A tensorial toolkit for quantum computing in lattice gauge theory

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Main ideas

- In most lattice simulations, the variables of integration are compact and character expansions (such as Fourier series) can be used to rewrite the partition function and average observables as discrete sums of contracted tensors.
- Example: the $O(2)$ model $\mathrm{e}^{\beta\cos(\theta_i-\theta_j)} = \sum_{n_{ij}=-\infty}^{+\infty} \mathrm{e}^{in_{ij}(\theta_i-\theta_j)} I_{n_{ij}}(\beta)$
- This reformulations have been used for RG blocking but they are also suitable for quantum computations/simulations when combined with truncations.
- We discuss FAQ about tensorial formulations:
  - effects of truncation on global symmetries
  - boundary conditions
  - Grassmann tensors

Tensor Renormalization Group (TRG)

- TRG: first implementation of Wilson program for lattice models without uncontrollable approximations
- Leads to universal fixed point equations but truncation methods need to be optimized
- Models we considered: Ising model, $O(2)$, $O(3)$, principal chiral models, gauge models (Ising, $U(1)$ and $SU(2)$))
- Quantum simulators, measurements of entanglement entropy, central charge ..
- Our group: PRB 87 064422 (2013), PRD 88 056005 (2013), PRD 89 016008 (2014), PRA90 063603 (2014), PRD 92 076003 (2015), PRE 93 012138 (2016) , PRA 96 023603 (2017), PRD 96 034514 (2017), arXiv:1803.11166, arXiv:1807.09186.
- Basic references for tensor methods for Lagrangian models: Levin and Nave, PRL 99 120601 (2007), Z.C. Gu et al. PRB 79 085118 (2009), Z. Y. Xie et al., PRB 86 045139 (2012)
- Schwinger model/fermions/CP(N): Yuya Shimizu, Yoshinobu Kuramashi; Ryo Sakai, Shinji Takeda; Hikaru Kawauchi.

TRG blocking: it's simple and exact!

2D Ising model: for each link we use the character expansion

$$\exp(\beta\sigma_1\sigma_2) = \cosh(\beta)(1 + \sqrt{\tanh(\beta)}\sigma_1\sqrt{\tanh(\beta)}\sigma_2) =$$
$$cosh(\beta)\sum_{n_{12}=0,1}(\sqrt{\tanh(\beta)}\sigma_1\sqrt{\tanh(\beta)}\sigma_2)^{n_{12}}.$$

Regroup the four terms involving a given spin σ_i and sum over its two values ± 1. The results can be expressed in terms of a tensor: $T^{(i)}_{xx'yy'}$ which can be visualized as a cross attached to the site i with the four legs covering half of the four links attached to i. The horizontal indices $x,\ x'$ and vertical indices $y,\ y'$ take the values 0 and 1 as the index n_{12}.

$$T^{(i)}_{xx'yy'} = f_x f_{x'} f_y f_{y'} \delta\left(\text{mod}[x + x' + y + y', 2]\right)\ ,$$

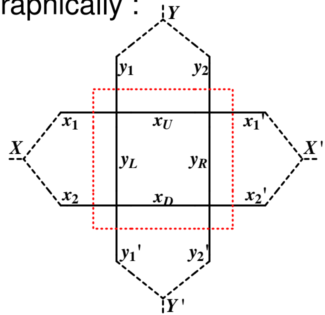
where $f_0 = 1$ and $f_1 = \sqrt{\tanh(\beta)}$. The delta symbol is 1 if $x + x' + y + y'$ is zero modulo 2 and zero otherwise.

TRG blocking (graphically)

Exact form of the partition function:

$$Z = 2^V (\cosh(\beta))^{2V} \mathrm{Tr} \prod_i T^{(i)}_{xx'yy'}$$

Tr mean contractions (sums over 0 and 1) over the links.
Reproduces the closed paths ("worms") of the HT expansion.
TRG blocking separates the degrees of freedom inside the block which are integrated over, from those kept to communicate with the neighboring blocks. Graphically :



TRG Blocking (formulas)

Blocking defines a new rank-4 tensor $T'_{XX'YY'}$ where each index now takes four values.

$$T'_{X(x_1,x_2)X'(x'_1,x'_2)Y(y_1,y_2)Y'(y'_1,y'_2)} = \sum_{x_U,x_D,x_R,x_L} T_{x_1 x_U y_1 y_L} T_{x_U x'_1 y_2 y_R} T_{x_D x'_2 y_R y'_2} T_{x_2 x_D y_L y'_1} \ ,$$

where $X(x_2, x_2)$ is a notation for the product states e. g. , $X(0,0) = 1,\ X(1,1) = 2,\ X(1,0) = 3,\ X(0,1) = 4$. The partition function can be written exactly as

$$Z = 2^V (\cosh(\beta))^{2V} \mathrm{Tr} \prod_{2i} T'^{(2i)}_{XX'YY'} \ ,$$

where $2i$ denotes the sites of the coarser lattice with twice the lattice spacing of the original lattice. Using a truncation in the number of “states" carried by the indices and rescaling all the tensors in such a way that one of them stays equal to 1, we can write a fixed point equation.

$O(2)$ model

$$Z = \int \prod_i \frac{d\theta_i}{2\pi} \mathrm{e}^{\beta \sum_{<ij>} \cos(\theta_i - \theta_j)} .$$

$$\mathrm{e}^{\beta \cos(\theta_i - \theta_j)} = \sum_{n_{ij}=-\infty}^{+\infty} \mathrm{e}^{i n_{ij}(\theta_i - \theta_j)} I_{n_{ij}}(\beta) ,$$

where the I_n are the modified Bessel functions of the first kind. In two dimensions, we obtain the factorizable expression:

$$T^i_{n_{ix}, n_{ix'}, n_{iy}, n_{iy'}} = \sqrt{I_{n_{ix}}(\beta)} \sqrt{I_{n_{iy}}(\beta)} \sqrt{I_{n_{ix'}}(\beta)} \sqrt{I_{n_{iy'}}(\beta)} \delta_{n_{ix}+n_{iy}, n_{ix'}+n_{iy'}} .$$

The partition function and the blocking of the tensor are similar to the Ising model but the sums run over all integers.
The $I_n(\beta)$ decay rapidly for large n and fixed β (namely like $1/n!$).
$\delta_{n_{ix}+n_{iy}, n_{ix'}+n_{iy'}}$ encodes the $O(2)$ symmetry.
The generalization to higher dimensions is straightforward.

TRG approach of the transfer matrix

The partition function can be expressed in terms of a transfer matrix:

$$Z = \mathrm{Tr}\, \mathbb{T}^{L_t} \ .$$

The matrix elements of $\mathbb{T}$ can be expressed as a product of tensors associated with the sites of a time slice (fixed t) and traced over the space indices (PhysRevA.90.063603)

$$\mathbb{T}_{(n_1,n_2,\dots n_{L_x})(n'_1,n'_2\dots n'_{L_x})} = \sum_{\tilde{n}_1 \tilde{n}_2 \dots \tilde{n}_{L_x}} T^{(1,t)}_{\tilde{n}_{L_x} \tilde{n}_1 n_1 n'_1} T^{(2,t)}_{\tilde{n}_1 \tilde{n}_2 n_2 n'_2 \dots} \dots T^{(L_x,t)}_{\tilde{n}_{L_{x-1}} \tilde{n}_{L_x} n_{L_x} n'_{L_x}}$$

with (for the $O(2)$ model with chemical potential)

$$T^{(x,t)}_{\tilde{n}_{x-1} \tilde{n}_x n_x n'_x} = \sqrt{I_{n_x}(\beta_\tau) I_{n'_x}(\beta_\tau) I_{\tilde{n}_{x-1}}(\beta_s) I_{\tilde{n}_x}(\beta_s) \mathrm{e}^{(\mu(n_x+n'_x))}}\, \delta_{\tilde{n}_{x-1}+n_x, \tilde{n}_x+n'_x}$$

The Kronecker delta function reflects the existence of a conserved current, a good quantum number (“particle number").

Transfer matrix for $O(2)$ with chemical potential

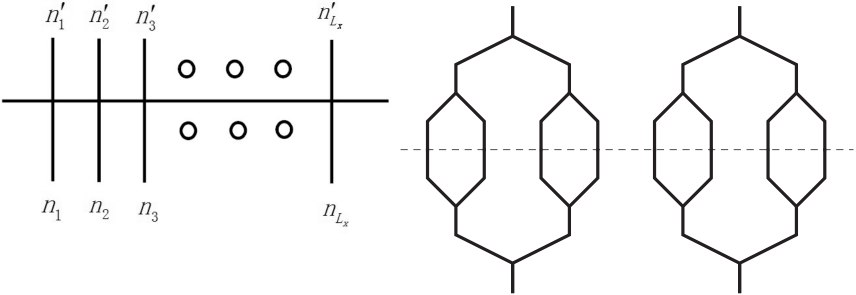
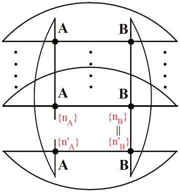


Figure: Graphical representation of the transfer matrix (left) and its successive coarse graining (right). See PRD 88 056005 and PRA 90, 063603 for explicit formulas.

Entanglement entropy S_E (PRE 93, 012138 (2016))

We consider the subdivision of AB into A and B (two halves in our calculation) as a subdivision of the spatial indices $\hat{\rho}_A \equiv \mathrm{Tr}_B \hat{\rho}_{AB}$; We use blocking methods until A and B are each reduced to a single site. In 1+1 dimensions:



The n -th order Rényi entanglement entropy is defined as:

$$S_n(A) \equiv \frac{1}{1-n} \ln(\mathrm{Tr}\,((\hat{\rho}_A)^n)) \ .$$

Quantum simulations with cold atoms: PRA 96 023603 (2017), Calabrese-Cardy scaling: PRD 96 034514 (2017)

Motivations for quantum simulations in lattice gauge theory and high energy physics

- Lattice QCD has been very successful at establishing that QCD is the theory of strong interactions, however some aspects remain inaccessible to classical computing.
- Some finite density calculations have a sign problem
- Real time evolution requires detailed information not available from conventional MC simulations at Euclidean time.
- Ambitious goal: collider jet physics from first principles
- Quantum simulations with optical lattices were successful in Condensed Matter (Bose-Hubbard), but so far no actual implementations for lattice gauge theory

Recent results (see J. Unmuth-Yockey talk + arxiv 1803.11166 and 1807.09186)

- We have reformulated the lattice Abelian Higgs model (scalar QED) in 1 space + 1 time dimension using the Tensor Renormalization Group method.
- The reformulation is gauge invariant and connects smoothly the classical Lagrangian formulation used by lattice gauge theorists and the quantum Hamiltonian method used in condensed matter.
- Calculations of the Polyakov loop show a remarkable data collapse that survives the time continuum limit. This can be tested with small volumes.
- We propose to use Bose-Hubbard (BH) Hamiltonians on a ladder as quantum simulators for the Abelian Higgs model which are being investigated by Johannes Zeiher in Immanuel Bloch's lab (MPQ, Garching).

Universal functions: the Polyakov loop

arXiv:1803.11166

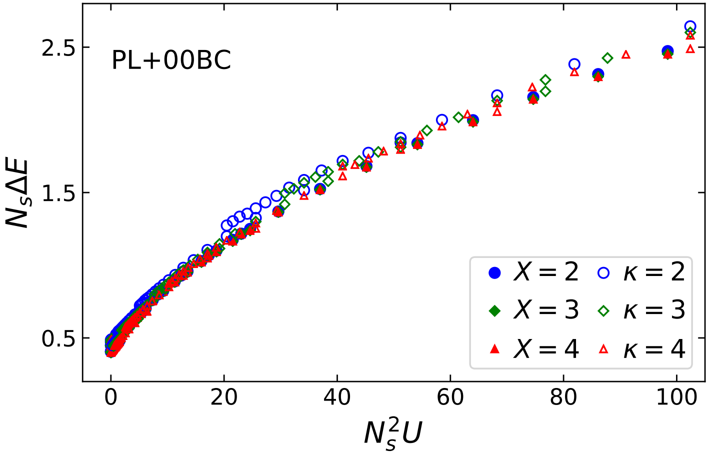


Figure: Data collapse of $N_s\Delta E$ defined from the insertion of the Polyakov loop, as a function of N_s^2U, or $(N_sg)^2$ (collapse of 24 datasets). Numerical work by Judah Unmuth-Yockey and Jin Zhang.

A first quantum calculator for the abelian Higgs model?

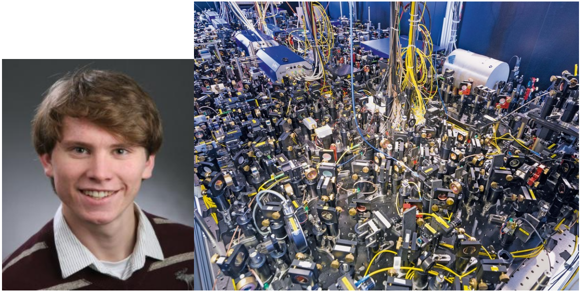


Figure: Left: Johannes Zeiher, a recent graduate from Immanuel Bloch's group can design ladder shaped optical lattices with nearest neighbor interactions. Right: an optical lattice experiment of Bloch's group.

Optical lattice implementation with a ladder

$$\bar{H} = \frac{\tilde{U}_g}{2}\sum_i \left(\bar{L}^z_{(i)}\right)^2 + \frac{\tilde{Y}}{2}\sum_i (\bar{L}^z_{(i)} - \bar{L}^z_{(i+1)})^2 - \tilde{X}\sum_i \bar{L}^x_{(i)}$$

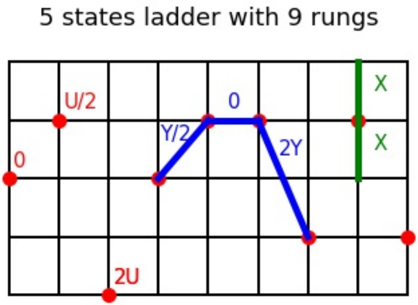
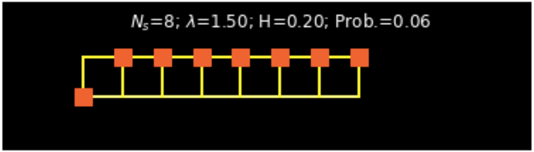
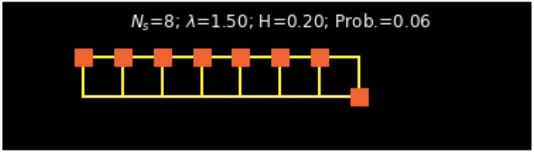
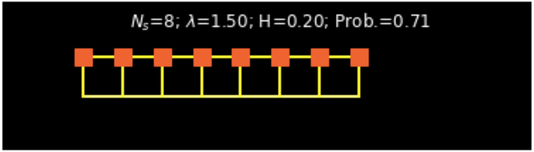


Figure: Ladder with one atom per rung: tunneling along the vertical direction, no tunneling in the the horizontal direction but short range attractive interactions. A parabolic potential is applied in the spin (vertical) direction.

Quantum Ising model (2 legs): Looking at the vacuum wavefunction: σ^z meas.







FAQ: Do truncation break global symmetries? No

For the $O(2)$ model, the action and the measure of integration are invariant under the global symmetry $\theta_{\mathbf{x}} \rightarrow \theta_{\mathbf{x}} + \Delta$, this implies

$$\langle f(\theta_{\mathbf{x}_1}, \dots, \theta_{\mathbf{x}_N}) \rangle = \langle f(\theta_{\mathbf{x}_1} + \Delta, \dots, \theta_{\mathbf{x}_N} + \Delta) \rangle$$

f is 2π-periodic and can be expressed in terms of Fourier modes

$$\langle \exp(i(n_1\theta_{\mathbf{x}_1} + \dots n_N\theta_{\mathbf{x}_N})) \rangle = \exp((n_1 + \dots n_N)\Delta) \langle \exp(i(n_1\theta_{\mathbf{x}_1} + \dots n_N\theta_{\mathbf{x}_N})) \rangle$$

If $\sum_{n=1}^{N} n_i \neq 0$ then $\langle \exp(i(n_1\theta_{\mathbf{x}_1} + \dots + n_N\theta_{\mathbf{x}_N})) \rangle = 0$

This selection rule is due to the quantum number selection rules at the sites and is independent of the particular values taken by the tensors. So if we set some of the tensor elements to zero as we do in a truncation, this does not affect the selection rule.

Boundary conditions

- Periodic boundary conditions (PBC) allow us to keep a discrete translational invariance. As a consequence the tensors themselves are translation invariant and assembled in the same way at every site, link etc.
- Open boundary conditions (OBC) can be implemented by introducing new tensors that can be placed at the boundary. The only difference is that there are missing links at sites or missing plaquettes a links. In all the examples we know, it is possible normalize the tensors in such a way that the missing elements can be taken into account by setting the corresponding indices to zero.
- It is also possible to define new boundary conditions that only make sense in the reformulation, for instance fixing some of the indices corresponding to missing elements to values different to zero, or summing over these.

PBC versus OBC with tensors (2D Ising)

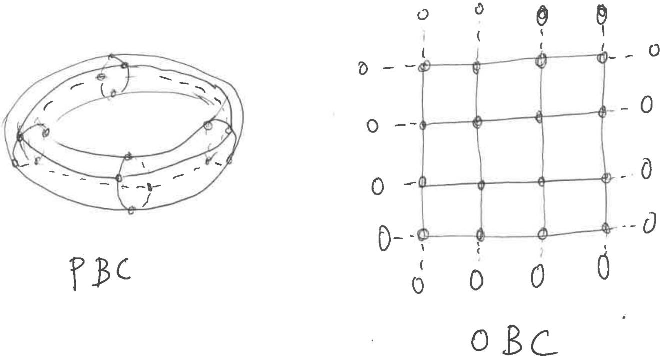


Figure: Assembling the translation invariant tensor with periodic BC (left), or using new tensors at the boundary for open BC (right).

Grassmann formulation (in progress)

The local tensor can be expressed as a function of four Grassmann variables attached to each site: $\eta_L,\ \eta_A,\ \eta_B\ \eta_R$. L means Left and R Right. LABR ordering present at each site when products are taken.

$$T(\eta) \equiv \sum_{n_L n_R n_A n_B = 0,1} T_{n_L n_R n_A n_B} \eta_L^{n_L} \eta_A^{n_A} \eta_B^{n_B} \eta_R^{n_R}$$

(η) is a short notation for $(\eta_L,\ \eta_A,\ \eta_B\ ,\eta_R)$
Because of the factorization properties ($\mathbb{Z}_2$ is abelian):

$$T_{1111} = T_{1100} \times T_{0011} = T_{0101} \times T_{1010} = T_{1001} \times T_{0110}$$

the tensor function exponentiates

$$T(\eta) = \mathrm{e}^{T_2(\eta)}$$

$$T_2(\eta) = t_s \eta_L \eta_R + t_t \eta_A \eta_B + \sqrt{t_s t_t}(\eta_L \eta_A + \eta_L \eta_B + \eta_A \eta_R + \eta_B \eta_R)$$

Grassmann form of the transfer matrix

The conventional transfer matrix element are all positive and should be read with a specific ordering of the Grassmann variables: first, left to right according to position $1,\ 2,\ldots,N_s$ and then A, B.

$$\mathbb{T}(\{\eta\}) \equiv \sum_{\{n\}} \mathbb{T}_{(n_{A1},\ldots,n_{AN_s},n_{B1},\ldots,n_{BN_s})} \eta_{A1}^{n_{A1}} \eta_{B1}^{n_{B1}} \ldots \eta_{AN_s}^{n_{AN_s}} \eta_{BN_s}^{n_{BN_s}}$$

where $\{n\}$ is a short notation for the summations
$\{n_{A1},\ldots,n_{AN_s},n_{B1},\ldots,n_{BN_s} = 0,1\}$
This ordering is achieved by inserting $\mathrm{e}^{+\eta_{R1}\eta_{L2}}$ between $T(\eta_1)$ and $T(\eta_2)$ and integrating with $\int d\eta_{L2} d\eta_{R1}$ etc. We use periodic boundary conditions in the σ formulation and the summation between the last site and the first site requires $\mathrm{e}^{-\eta_{RN_s}\eta_{L1}}$ (antiperiodic in the Grassmann formulation).
This Gaussian integration can be performed exactly, can we reproduce Kaufman result (eigenvalues of the transfer matrix at finite volume)?

Conclusions

- We have proposed a gauge-invariant approach for the quantum simulation of the abelian Higgs model.
- The tensor renormalization group approach provides a discrete formulation in the limit $\lambda \to \infty$ (no Higgs mode) suitable for quantum computing
- Calculations of the Polyakov loop at finite N_x and small gauge coupling show a universal behavior (collapse related to the KT transition of the limiting $O(2)$ model)
- A ladder of cold atoms with N_s rungs, one atom per rung, and $2s + 1$ long sides seems to be the most promising realization
- Proof of principle: data collapse for the quantum Ising model.
- Thanks for listening!

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The Hamiltonian (time continuum limit)

- For $1 << \beta_{pl} << \kappa_\tau$, we obtain the time continuum limit.
- For practical implementation, we need a truncation of the plaquette quantum number ("finite spin")
- We use the notation $\bar{L}^x_{(i)}$ to denote a matrix with equal matrix elements on the first off-diagonal (like the first generator of the rotation algebra in the spin-1 representation)
- Parameters: $\tilde{Y} \equiv (\beta_{pl}/(2\kappa_\tau))\tilde{U}_g$ and $\tilde{X} \equiv (\beta_{pl}\kappa_s\sqrt{2})\tilde{U}_g$ which are the (small) energy scales.
- The final form of the Hamiltonian $\bar{H}$ is

$$\bar{H} = \frac{\tilde{U}_g}{2}\sum_i \left(\bar{L}^z_{(i)}\right)^2 + \frac{\tilde{Y}}{2}\sum_i (\bar{L}^z_{(i)} - \bar{L}^z_{(i+1)})^2 - \tilde{X}\sum_i \bar{L}^x_{(i)} \ .$$

Gaussian integration

$$\mathbb{T}(\{\eta\}) = \int d\eta_{L1} d\eta_{RN_s} d\eta_{L2} d\eta_{R1}\, T(\eta_1) \mathrm{e}^{+\eta_{R1}\eta_{L2}} d\eta_{3L} d\eta_{R2}\, T(\eta_2)\mathrm{e}^{+\eta_{R2}\eta_{L3}} \dots$$
$$\dots \quad d\eta_{N_s-1L} d\eta_{RN_s}\, T(\eta_{N_s-1})\mathrm{e}^{+\eta_{RN_s-1}\eta_{LN_s}}\, T(\eta_{N_s})\mathrm{e}^{-\eta_{RN_s}\eta_{L1}}$$

$$\mathbb{T}(\{\eta\}) = \int [\mathcal{D}\eta_{L,R}] \mathrm{e}^{[\eta_{R1}\eta_{L2}+\eta_{R2}\eta_{L3}+\cdots+\eta_{RN_s-1}\eta_{LN_s}-\eta_{RN_s}\eta_{L1}]} \prod_{x=1}^{N_s} T(\eta_x)$$

$$[\mathcal{D}\eta_{L,R}] \equiv d\eta_{L1} d\eta_{RN_s} \dots d\eta_{3L} d\eta_{R2} d\eta_{L2} d\eta_{R1}$$

$$\mathbb{T}(\{\eta\}) = \int [\mathcal{D}\eta_{L,R}] \mathrm{e}^{[T_2(\eta_1)+\eta_{R1}\eta_{L2}+T_2(\eta_2)+\eta_{R2}\eta_{L3}+\cdots+\eta_{RN_s-1}\eta_{LN_s}+T_2(\eta_{N_s})-\eta_{RN_s}\eta_{L1}]}$$