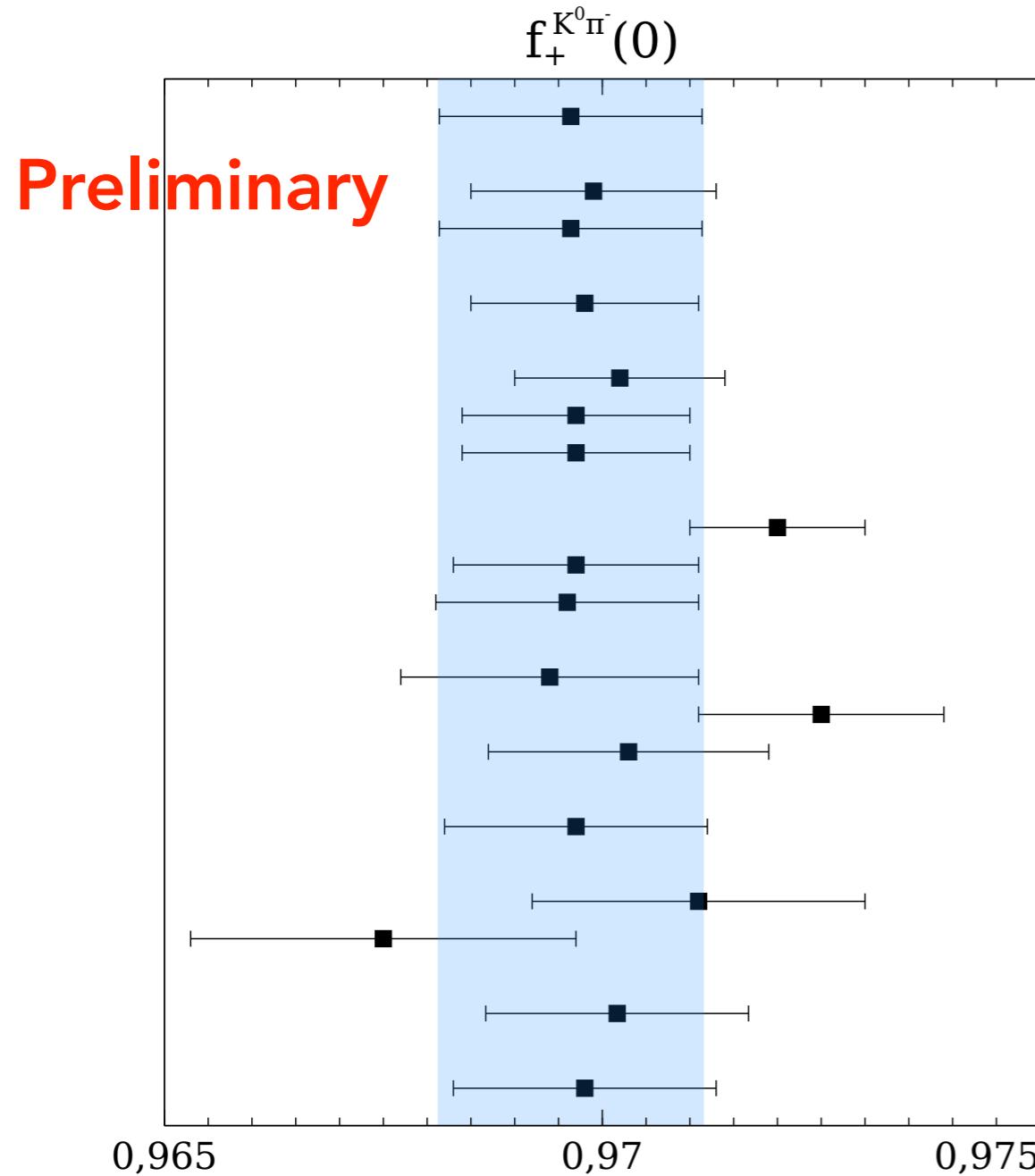
$$h_{m_\pi} = \tilde{C}_6 m_\pi^2 + \tilde{C}_8 m_\pi^4$$

Chiral-continuum fit



- Fit error: statistical + chiral interpolation + discretization + fit parameters [$O(p^4)$ LECs, ...]
- Interpolation line and band: full QCD, phys. m_s , with NNLO isospin breaking effects included
- data points: corrected for FV effects

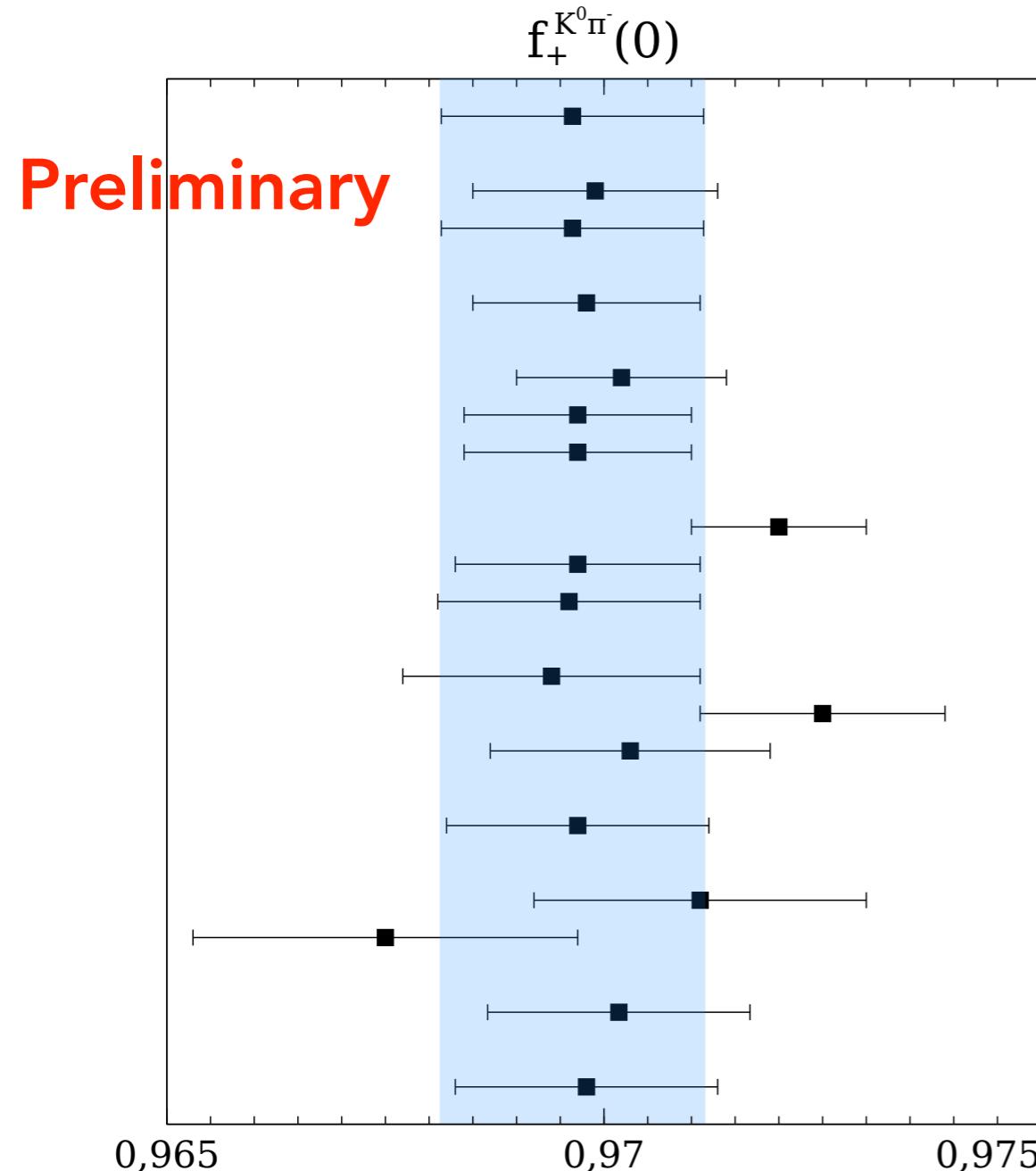
Systematic error analysis



base
NNLO
N³LO
 f_K vs f_π at two loops
NNLO analyt.
N³LO analyt.
N⁴LO analyt.
no analyt. a^2
 $\alpha_s^2 a^2 (m_\pi^2 - m_K^2)$
 $\alpha_s^2 a^2 (m_\pi^2 - m_K^2) + \alpha_s a^2$
no $a \approx 0.15$ fm
continuum, no $a \approx 0.15$ fm
continuum + analyt. a^2
no $a \approx 0.042$ fm
no phys. mass data
only phys. mass data
no FV
 m_s^{sea} vs m_s^{val}

chiral
truncation

Systematic error analysis



base
NNLO
N³LO

f_K vs f_π at two loops

NNLO analyt.
N³LO analyt.
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no analyt. a^2
 $\alpha_s^2 a^2 (m_\pi^2 - m_K^2)$
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no $a \approx 0.15$ fm
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continuum + analyt. a^2

no $a \approx 0.042$ fm

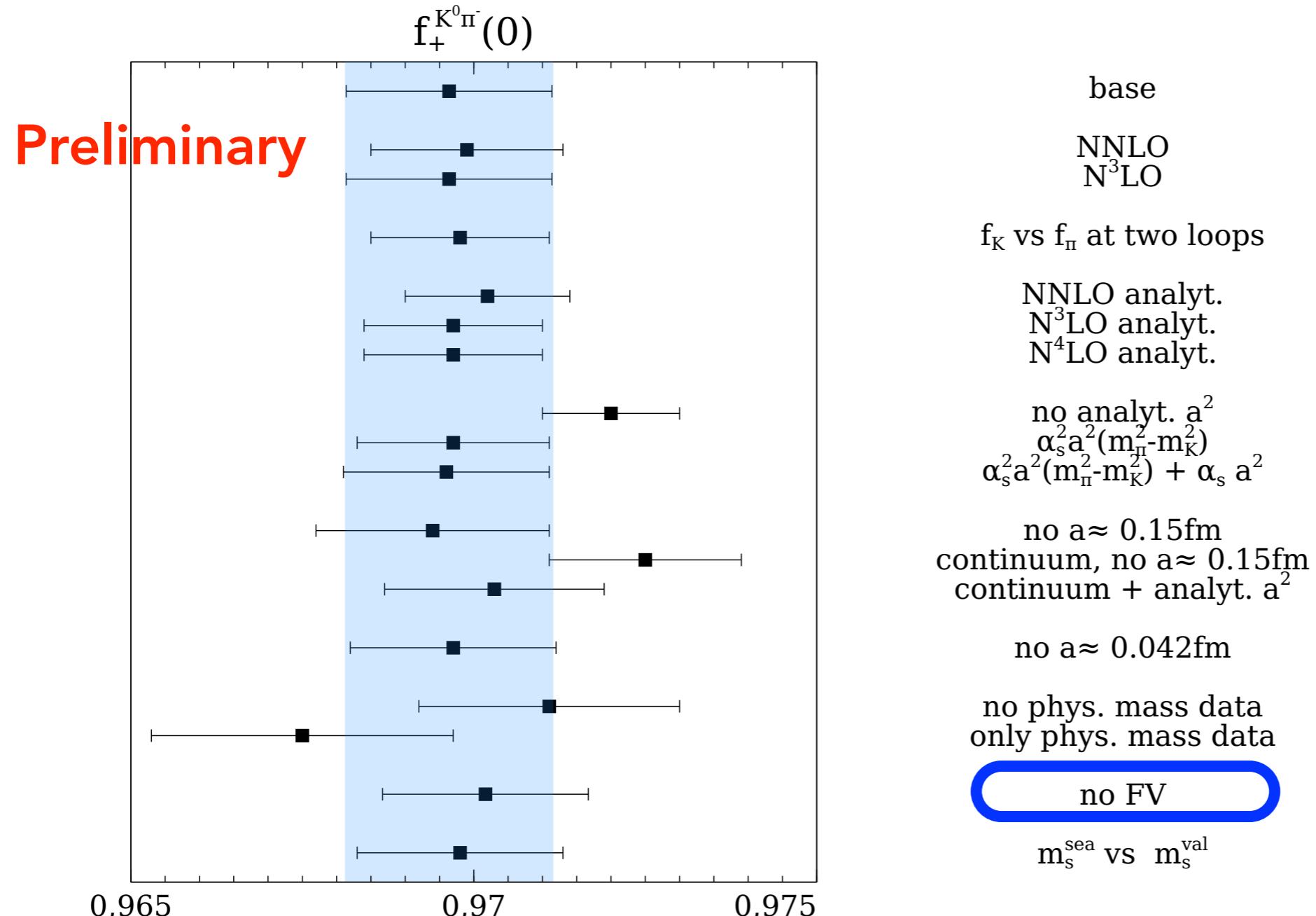
no phys. mass data
only phys. mass data

no FV

m_s^{sea} vs m_s^{val}

discretization

Systematic error analysis



NNLO/NLO contributions ~ 0.26

FVE: $\Delta(\text{no FV} - \text{NLO FV}) \times \text{NNLO/NLO factor}$

Systematic error analysis

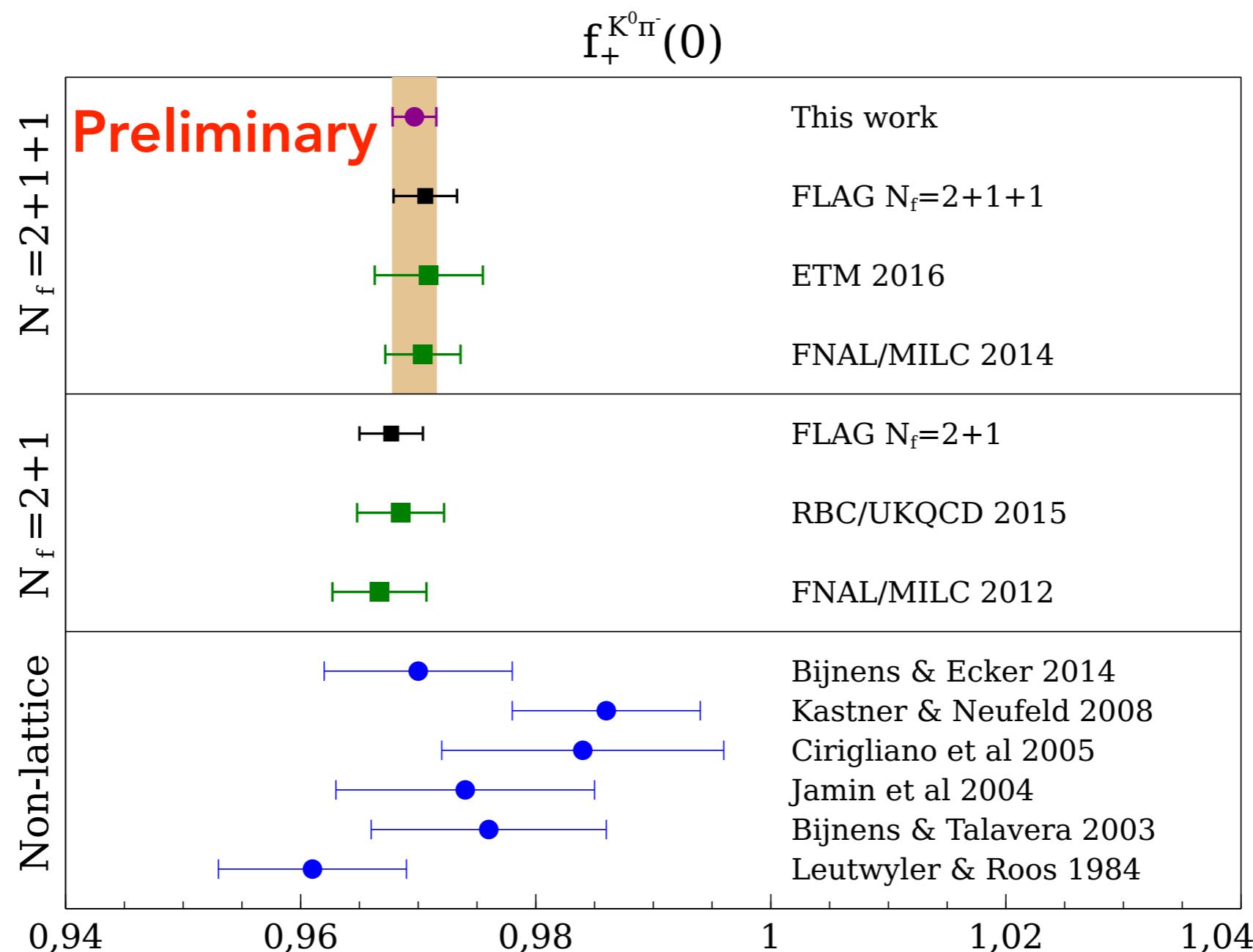
Preliminary

Source of uncertainty	Error $f_+^{K^0\pi^-}(0)$ (%)	2014 error
Stat. + disc. + chiral inter.	0.154	0.24
$L_{7,8}^r$	0.079	-
Scale r_1	0.080	0.08
$m_s^{\text{val}} \neq m_s^{\text{sea}}$	0.013	0.03
Finite volume	0.014	0.2
Higher-order isospin corrections	0.015	0.016
Isospin-breaking parameter R	0.002	-
Total Error	0.193	0.33

$$f_+^{K\pi}(0) = 0.9696(15)_{\text{stat}}(11)_{\text{sys}} = 0.9696(19)$$

cf 2014 PRL: $f_+^{K\pi}(0) = 0.9704(24)_{\text{stat}}(22)_{\text{sys}} = 0.9704(32)$

Kaon form factor in comparison



Implications for $|V_{us}|$

experimental averages:

(M. Moulson @ CKM 2016, [arXiv:1704.04104](#))

Preliminary

All modes: $|V_{us}| f_+^{K^0\pi^-}(0) = 0.21654(41)$

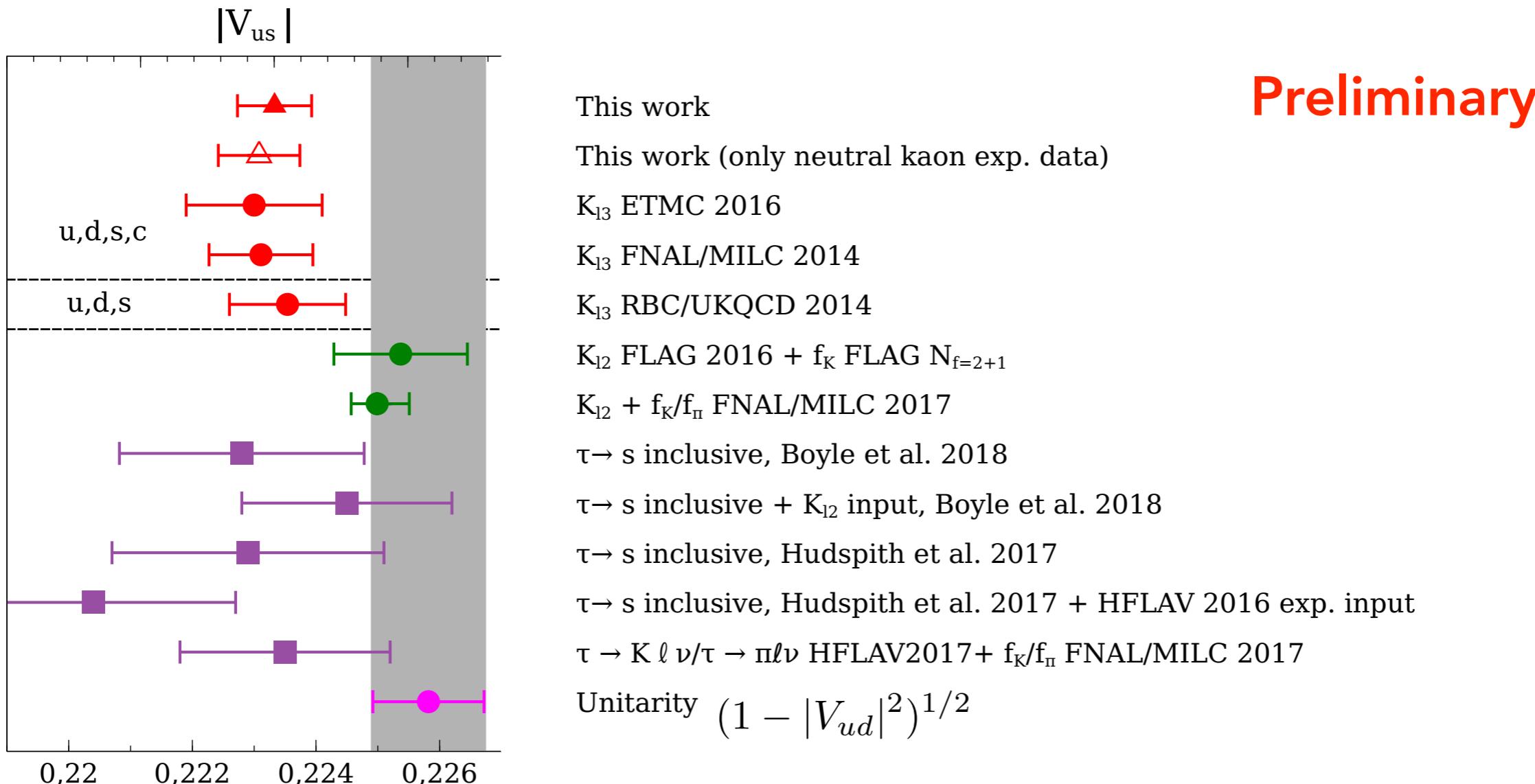
→ $|V_{us}| = 0.22333(42)_{\text{exp}}(43)_{f_+(0)}$

Theory error commensurate with experiment

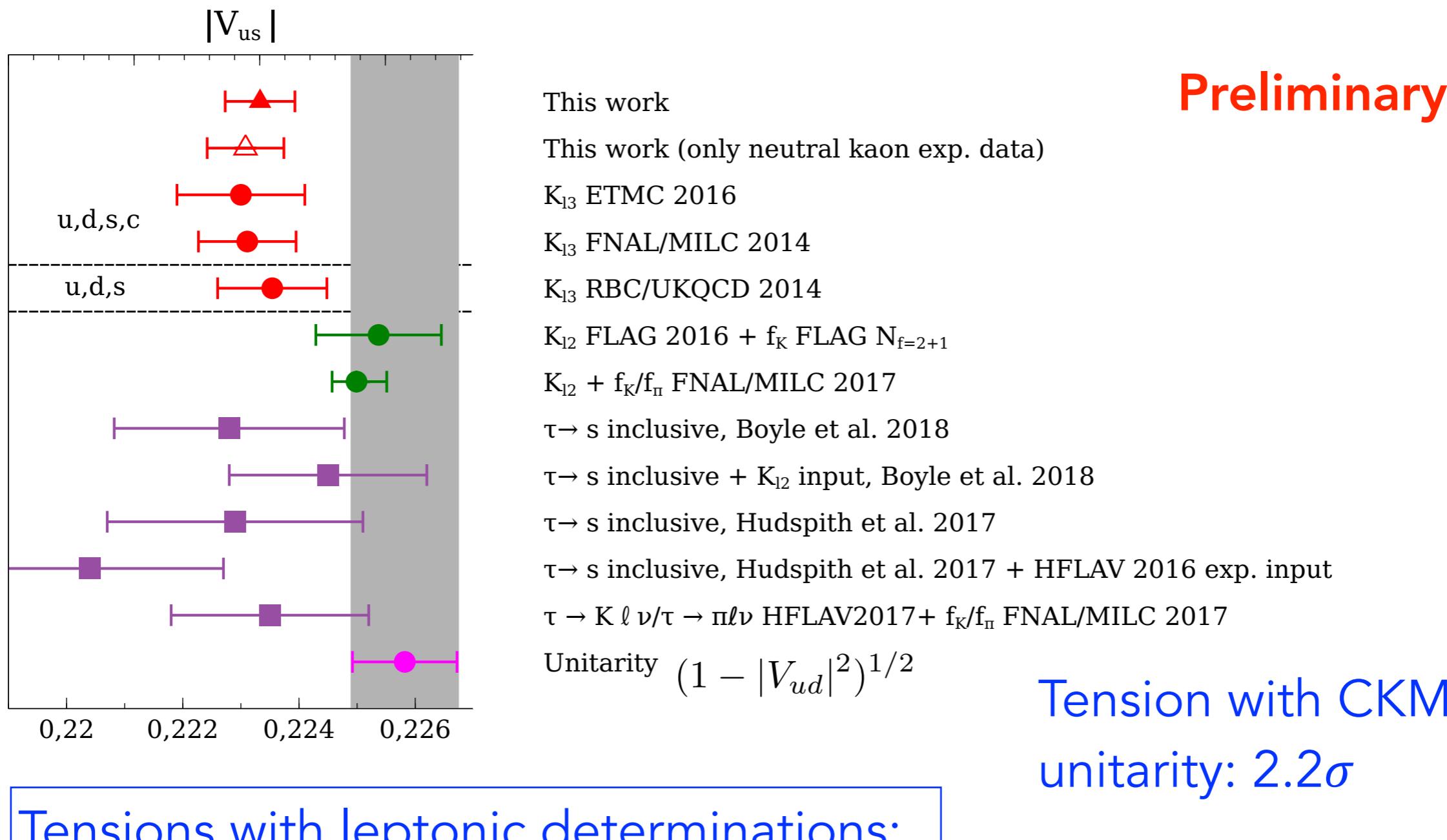
K^0 only: $|V_{us}| f_+^{K^0\pi^-}(0) = 0.21633(44)_{\text{exp}}(24)_{\delta_{\text{EM}}^{K^0\ell}}$

→ $|V_{us}|_{K^0\pi} = 0.22309(44)_{\text{exp}}(25)_{\delta_{\text{EM}}^{K\ell}}(43)_{f_+(0)}$

Implications for $|V_{us}|$



Implications for $|V_{us}|$

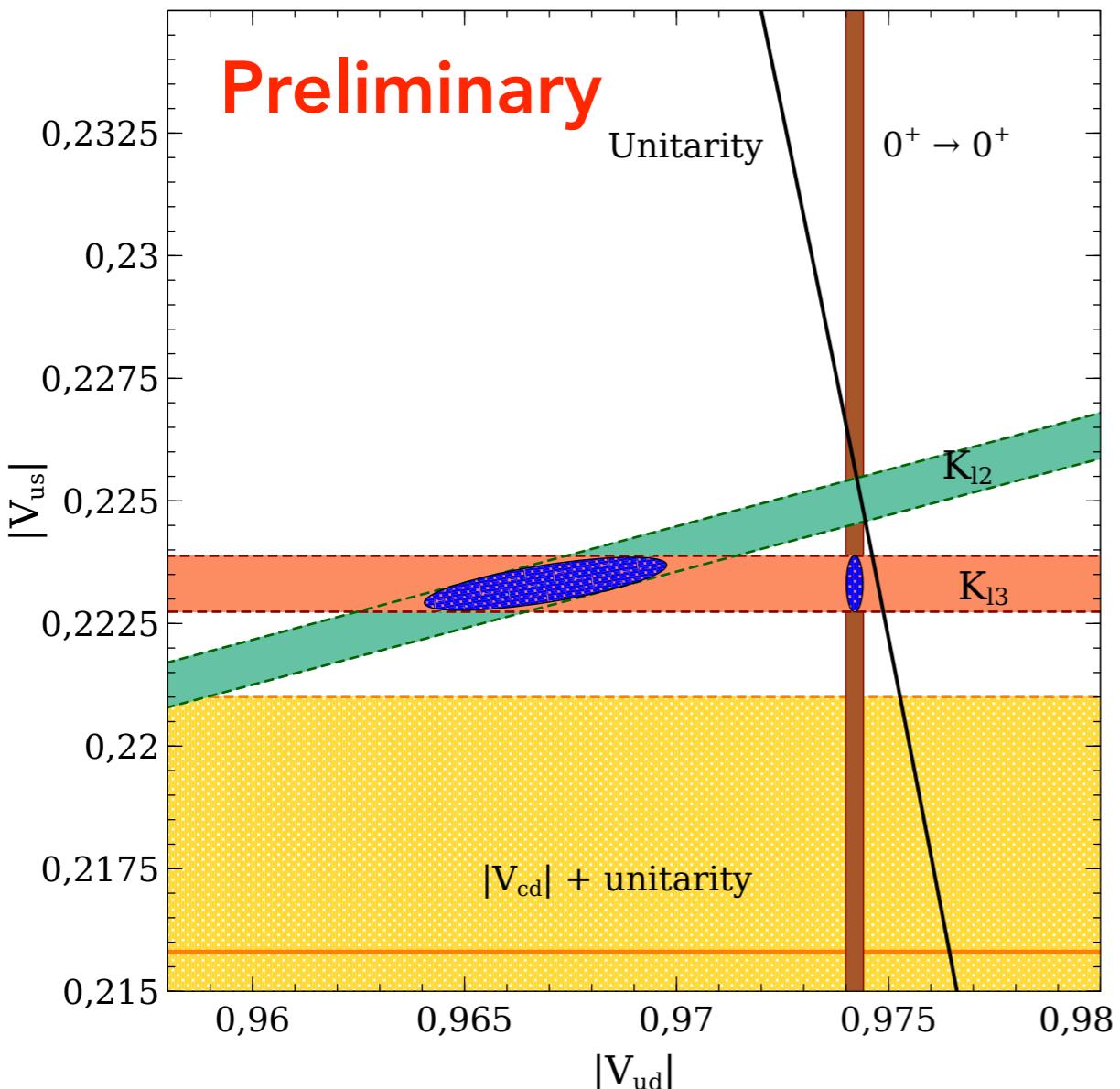


Tensions with leptonic determinations:

- $\Gamma_{K_{\ell 2}}^{\text{exp}} + f_{K^\pm}: 1.7\sigma$
- $\Gamma_{K_{\ell 2}}^{\text{exp}} + f_{K^\pm} / f_{\pi^\pm} + |V_{ud}|: 2.3\sigma$

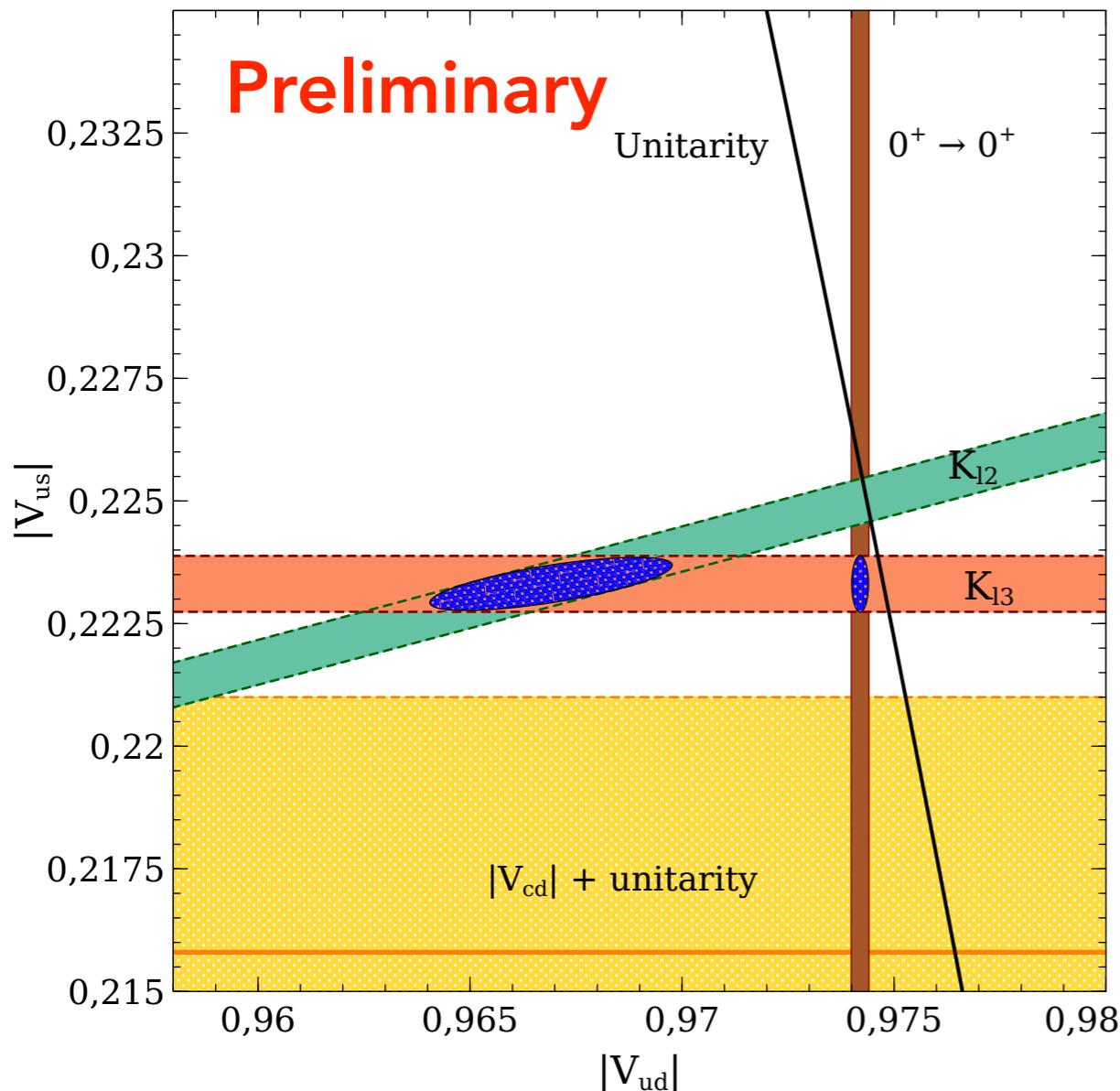
First row CKM unitarity

$$\Delta_u \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1 \quad V_{ub} \sim 0$$



First row CKM unitarity

$$\Delta_u \equiv |V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 - 1 \quad V_{ub} \sim 0$$



$$K_{\ell 3}: \quad \Delta_u = -0.00110(27)_{V_{us}}(41)_{V_{ud}}$$

$$K_{\ell 3}, K_{\ell 2}, f_+(0), f_K/f_\pi:$$

$$\Delta_u = -0.00151(38)_{f_+(0)}(35)_{f_K/f_\pi}(36)_{\text{exp}}(27)_{\text{EM}}$$

$|V_{ud}| = 0.97420(21)$ from nuclear β -decay
(Hardy & Towner @ CIPANP 2018, arXiv:1808.01146)

Conclusions and Outlook



Conclusions and Outlook

- ★ We present a LQCD calculation of $f_+^{K\pi}(0)$ with 0.20% total error — most precise result to date.
Our results are still preliminary, but paper is nearly final.
- ★ The resulting determination of $|V_{us}|$ is in tension with determinations from leptonic decays and with first row CKM unitarity (and $|V_{ud}|$ determined from β -decay) at the $2\text{-}2.6\sigma$ level.
- ★ We also obtain the LEC combination $[C_{12}^r + C_{34}^r - (L_5^r)^2](M_\rho) = (2.92 \pm 0.30) \cdot 10^{-6}$
- ★ We plan to calculate the ratio of $f_+^{K\pi}(0)$ and f_{K^\pm}/f_{π^\pm} with correlations. This will sharpen our calculation of $f_+^{K\pi}(0)$ and yield more precise unitarity tests from only kaon decay.



Thank you!

Appendix

Fermilab Lattice and MILC collaboration

- Fermilab Lattice Collaboration:

AXK, E. Freeland, E. Gámiz, S. Gottlieb, A. Kronfeld, J. Laiho, P. Mackenzie, E. Neil, J. Simone, R. Van de Water
Z. Gelzer, Y. Liu, A. Veernala, C. Bouchard, C.-C. Chang, D. Du, R. Zhou

- MILC:

A. Bazavov, C. Bernard, C. DeTar, S. Gottlieb, U. Heller, J. Osborn, R. Sugar, D. Toussaint,
S. Basak, N. Brown, J. Komijani, R. Li, Y. Liu, L. Levkova, T. Primer

Simulation details

$\approx a(\text{fm})$	m_l/m_s^{sea}	$m_\pi^P L$	$N_{conf} \times N_{src}$	am_s^{sea}	am_s^{val}	
0.15	0.035	3.2	1000×4	0.0647	0.06905	
0.12	0.2	4.5	1053×8	0.0509	0.0535	
	0.1	3.2	1020×8	0.0507	0.053	†
	0.1	4.3	993×4	0.0507	0.053	
	0.1	5.4	1029×8	0.0507	0.053	*
	0.035	3.9	945×8	0.0507	0.0531	
0.09	0.2	4.5	773×4	0.037	0.038	
	0.1	4.7	853×4	0.0363	0.038	
	0.035	3.7	950×8	0.0363	0.0363	*
0.06	0.2	4.5	1000×8	0.024	0.024	*
	0.035	3.7	692×6	0.022	0.022	†
0.042	0.2	4.3	432×12	0.0158	0.0158	†

†: New ensembles since our 2014 paper.

*: Ensembles with increased statistics since 2014.

Poorly sampled topology effects

- affects only the 0.042 fm ensemble at $m_\ell = 0.2m_s$
- following Bernard and Toussaint ([arXiv:1707.05430](#), 2017 PRD) the

correction

$$f_+^{K\pi}(0)_{\text{corrected}} = f_+^{K\pi}(0)_{\text{sampled}} - \frac{1}{2\chi_T V} (f_+^{K\pi}(0))'' \left(1 - \frac{\langle Q^2 \rangle_{\text{sample}}}{\chi_T V} \right)$$

can be obtained by calculating $(f_+^{K\pi}(0))'' \equiv \frac{d^2 f}{d\theta^2} \Big|_{\theta=0}$ in ChPT:

$$f_+^{K\pi}(0)'' = -\frac{1}{4} \frac{(m_l - m_s)^2}{(m_l + 2m_s)^2}$$

- The correction is $< 1/2$ stat error

Chiral-continuum fit function

Fit parameters: $\delta'_{A,V}$, $K_{1,2,3}$, K'_2 , $\tilde{C}_{4,6,8}$, $L_{1,2,3,5,6}$, $2L_6 - L_4$

Fixed parameters: f_π , taste splittings, r_1/a , $L_{7,8}$.

3. Take the continuum limit and interpolate to the QCD meson masses including NNLO isospin corrections [Gasser & Leutwyler, NPB250, 517 \(1985\)](#), [Bijnens & Ghorbani, 0711.0148](#)

$$f_+^{K^0\pi^-}(0) = 1 + f_2^{\text{cont. isospin-break.}} + f_4^{\text{cont. isospin-break.}} + (m_{\pi^+}^2 - m_{K^0}^2) [\tilde{C}_4 + h_{m_\pi}]$$

$$\text{with } h_{m_\pi} = \tilde{C}_6 m_\pi^2 + \tilde{C}_8 m_\pi^4$$

* Isospin breaking contributions: depend on $R \equiv \frac{m_s - \hat{m}}{m_u - m_d} = 34.7(5)_{\text{stat}} ({}^{+1.0}_{-0.6})_{\text{syst}}$
with m_s/\hat{m} and m_u/m_d from [FNAL/MILC 1712.09262](#) without correlations