Lattice Calculation of Partin Distribution Function from LaMET

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Parton Distribution Function

- Defined on the lightcone coordinate $\xi^{\pm} = \frac{t \pm z}{\sqrt{2}}$

$$q(x,\mu) \equiv \int \frac{d\xi^-}{4\pi} e^{-ixP^+\xi^-} \langle P|\bar{\psi}(\xi^-)\gamma^+ U(\xi^-,0)\psi(0)|P\rangle$$

where $-1 \leq x \leq 1$ and

$$U(\xi^-,0) = P\exp\left(-ig\int_0^{\xi^-} d\eta^- A^+(\eta^-)\right)$$

- $q(x,\mu) = -\bar{q}(-x,\mu)$ for $x < 0$
- $q(x,\mu)$ has intrinsic real-time dependence, inaccessible on the lattice.
- Only moments can be calculated on the lattice [1-4].

[1] G. Martinelli and C. T. Sachrajda, Phys. Lett. B196, 184 (1987).
[2] G. Martinelli and C. T. Sachrajda, Phys. Lett. B217, 319 (1989).
[3] W. Detmold, W. Melnitchouk, and A. W. Thomas, Eur. Phys. J.direct 3, 13 (2001), arXiv:hep-lat/0108002 [heplat].
[4] D. Dolgov et al. (LHPC, TXL), Phys. Rev. D66, 034506 (2002), arXiv:hep-lat/0201021 [hep-lat].

Quasi-PDF

- Defined by an equal-time correlator [1]

$$\tilde{q}_\Gamma(x, P_z, \tilde{\mu}) \equiv \int_{-\infty}^{\infty} \frac{dz}{4\pi} e^{ixP_z z} \langle P|\bar{\psi}(z)\Gamma U(z,0)\psi(0)|P\rangle$$

where $-\infty < x < \infty$, $\Gamma = \gamma_z$ or γ_t [2], and

$$U(z,0) = P\exp\left(-ig\int_0^z dz' A_z(z')\right)$$

- $\tilde{q}_\Gamma$ has the same IR but different UV physics comparing with PDF q.
- The UV difference is controllable and calculable
  → factorization theorem

[1] X. Ji, Phys. Rev. Lett. 110, 262002 (2013), arXiv:1305.1539 [hep-ph].
[2] Y. Hatta, X. Ji, and Y. Zhao, Phys. Rev. D89, 085030 (2014), arXiv:1310.4263 [hep-ph].

Large Momentum ET (LaMET)

- Relating parton physics observables to equal-time correlators in a large momentum nucleon state (quasi-observables).
- PDF: $P_z \to \infty$, then $\Lambda \to \infty$
- Quasi-PDF: $\Lambda \to \infty$, then $P_z \to \infty$
- The two limits do not commute.
- The factorization theorem in RI/MOM scheme [1-7]

$$\tilde{q}_R(x, P_z, p_z^R, \mu_R) = \int_{-1}^{1} \frac{dy}{|y|} C\left(\frac{x}{y}, r, \frac{yP_z}{\mu}, \frac{yP_z}{p_z^R}\right) q(y, \mu) + \mathcal{O}\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\rm QCD}^2}{P_z^2}\right)$$

$$r = \mu_R^2 / {p_z^R}^2$$

[1] X. Ji, Phys. Rev. Lett. 110, 262002 (2013), arXiv:1305.1539 [hep-ph].
[2] X. Ji, Sci. China Phys. Mech. Astron. 57, 1407 (2014), arXiv:1404.6680 [hep-ph].
[3] Y.-Q. Ma and J.-W. Qiu, (2014), arXiv:1404.6860 [hep-ph].
[4] Y.-Q. Ma and J.-W. Qiu, Phys. Rev. Lett. 120, 022003 (2018), arXiv:1709.03018 [hep-ph].
[5] I. W. Stewart and Y. Zhao, Phys. Rev. D97, 054512 (2018), arXiv:1709.04933 [hep-ph].
[6] T. Izubuchi, X. Ji, L. Jin, I. W. Stewart, and Y. Zhao, (2018), arXiv:1801.03917 [hep-ph].
[7] Y. S. Liu, J. W. Chen, L. Jin, H. W. Lin, Y. B. Yang, J. H. Zhang and Y. Zhao, (2018), arXiv:1807.06566 [hep-lat].

Lattice Calculation [1]

- Lattice space $\mathrm{a} = 0.06$ fm
- Box size $48^3 \times 96$ $(L = 2.9\ \mathrm{fm})$
- $m_\pi = 310$ MeV $(m_\pi L \approx 4.5)$
- The nucleon momentum $P_z = \{1.7, 2.15, 2.6\}$ GeV
- clover valence fermions, gauge configurations with $N_f = 2 + 1 + 1$ flavors of highly improved staggered quarks (HISQ) [2] generated by MILC Collaboration [3]
- the gauge links are hypercubic (HYP)-smeared [4]
- Unpolarized isovector PDF: $q^{u-d}(x) = \begin{cases} u(x) - d(x) & x > 0 \\ -\bar{u}(-x) + \bar{d}(-x) & x < 0 \end{cases}$

[1] Y. S. Liu, J. W. Chen, L. Jin, H. W. Lin, Y. B. Yang, J. H. Zhang and Y. Zhao, (2018), arXiv:1807.06566 [hep-lat].
[2] E. Follana, Q. Mason, C. Davies, K. Hornbostel, G. P. Lepage, J. Shigemitsu, H. Trottier, and K. Wong (HPQCD, UKQCD), Phys. Rev. D75, 054502 (2007), arXiv:hep-lat/0610092 [hep-lat].
[3] A. Bazavov et al. (MILC), Phys. Rev. D87, 054505 (2013), arXiv:1212.4768 [hep-lat].
[4] A. Hasenfratz and F. Knechtli, Phys. Rev. D64, 034504 (2001), arXiv:hep-lat/0103029 [hep-lat].

Operator Mixing

- The quasi-PDF operator might mix with the scalar operator ($\Gamma = 1$) for some choice of Γ [1-4].
- To avoid operator mixing, we choose $\Gamma = \gamma_t$ which is free from such mixing at $\mathcal{O}(a^0)$.

$$\tilde{q}_{\gamma^t}(x, P_z, \tilde{\mu}) \equiv \int_{-\infty}^{\infty} \frac{dz}{4\pi} e^{ixP_z z} \langle P|\bar{\psi}(z)\gamma^t U(z,0)\psi(0)|P\rangle$$

- The nonlocal operator mixing pattern is classified in [5].

[1] M. Constantinou and H. Panagopoulos, Phys. Rev. D96, 054506 (2017), arXiv:1705.11193 [hep-lat].
[2] J. Green, K. Jansen, and F. Steens, Phys. Rev. Lett. 121, 022004 (2018), arXiv:1707.07152 [hep-lat].
[3] C. Alexandrou, K. Cichy, M. Constantinou, K. Hadjiyiannakou, K. Jansen, H. Panagopoulos, and F. Steffens, Nucl. Phys. B923, 394 (2017), arXiv:1706.00265 [hep-lat].
[4] J.-W. Chen, T. Ishikawa, L. Jin, H.-W. Lin, Y.-B. Yang, J.-H. Zhang, and Y. Zhao, Phys. Rev. D97, 014505 (2018), arXiv:1706.01295 [hep-lat].
[5] J.-W. Chen, T. Ishikawa, L. Jin, H.-W. Lin, Y.-B. Yang, J.-H. Zhang, and Y. Zhao, (2017), arXiv:1710.01089 [hep-lat].

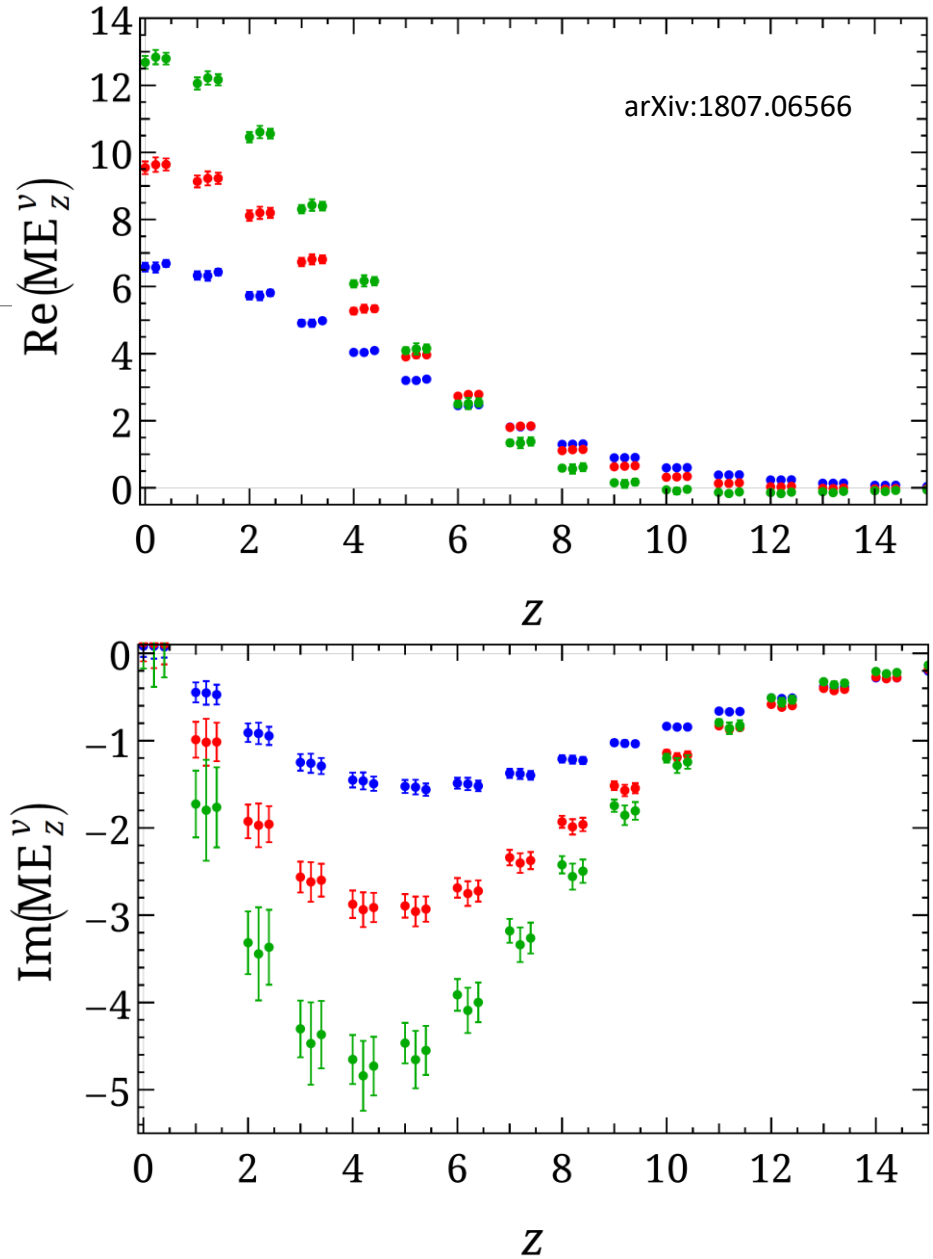
Bare M.E.

- The bare matrix element

$$\tilde{h}(z, P_z, a^{-1}) = \frac{1}{2P^0}\langle P|O_{\gamma^t}(z)|P\rangle$$

$$O_\Gamma(z) = \bar{\psi}(z)\Gamma U(z,0)\psi(0)$$

- Blue, red, green data correspond to P_z =1.7, 2.15, 2.6 GeV.
- Five source-sink separations: 0.60, 0.72, 0.84,0.96, 1.08 fm
- Ground state extraction [1] from left to right: all, largest 4, 3 t_{sep}.



[1] T. Bhattacharya, S. D. Cohen, R. Gupta, A. Joseph, H.-W. Lin, and B. Yoon, Phys. Rev. D89, 094502 (2014), arXiv:1306.5435 [hep-lat].

Renormalization

- Linear divergence of quark quasi-PDF [1] can be renormalized by Wilson line self-energy [2-7]
- Multiplicative renormalizability to all order in coordinate space [8,9]
- Nonperturbative renormalization [10-12] in RI/MOM scheme [13]
- Matching between RI/MOM quasi-PDF and $\overline{\mathrm{MS}}$ PDF [14,15].

[1] X. Xiong, X. Ji, J.-H. Zhang, and Y. Zhao, Phys. Rev. D90, 014051 (2014), arXiv:1310.7471 [hep-ph].
[2] X. Ji and J.-H. Zhang, Phys. Rev. D92, 034006 (2015), arXiv:1505.07699 [hep-ph].
[3] T. Ishikawa, Y.-Q. Ma, J.-W. Qiu, and S. Yoshida, (2016), arXiv:1609.02018 [hep-lat].
[4] J.-W. Chen, X. Ji, and J.-H. Zhang, Nucl. Phys. B915, 1 (2017), arXiv:1609.08102 [hep-ph].
[5] X. Xiong, T. Luu, and U.-G. Meiner, (2017), arXiv:1705.00246 [hep-ph].
[6] M. Constantinou and H. Panagopoulos, Phys. Rev. D96, 054506 (2017), arXiv:1705.11193 [hep-lat].
[7] G. Spanoudes and H. Panagopoulos, (2018), arXiv:1805.01164 [hep-lat].
[8] X. Ji, J.-H. Zhang, and Y. Zhao, Phys. Rev. Lett. 120, 112001 (2018), arXiv:1706.08962 [hep-ph].
[9] T. Ishikawa, Y.-Q. Ma, J.-W. Qiu, and S. Yoshida, Phys. Rev. D96, 094019 (2017), arXiv:1707.03107 [hep-ph].
[10] J. Green, K. Jansen, and F. Steens, Phys. Rev. Lett. 121, 022004 (2018), arXiv:1707.07152 [hep-lat].
[11] C. Alexandrou, K. Cichy, M. Constantinou, K. Hadjiyiannakou, K. Jansen, H. Panagopoulos, and F. Steffens, Nucl. Phys. B923, 394 (2017), arXiv:1706.00265 [hep-lat].
[12] J.-W. Chen, T. Ishikawa, L. Jin, H.-W. Lin, Y.-B. Yang, J.-H. Zhang, and Y. Zhao, Phys. Rev. D97, 014505 (2018), arXiv:1706.01295 [hep-lat].
[13] G. Martinelli, C. Pittori, C. T. Sachrajda, M. Testa, and A. Vladikas, Nucl. Phys. B445, 81 (1995), arXiv:heplat/9411010 [hep-lat].
[14] I. W. Stewart and Y. Zhao, Phys. Rev. D97, 054512 (2018), arXiv:1709.04933 [hep-ph].
[15] M. Constantinou and H. Panagopoulos, Phys. Rev. D96, 054506 (2017), arXiv:1705.11193 [hep-lat].

RI/MOM Scheme

$$O_\Gamma(z) = \bar{\psi}(z)\Gamma U(z,0)\psi(0)$$

- The quantum corrections of quasi-PDF matrix element in an off-shell quark state vanish at a given momentum

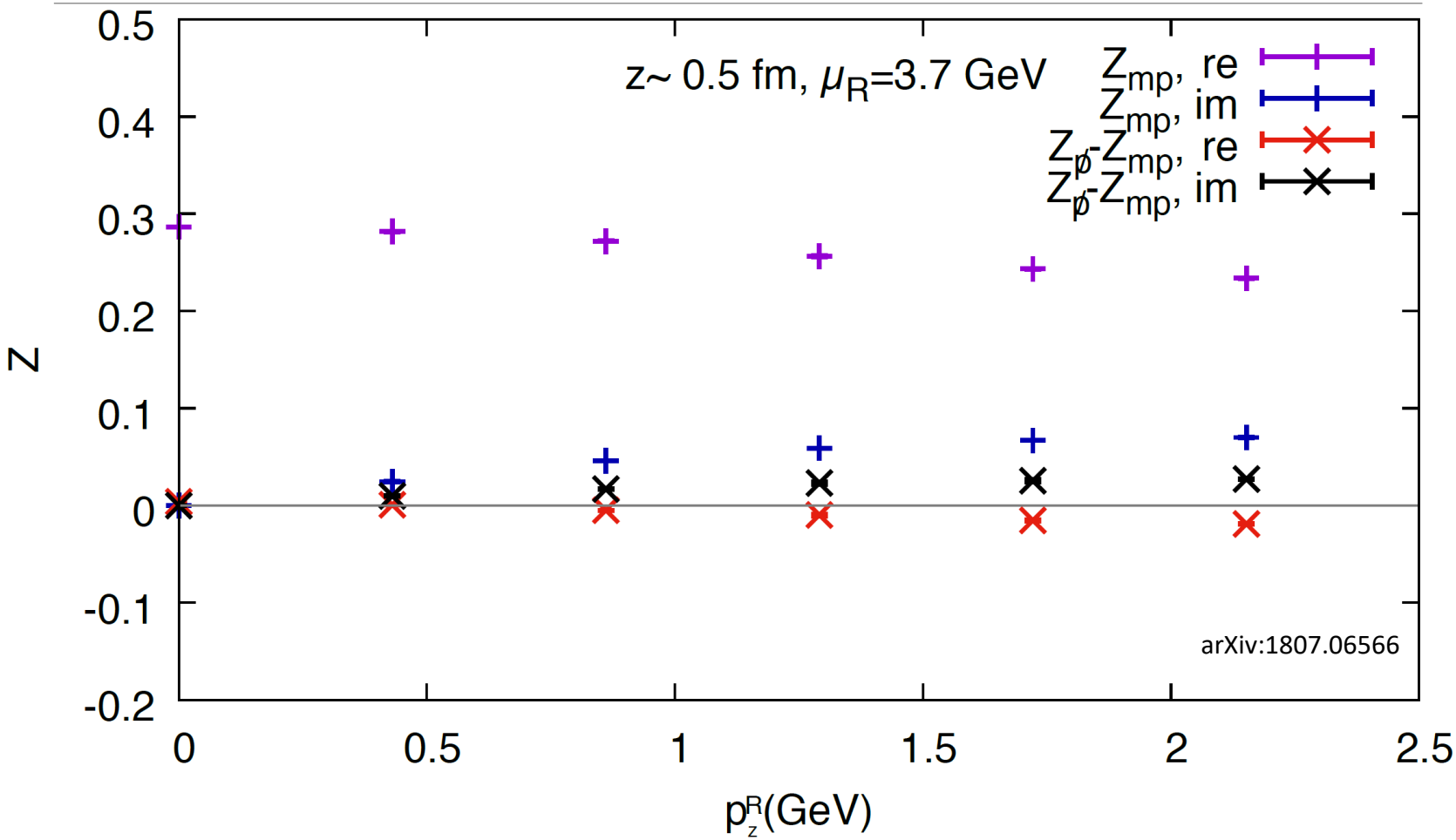
$$Z(z, p_z^R, a^{-1}, \mu_R) = \left.\frac{\langle p, s|O_{\gamma^t}(z)|p, s\rangle}{\langle p, s|O_{\gamma^t}(z)|p, s\rangle_{\text{tree}}}\right|_{\substack{p^2 = -\mu_R^2 \\ p_z = p_z^R}}$$

- The subtraction point is specified by two scales μ_R and p_z^R.
- The RI/MOM quasi-PDF is obtained by

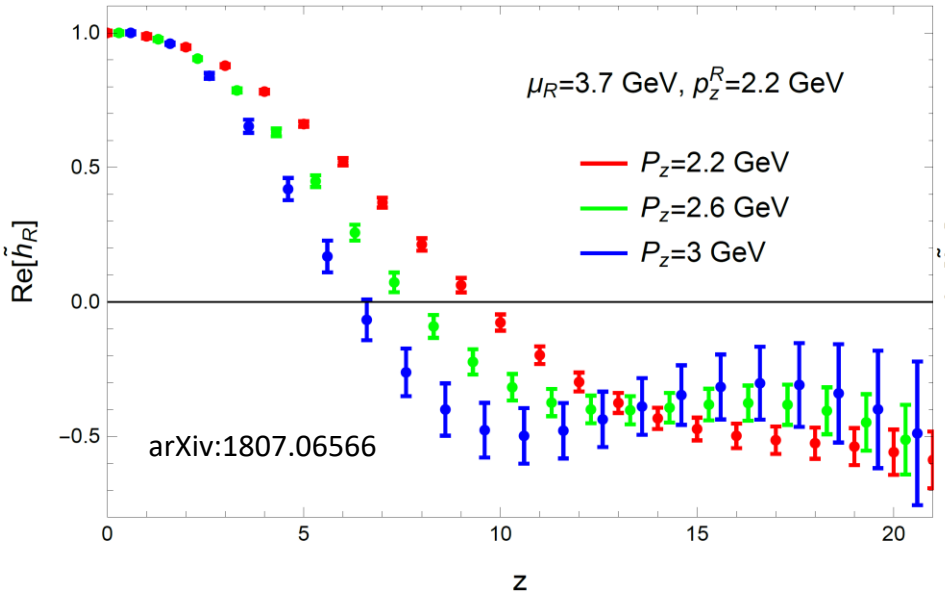
$$\tilde{h}_R(z, P_z, p_z^R, \mu_R) = Z^{-1}(z, p_z^R, a^{-1}, \mu_R)\tilde{h}(z, P_z, a^{-1})\Big|_{a\to 0}$$

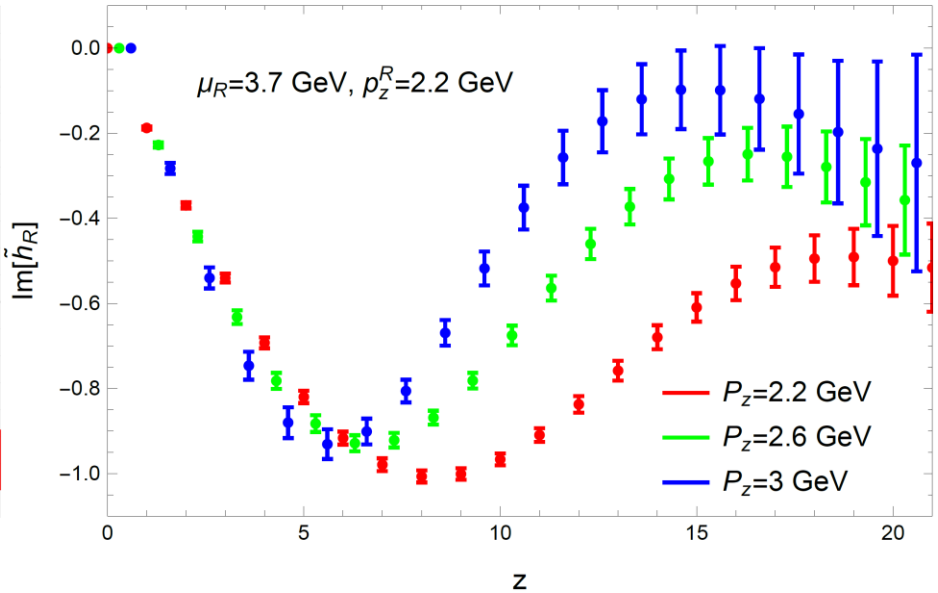
where $\tilde{h}(z, P_z, a^{-1}) = \frac{1}{2P^0}\langle P|O_{\gamma^t}(z)|P\rangle$ is the bare matrix element.

Renormalization Constant



Renormalized M.E.





Fourier Transformation

- Regular FT:

$$\tilde{q}_R(x, P_z, p_z^R, \mu_R) = P_z \int \frac{dz}{2\pi}\, e^{ixP_z z} \tilde{h}_R(z, P_z, p_z^R, \mu_R)$$

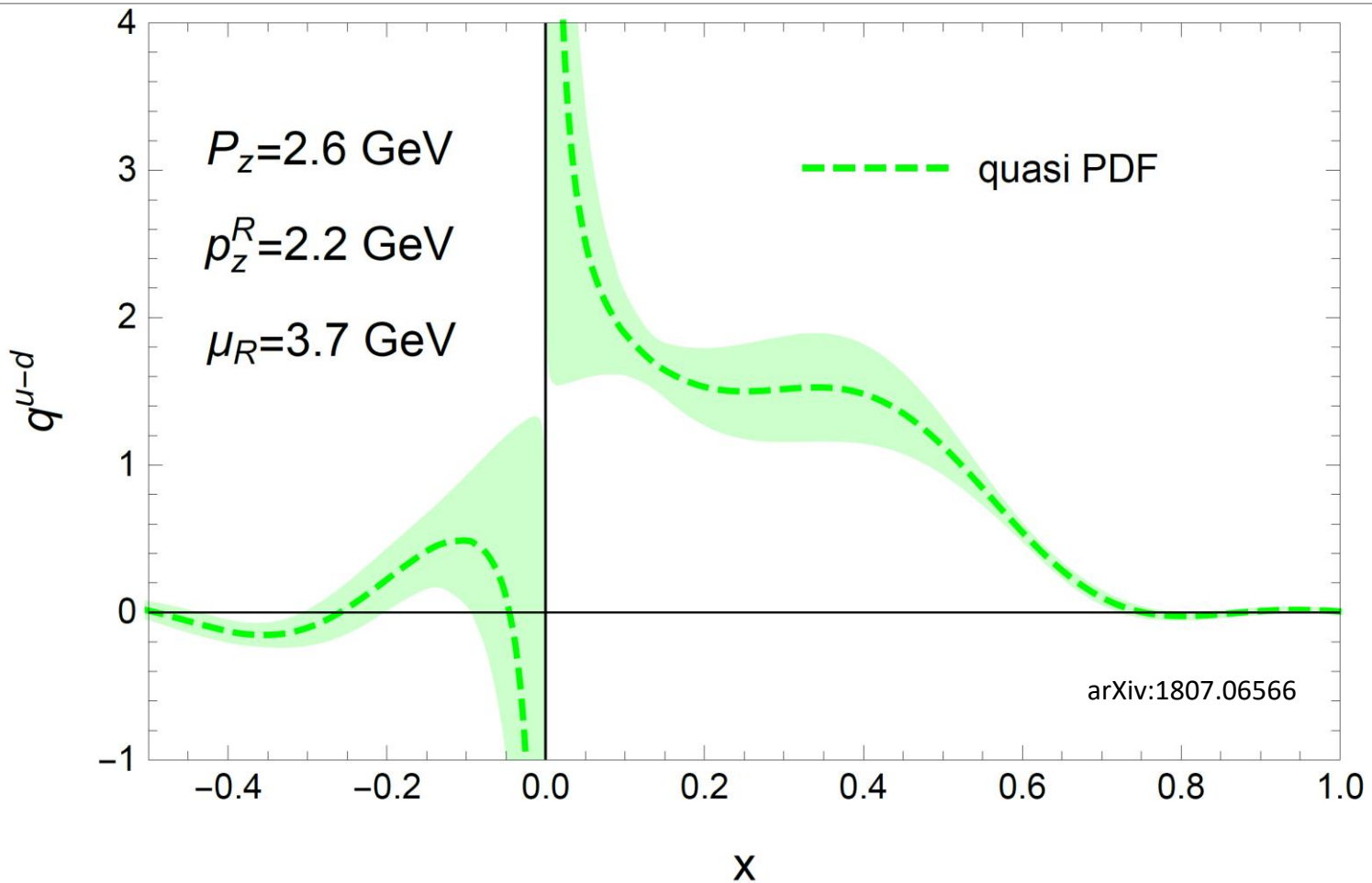
- Derivative method [1]:

$$\tilde{q}(x) = \int_{-z_{\max}}^{+z_{\max}} \frac{dz}{2\pi} \frac{i e^{ixP_z z}}{x} \partial_z \tilde{h}_R(z)$$

- Equivalent to set $\tilde{h}_R(z) = \tilde{h}_R(z_{max})$ if $|z| \geq z_{max}$.
- $\partial_z \tilde{h}_R(z)$ is consistent with zero for $|z| \geq 15a$ and we take $z_{max} = 20a$.

[1] H.-W. Lin, J.-W. Chen, T. Ishikawa, and J.-H. Zhang, (2017), arXiv:1708.05301 [hep-lat].

FT with Derivative Method



Matching

- Factorization

$$\tilde{q}_R(x, P_z, p_z^R, \mu_R) = \int_{-1}^{1} \frac{dy}{|y|} C\left(\frac{x}{y}, r, \frac{yP_z}{\mu}, \frac{yP_z}{p_z^R}\right) q(y, \mu) + \mathcal{O}\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\rm QCD}^2}{P_z^2}\right)$$

- Matching coefficient

$$r = \mu_R^2 / {p_z^R}^2$$

$$C\left(x, r, \frac{p_z}{\mu}, \frac{p_z}{p_z^R}\right) = \delta(1-x) + \left[f_{1,mp}\left(x, \frac{p_z}{\mu}\right) - \left|\frac{p_z}{p_z^R}\right| f_{2,mp}\left(1 + \frac{p_z}{p_z^R}(x-1), r\right)\right]_+ + \mathcal{O}(\alpha_s^2)$$

- The generalized plus function

$$\int dx\, [h(x)]_+ \, g(x) = \int dx\, h(x)\, [g(x) - g(1)]$$

Matching in Landau Gauge [1]

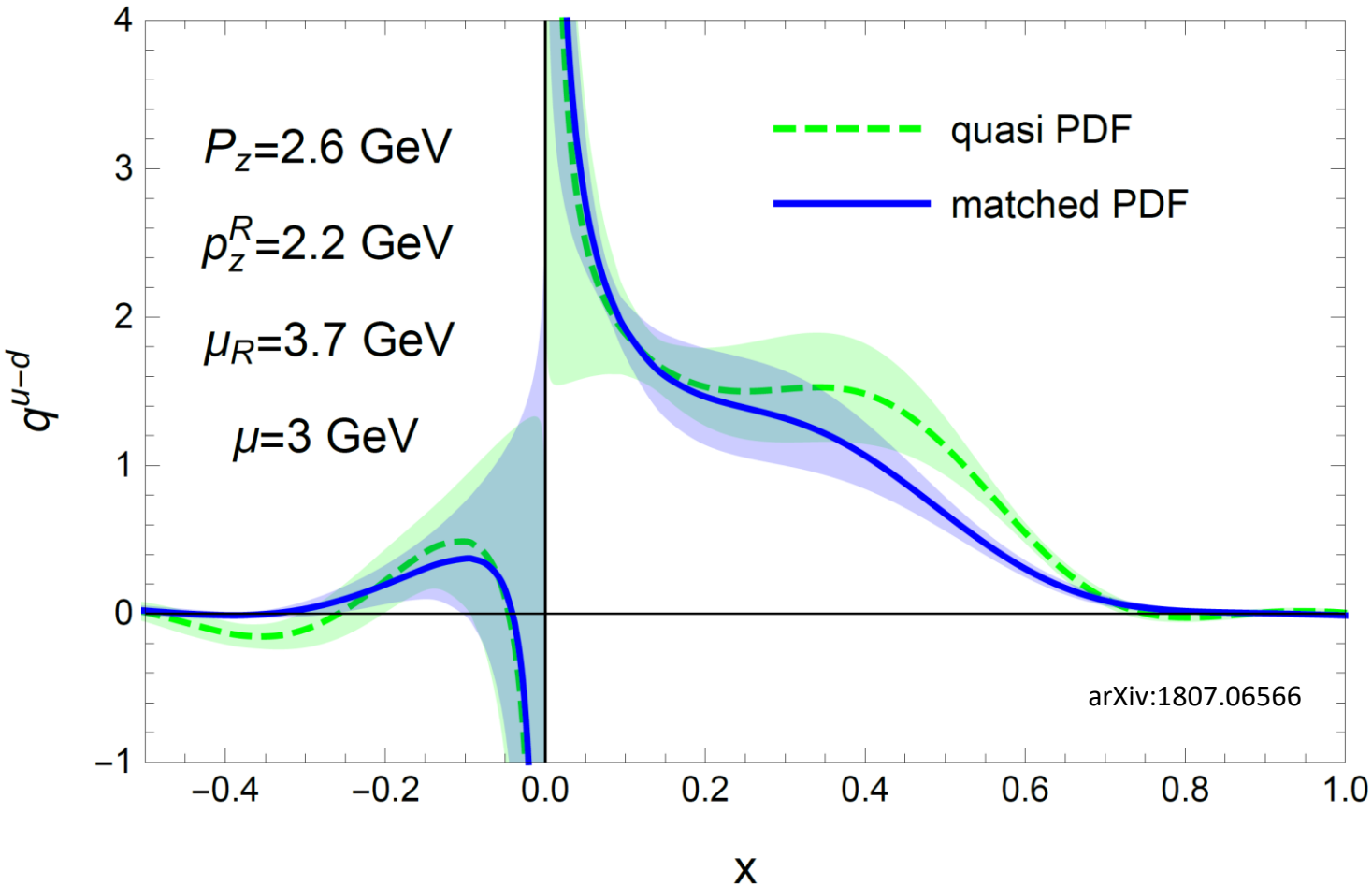
$$f_{1,mp}\left(x,\frac{p_z}{\mu}\right)=f_{1,\not{p}}\left(x,\frac{p_z}{\mu}\right)=\frac{\alpha_s C_F}{2\pi}\begin{cases}\frac{1+x^2}{1-x}\ln\frac{x}{x-1}+1 & x>1\\ \frac{1+x^2}{1-x}\ln\frac{4x(1-x)p_z^2}{\mu^2}-\frac{x(1+x)}{1-x} & 0<x<1\\ -\frac{1+x^2}{1-x}\ln\frac{x}{x-1}-1 & x<0\end{cases}$$

$$f_{2,mp}(x,r)=\frac{\alpha_s C_F}{2\pi}\begin{cases}\frac{-3r^2+13rx-8x^2-10rx^2+8x^3}{2(r-1)(x-1)(r-4x+4x^2)}+\frac{-3r+8x-rx-4x^2}{2(r-1)^{3/2}(x-1)}\tan^{-1}\frac{\sqrt{r-1}}{2x-1} & x>1\\ \frac{-3r+7x-4x^2}{2(r-1)(1-x)}+\frac{3r-8x+rx+4x^2}{2(r-1)^{3/2}(1-x)}\tan^{-1}\sqrt{r-1} & 0<x<1\\ -\frac{-3r^2+13rx-8x^2-10rx^2+8x^3}{2(r-1)(x-1)(r-4x+4x^2)}-\frac{-3r+8x-rx-4x^2}{2(r-1)^{3/2}(x-1)}\tan^{-1}\frac{\sqrt{r-1}}{2x-1} & x<0\end{cases}$$

$$f_{2,\not{p}}(x,r)=\frac{\alpha_s C_F}{2\pi}\begin{cases}\frac{3-3r-2x}{2(r-1)(x-1)}+\frac{4rx-8x^2+8x^3}{(r-4x+4x^2)^2}+\frac{2-2r-rx+2x^2}{(r-1)^{3/2}(x-1)}\tan^{-1}\frac{\sqrt{r-1}}{2x-1} & x>1\\ \frac{3-3r-2x+4x^2}{2(r-1)(1-x)}+\frac{-2+2r+rx-2x^2}{(r-1)^{3/2}(1-x)}\tan^{-1}\sqrt{r-1} & 0<x<1\\ -\frac{3-3r-2x}{2(r-1)(x-1)}-\frac{4rx-8x^2+8x^3}{(r-4x+4x^2)^2}-\frac{2-2r-rx+2x^2}{(r-1)^{3/2}(x-1)}\tan^{-1}\frac{\sqrt{r-1}}{2x-1} & x<0\end{cases}$$

[1] Y. S. Liu, J. W. Chen, L. Jin, H. W. Lin, Y. B. Yang, J. H. Zhang and Y. Zhao, (2018), arXiv:1807.06566 [hep-lat].

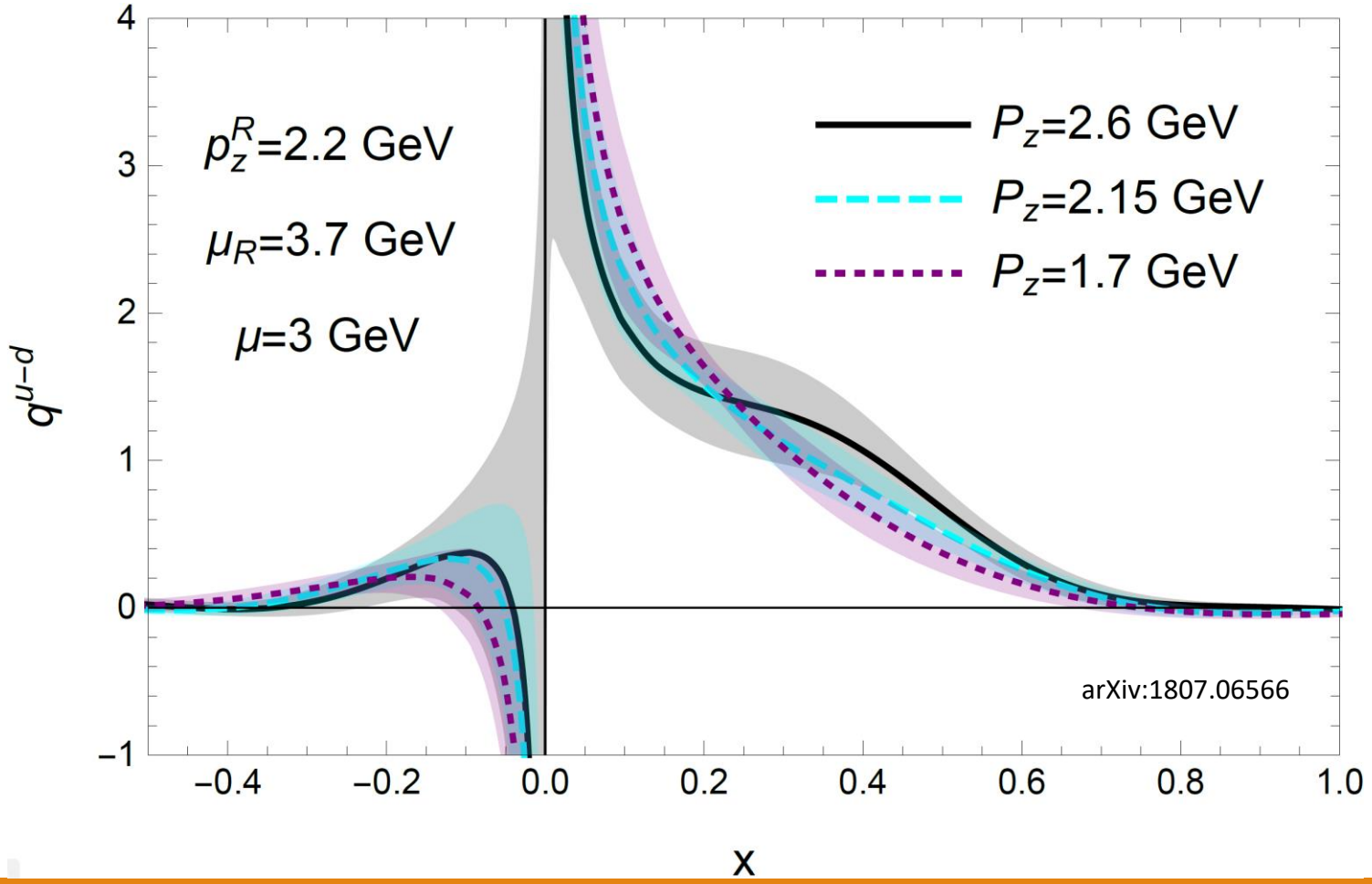
Matched PDF


P_z=2.6 GeV
p_z^R=2.2 GeV
μ_R=3.7 GeV
μ=3 GeV
quasi PDF
matched PDF
arXiv:1807.06566
q^{u-d}
x
4
3
2
1
0
−1
−0.4
−0.2
0.0
0.2
0.4
0.6
0.8
1.0

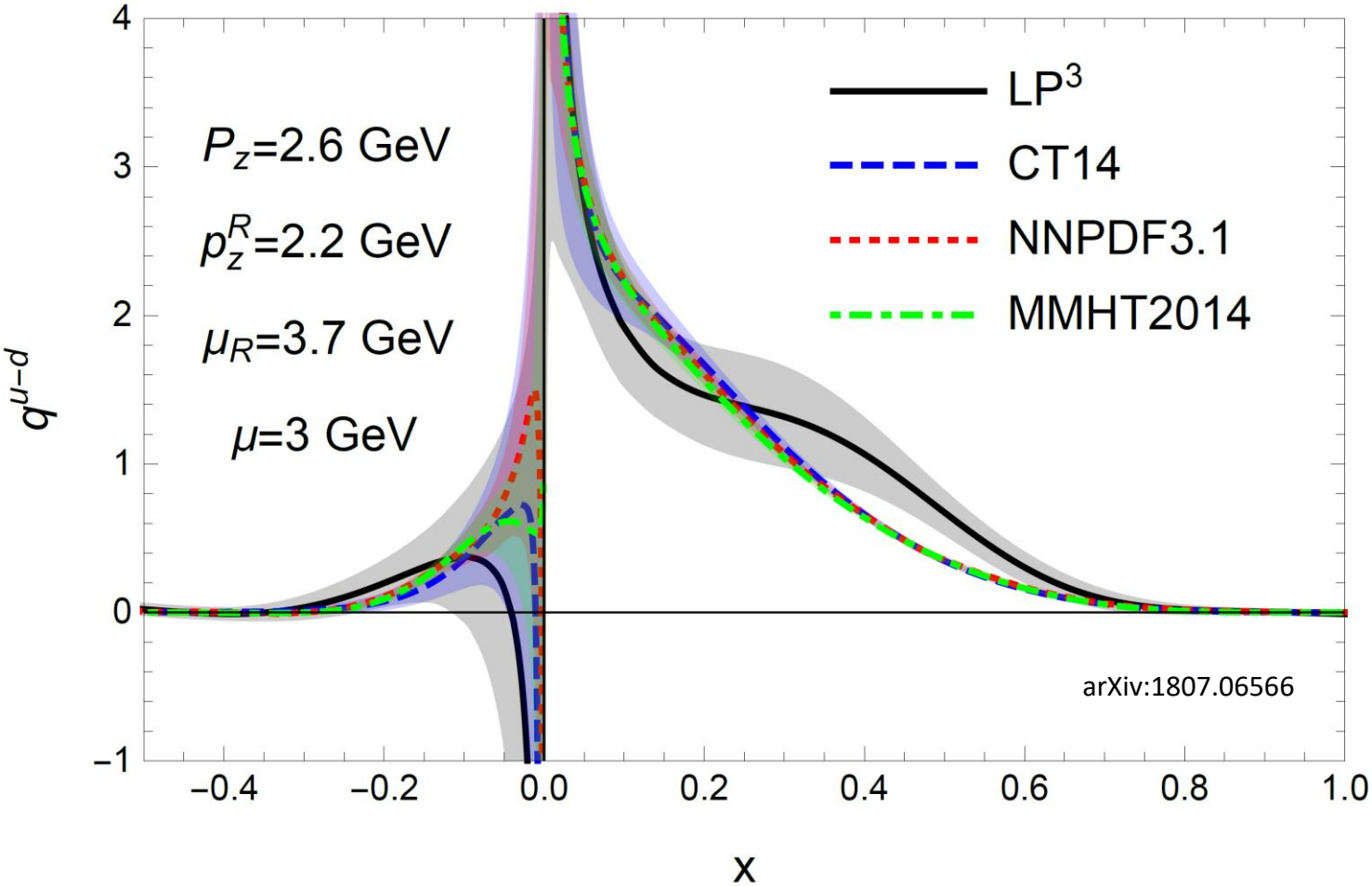
Error Analysis

- Statistical Error
- Excited State Contamination
- Mass correction
- Inverting the factorization formula
- Dependence of unphysical energy scale μ_R and p_z^R
  - Changing μ_R from 2.3 to 3.7 GeV
  - Varying p_z^R from 1.3 to 3 GeV
- Study different projections
- And more

Different Nucleon Momentum



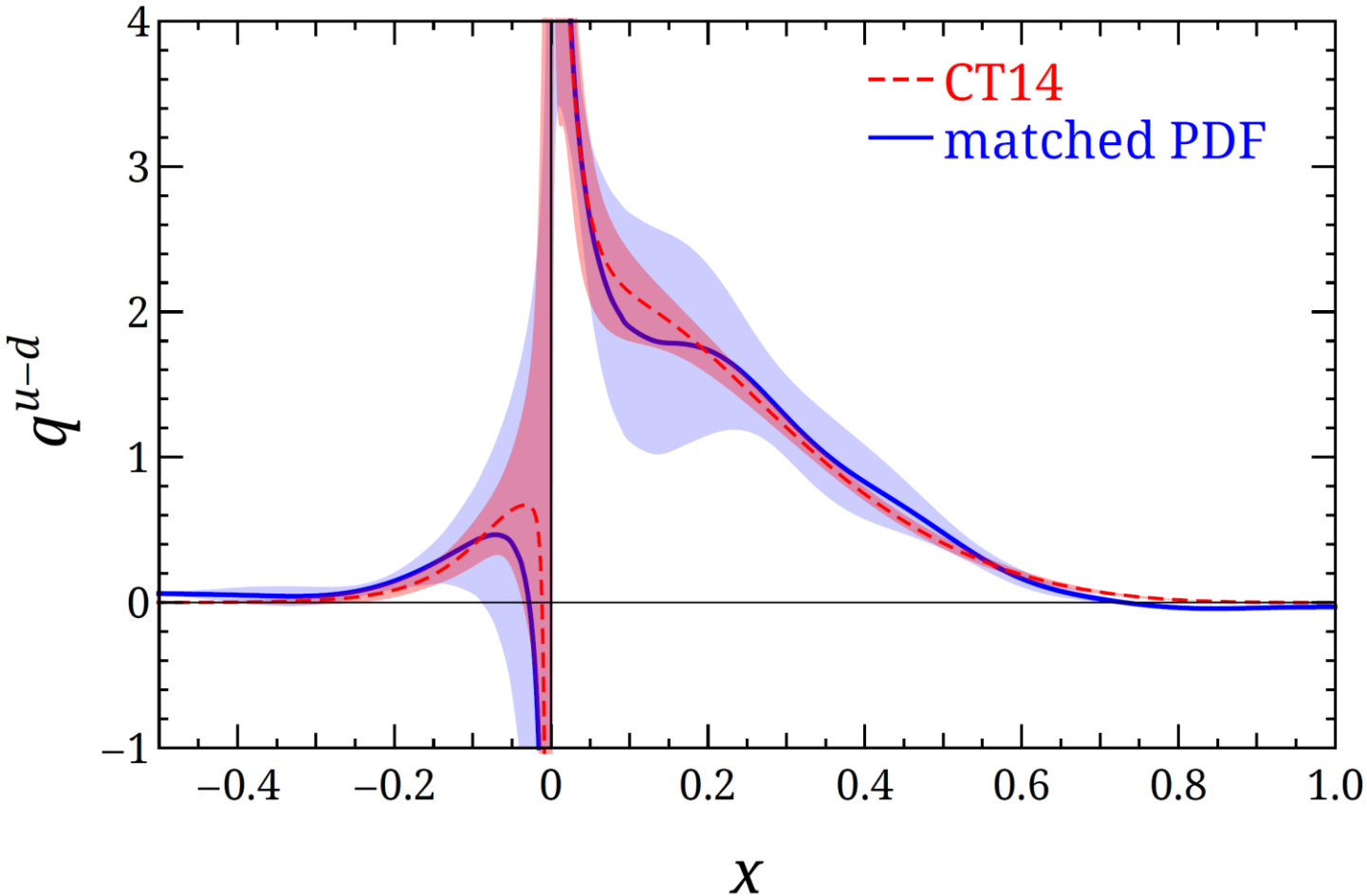
Comparing with Global-Fit



Summary and Outlook

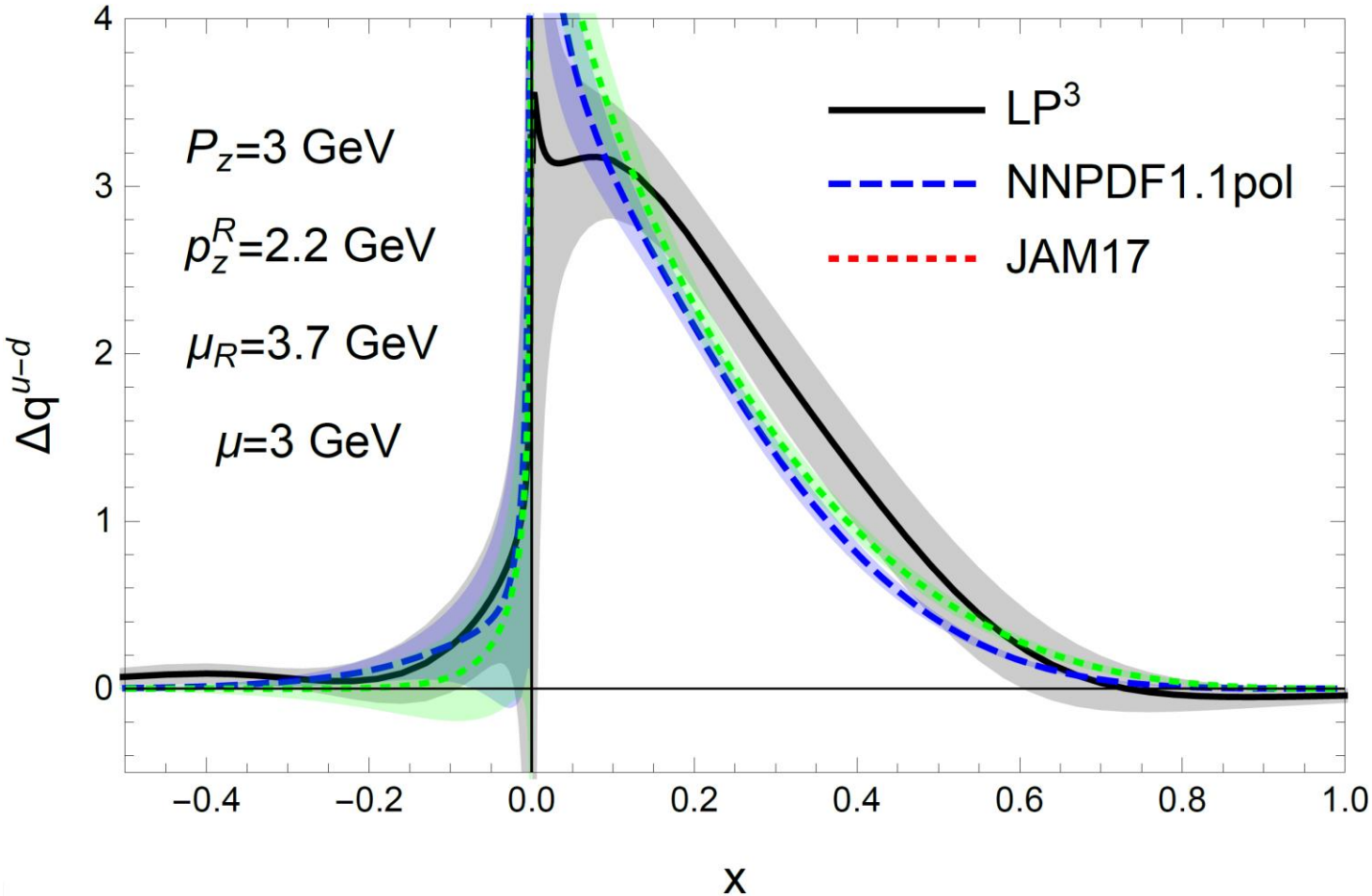
- A breakthrough has been made to directly access x-dependence of PDFs using lattice calculation.
- Studying parton physics using LaMET is a fast growing new field: lattice simulation, renormalization, matching coefficient calculation, more application of LaMET on partonic observable, etc.
- Future work:
  Finer Lattice Spacing (Higher Nucleon Momentum)
  Higher Order Loop Matching Kernel
  Other Physical Observables: DA, GPD, TMD, etc.

Unpol. PDF at physical m_π [1]



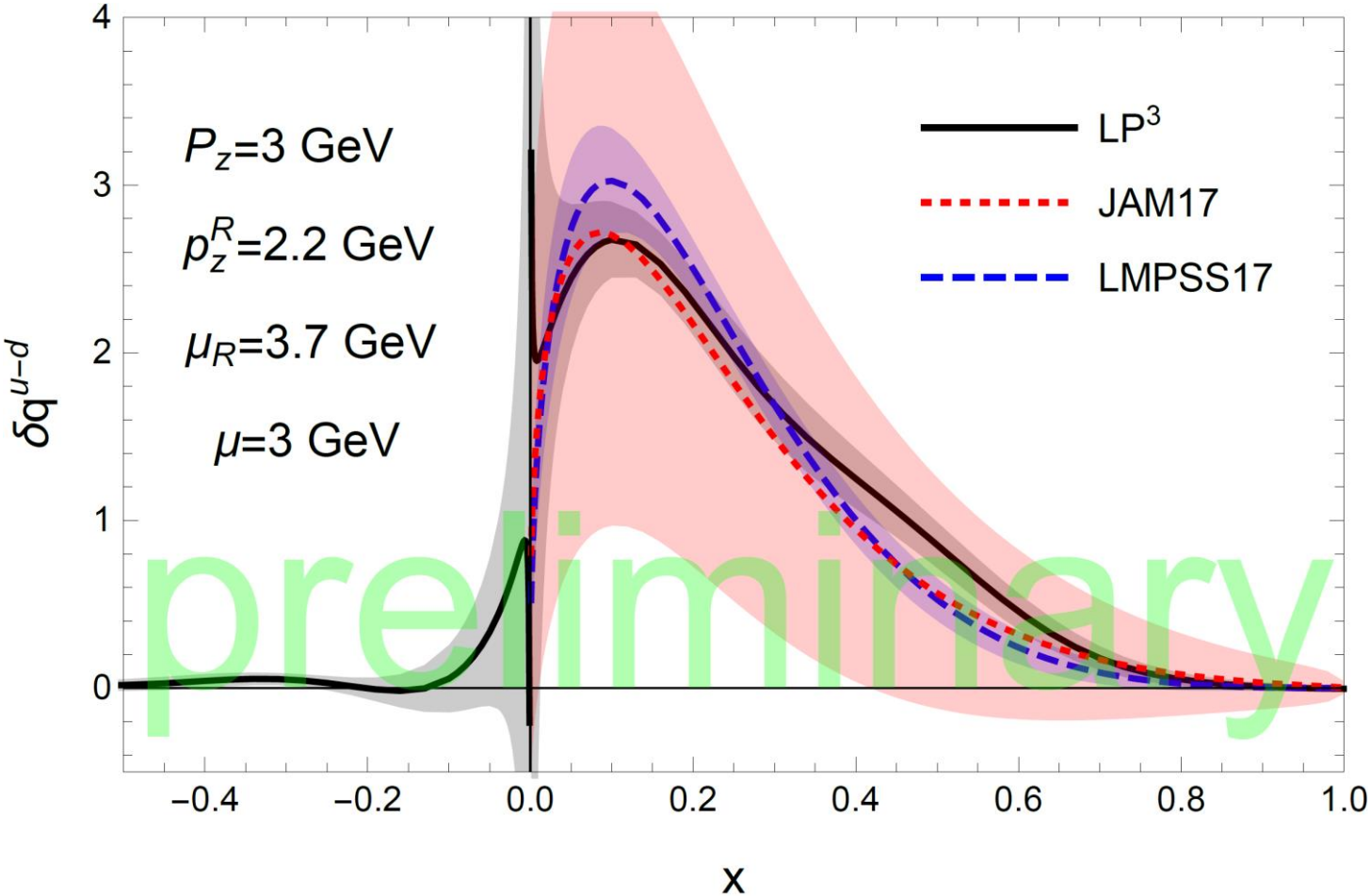
[1] J. W. Chen, L. Jin, H. W. Lin, Y. S. Liu, Y. B. Yang, J. H. Zhang and Y. Zhao, arXiv:1803.04393 [hep-lat].

Helicity at physical m_π [1]



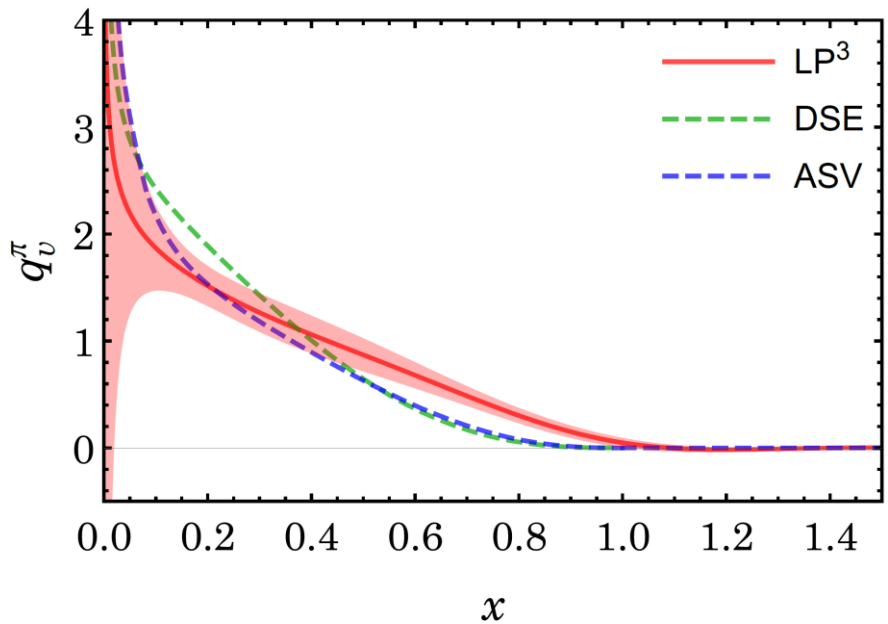
[1] H. W. Lin, J. W. Chen, L. Jin, Y. S. Liu, Y. B. Yang, J. H. Zhang and Y. Zhao, arXiv:1807.07431 [hep-lat].

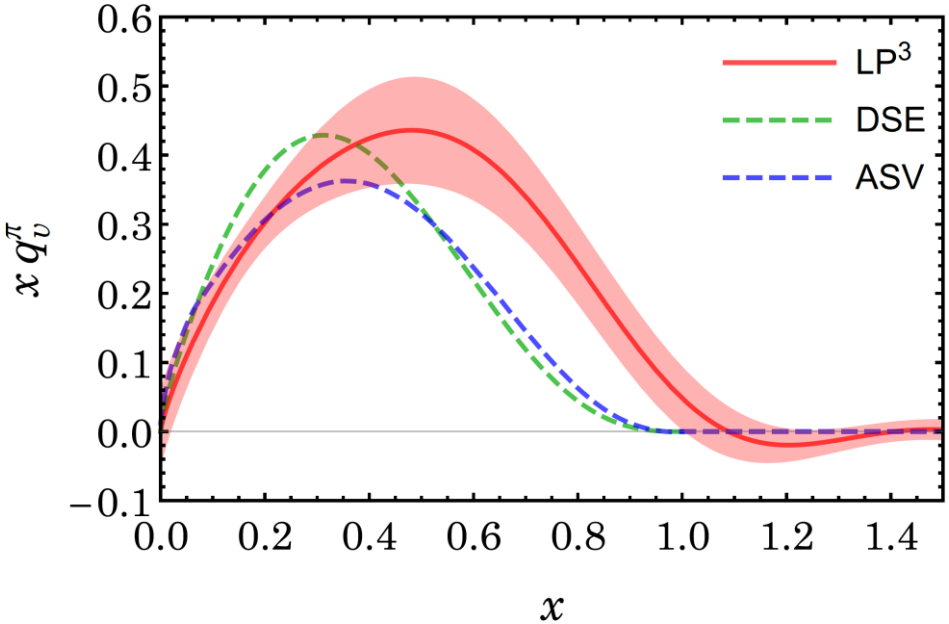
Transversity at physical m_π



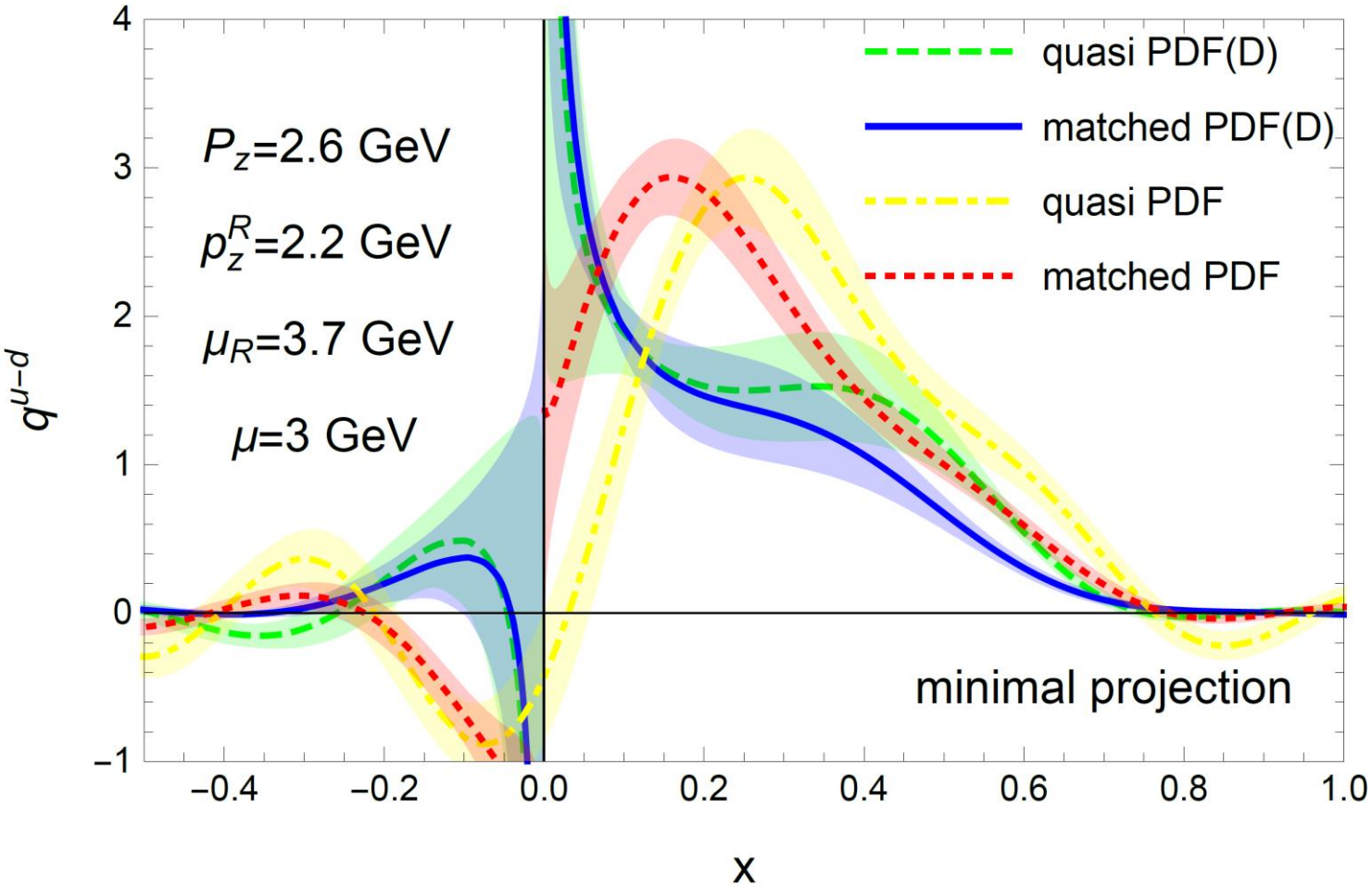
Pion valence quark PDF

- $m_\pi = 310$ MeV





Derivative Method



Projection

$$O_\Gamma(z) = \bar{\psi}(z)\Gamma U(z,0)\psi(0)$$

$$Z(z, p_z^R, a^{-1}, \mu_R) = \left.\frac{\langle p, s|O_{\gamma^t}(z)|p, s\rangle}{\langle p, s|O_{\gamma^t}(z)|p, s\rangle_{\text{tree}}}\right|_{\substack{p^2 = -\mu_R^2 \\ p_z = p_z^R}}$$

$$\langle p, s|O_{\gamma^t}(z)|p, s\rangle = \text{Tr}\left[\Lambda_{\gamma^t}(z,p)\mathcal{P}\right]$$

$$\Lambda_{\gamma^t}(p,z) = \tilde{F}_t(p,z)\gamma^t + \tilde{F}_z(p,z)\frac{p_t\gamma^z}{p_z} + \tilde{F}_p(p,z)\frac{p_t\not{p}}{p^2}$$

$$Z_{\not{p}}(z, p_z^R, a^{-1}, \mu_R) \equiv \left.\left[\tilde{F}_t(p,z) + \tilde{F}_z(p,z) + \tilde{F}_p(p,z)\right]\right|_{\substack{p^2 = -\mu_R^2 \\ p_z = p_z^R}}$$

$$Z_{mp}(z, p_z^R, a^{-1}, \mu_R) \equiv \tilde{F}_t(p,z)\Big|_{\substack{p^2 = -\mu_R^2 \\ p_z = p_z^R}}$$

Matching Coefficient

$$C\left(x, r, \frac{p_z}{\mu}, \frac{p_z}{p_z^R}\right) = \delta(1-x) + \left[\tilde{q}_B^{(1)}(x,\rho) - q^{(1)}(x,p,\mu) - \tilde{q}_{CT}^{(1)}\left(x, r, \frac{p_z}{p_z^R}\right)\right]\Bigg|_{proj} + \mathcal{O}(\alpha_s^2)$$

$$= \delta(1-x) + \left[f_{1,proj}\left(x, \frac{p_z}{\mu}\right) - \left|\frac{p_z}{p_z^R}\right| f_{2,proj}\left(1 + \frac{p_z}{p_z^R}(x-1), r\right)\right]_+ + \mathcal{O}(\alpha_s^2)$$

$$\tilde{q}_B^{(1)}(x,\rho) = \tilde{q}^{(1)}(x, (p_t \to p_z, \vec{p}_\perp, p_z), \rho \to 0)$$

$$\tilde{q}_B^{(1)}(x,\rho)\Big|_{proj} - q^{(1)}(x,p,\mu)\Big|_{proj} = f_{1,proj}\left(x, \frac{p_z}{\mu}\right)_+$$

$$\tilde{q}_{CT}^{(1)}\left(x, r, \frac{p_z}{p_z^R}\right)\Bigg|_{proj} = \left|\frac{p_z}{p_z^R}\right| \tilde{q}^{(1)}\left(1 + \frac{p_z}{p_z^R}(x-1), p_z = p_z^R, \rho = r\right)\Bigg|_{proj}$$

$$= \left|\frac{p_z}{p_z^R}\right| f_{2,proj}\left(1 + \frac{p_z}{p_z^R}(x-1), r\right)_+$$

Projections

- $\not{p}$ projection

$$\mathcal{P} = \not{p}/(4p^t)$$

$$\tilde{q}_B^{(1)}(x,\rho)\Big|_{\not{p}} = \left[\tilde{f}_t(x,\rho) + \tilde{f}_z(x,\rho) + \tilde{f}_p(x,\rho)\right]_+$$

$$q^{(1)}(x,p,\mu)\Big|_{\not{p}} = \left[f_+\left(x,\frac{\mu^2}{p^2}\right) + f_p(x)\right]_+$$

- Minimal projection

$$\tilde{q}_B^{(1)}(x,\rho)\Big|_{mp} = \left[\tilde{f}_t(x,\rho) + \tilde{f}_z(x,\rho)\right]_+\Bigg|_{\rho\to 0}$$

$$q^{(1)}(x,p,\mu)\Big|_{mp} = f_+\left(x,\frac{\mu^2}{p^2}\right)_+$$

PDF One-Loop

$$q^{(0)}(x) = \delta(1-x)$$

$$q^{(1)}(x,p,\mu) = \mathrm{Tr}\left[\left(\left[f_+\left(x,\frac{\mu^2}{p^2}\right)\right]_+ \gamma^+ + \left[f_p(x)\frac{p^+\not{p}}{p^2}\right]_+\right)\mathcal{P}\right]$$

$$f_+\left(x,\frac{\mu^2}{p^2}\right) = \frac{\alpha_s C_F}{2\pi}\left[\frac{-5+10x-6x^2}{2(1-x)} + \frac{1+x^2}{1-x}\ln\frac{\mu^2}{-x(1-x)p^2}\right]\theta(x)\theta(1-x)$$

$$f_p(x) = \frac{\alpha_s C_F}{2\pi}(1-2x)\theta(x)\theta(1-x)$$

Quasi-PDF One-Loop $\tilde{q}^{(0)}(x) = \delta(1-x)$

$$\tilde{q}^{(1)}(x,p,\rho) = \mathrm{Tr}\left[\left(\left[\tilde{f}_t(x,\rho)\right]_+ \gamma^t + \left[\tilde{f}_z(x,\rho)\right]_+ \frac{p_t}{p_z}\gamma^z + \left[\tilde{f}_p(x,\rho)\right]_+ \frac{p_t \not{p}}{p^2}\right)\mathcal{P}\right] \qquad \rho = \frac{-p^2 - i\epsilon}{p_z^2}$$

$$\tilde{f}_t(x,\rho) = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{8x^2(1-x)-x\rho(13-10x)+3\rho^2}{2(1-x)(1-\rho)(4x-4x^2-\rho)} + \frac{16x^2(2-3x+x^2)-4x\rho(5-3x-x^2)+\rho^2(3+x)}{4(1-x)(1-\rho)^{3/2}(4x-4x^2-\rho)}\ln\frac{2x-1+\sqrt{1-\rho}}{2x-1-\sqrt{1-\rho}} & x>1 \\ \frac{x(-7+4x)+3\rho}{2(1-x)(1-\rho)} + \frac{4x(2-x)-\rho(3+x)}{4(1-x)(1-\rho)^{3/2}}\ln\frac{1+\sqrt{1-\rho}}{1-\sqrt{1-\rho}} & 0<x<1 \\ -\frac{8x^2(1-x)-x\rho(13-10x)+3\rho^2}{2(1-x)(1-\rho)(4x-4x^2-\rho)} - \frac{16x^2(2-3x+x^2)-4x\rho(5-3x-x^2)+\rho^2(3+x)}{4(1-x)(1-\rho)^{3/2}(4x-4x^2-\rho)}\ln\frac{2x-1+\sqrt{1-\rho}}{2x-1-\sqrt{1-\rho}} & x<0 \end{cases}$$

$$\tilde{f}_z(x,\rho) = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{-32x^2(1-x)^2(2x-1)-4x\rho(8-43x+65x^2-38x^3+8x^4)+\rho^2(5-41x+42x^2-8x^3)+2\rho^3(2-x)}{2(1-x)(1-\rho)^2(4x-4x^2-\rho)^2} \\ \quad + \frac{4-8x+8x^2+\rho(3-13x+4x^2)+2\rho^2}{4(1-x)(1-\rho)^{5/2}}\ln\frac{2x-1+\sqrt{1-\rho}}{2x-1-\sqrt{1-\rho}} & x>1 \\ \frac{-5+15x-12x^2-2\rho(2-3x)}{2(1-x)(1-\rho)^2} + \frac{4-8x+8x^2+\rho(3-13x+4x^2)+2\rho^2}{4(1-x)(1-\rho)^{5/2}}\ln\frac{1+\sqrt{1-\rho}}{1-\sqrt{1-\rho}} & 0<x<1 \\ -\frac{-32x^2(1-x)^2(2x-1)-4x\rho(8-43x+65x^2-38x^3+8x^4)+\rho^2(5-41x+42x^2-8x^3)+2\rho^3(2-x)}{2(1-x)(1-\rho)^2(4x-4x^2-\rho)^2} \\ \quad - \frac{4-8x+8x^2+\rho(3-13x+4x^2)+2\rho^2}{4(1-x)(1-\rho)^{5/2}}\ln\frac{2x-1+\sqrt{1-\rho}}{2x-1-\sqrt{1-\rho}} & x<0 \end{cases}$$

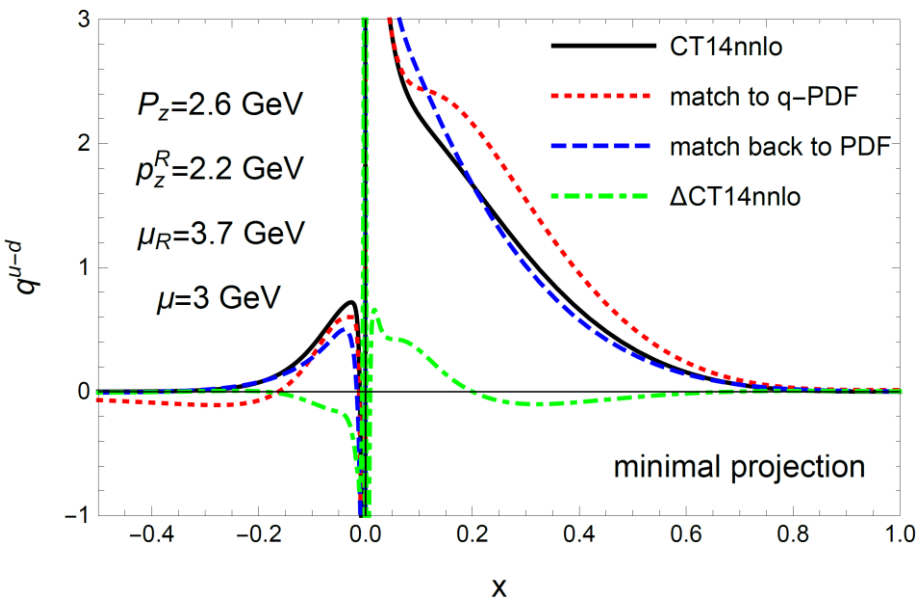
$$\tilde{f}_p(x,\rho) = \frac{\alpha_s C_F}{2\pi}\begin{cases} \frac{16x\rho(1-x)^2(1-6x)-2\rho^2(1-22x+26x^2-4x^3)-\rho^3(7-6x)}{2(1-\rho)^2(4x-4x^2-\rho)^2} + \frac{-\rho(8-12x+\rho)}{4(1-\rho)^{5/2}}\ln\frac{2x-1+\sqrt{1-\rho}}{2x-1-\sqrt{1-\rho}} & x>1 \\ \frac{2-4x+\rho(7-8x)}{2(1-\rho)^2} + \frac{-\rho(8-12x+\rho)}{4(1-\rho)^{5/2}}\ln\frac{1+\sqrt{1-\rho}}{1-\sqrt{1-\rho}} & 0<x<1 \\ -\frac{16x\rho(1-x)^2(1-6x)-2\rho^2(1-22x+26x^2-4x^3)-\rho^3(7-6x)}{2(1-\rho)^2(4x-4x^2-\rho)^2} - \frac{-\rho(8-12x+\rho)}{4(1-\rho)^{5/2}}\ln\frac{2x-1+\sqrt{1-\rho}}{2x-1-\sqrt{1-\rho}} & x<0 \end{cases}$$

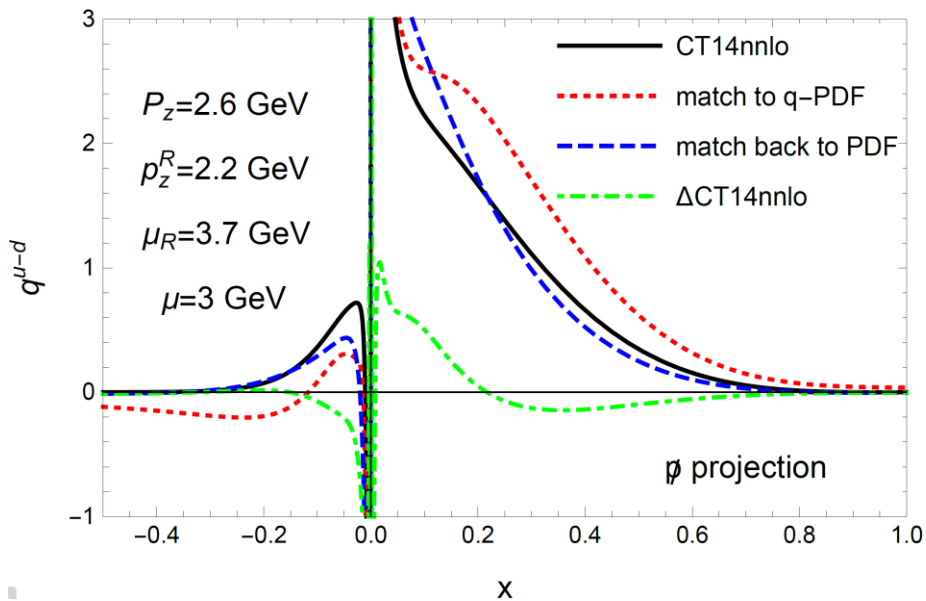
Inversion of Factorization

- Simplest way to invert the factorization formula

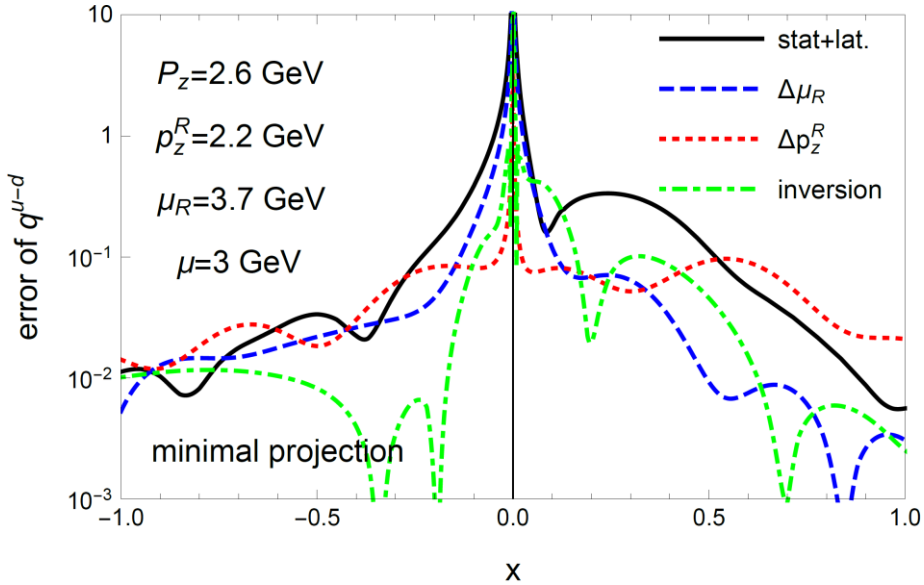
$$q(x,\mu) = \int_{-\infty}^{\infty} \frac{dy}{|y|}\, C'\left(\frac{x}{y}, r, \frac{yP_z}{\mu}, \frac{yP_z}{p_z^R}\right) \tilde{q}(y, P_z, p_z^R, \mu_R) + \mathcal{O}\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\rm QCD}^2}{P_z^2}, \alpha_s^2\right)$$

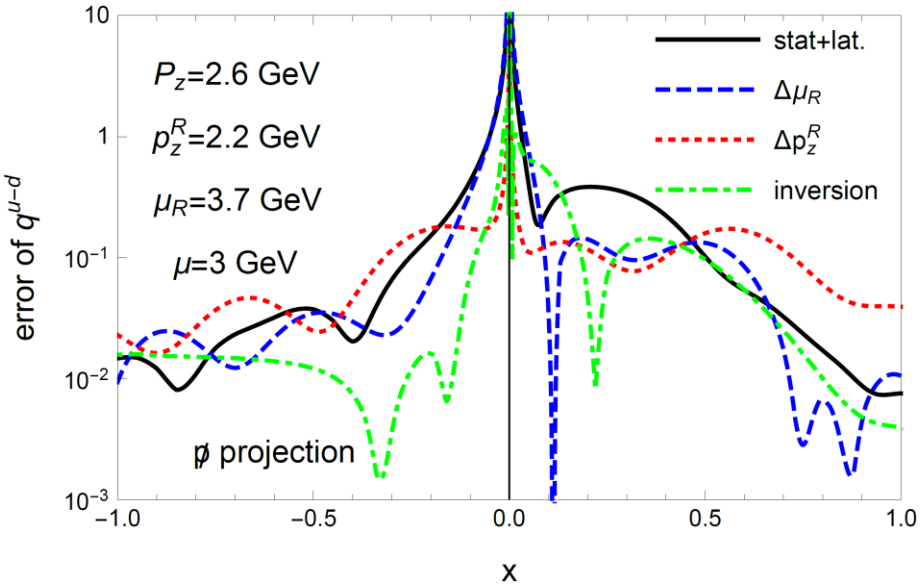
$$C' = C(\alpha_s \to -\alpha_s)$$





Errors


P_z=2.6 GeV
p_z^R=2.2 GeV
μ_R=3.7 GeV
μ=3 GeV
minimal projection
stat+lat.
$\Delta\mu_R$
Δp_z^R
inversion
error of $q^{\mu-d}$
x


P_z=2.6 GeV
p_z^R=2.2 GeV
μ_R=3.7 GeV
μ=3 GeV
$\not{p}$ projection
stat+lat.
$\Delta\mu_R$
Δp_z^R
inversion
error of $q^{\mu-d}$
x

Rossi and Testa I [1]

In eqs. (4) and (5) the bilocal operator can be Taylor expanded around $\xi = 0$, yielding

$$\langle P|\phi(0)\phi(\xi)|P\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \langle P|\phi(0) \frac{\partial}{\partial \xi^{\mu_1}} \frac{\partial}{\partial \xi^{\mu_2}} \cdots \frac{\partial}{\partial \xi^{\mu_n}} \phi(\xi)\Big|_{\xi=0} |P\rangle \xi^{\mu_1} \xi^{\mu_2} \ldots \xi^{\mu_n} \equiv$$

$$\equiv \sum_{n=0}^{\infty} \langle P|O_{\mu_1\mu_2\ldots\mu_n}(0)|P\rangle \xi^{\mu_1} \xi^{\mu_2} \ldots \xi^{\mu_n} \,. \tag{8}$$

The matrix elements of $O_{\mu_1\mu_2\ldots\mu_n}(0)$ are of the form

$$\langle P|O_{\mu_1\mu_2\ldots\mu_n}(0)|P\rangle = A_n P_{\mu_1} P_{\mu_2} \ldots P_{\mu_n} + traces \,, \tag{9}$$

where *traces* denote form factors containing some $g_{\mu_i\mu_j}$ tensor. For example, in the case of $O_{\mu_1\mu_2}(0)$, we have

$$\langle P|O_{\mu_1\mu_2}(0)|P\rangle = A_2 P_{\mu_1} P_{\mu_2} + B_2 g_{\mu_1\mu_2} \,. \tag{10}$$

[1] G. Rossi and M. Testa, (2018), arXiv:1806.00808 [hep-lat].

Rossi and Testa II [1]

where Λ is an UV cutoff. Renormalization is carried out by means of the so-called "matching procedure" which consists in writing

$$\tilde{F}(x, P_z; \Lambda) = \int_x^{+\infty} \frac{dx'}{x'} Z(\frac{x}{x'}; \Lambda, \mu) F(x', P_z; \mu) \,, \tag{14}$$

where $Z(\frac{x}{x'}; \Lambda, \mu)$ is a logarithmically divergent renormalization function (computed in perturbation theory) which is needed to make $F(x, P_z; \mu)$ UV finite.

We observe that the convolution property of the Mellin transform implies

$$\int_{-\infty}^{+\infty} dx \tilde{F}(x, P_z; \Lambda) x^n = \int_{-\infty}^{+\infty} dx' x'^{\,n} Z(x'; \Lambda, \mu) \int_{-\infty}^{+\infty} dx\, x^n F(x, P_z; \mu) \equiv$$
$$\equiv Z_n \left(\frac{\Lambda}{\mu}\right) \int_{-\infty}^{+\infty} dx\, x^n F(x, P_z; \mu) \,, \tag{15}$$

showing that moments of $\tilde{F}$ renormalizes multiplicatively and independently one from the others.

[1] G. Rossi and M. Testa, (2018), arXiv:1806.00808 [hep-lat].

Lattice Parton Physics Project-$\mathbf{LP}^3$

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