Simulations of gaussian systems in Minkowski time

B. Ruba J. Wosiek

Jagiellonian University (Cracow)

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Outline

- Beyond Complex Langevin approach.
- Free particles.
- Relation with thimbles.
- Oscillators and scalar fields.

Sign problem

Dream: compute efficiently the "complex averages"

$$\langle \mathcal{O} \rangle_\rho = \frac{\int_{\mathbb{R}^n} \mathrm{d}^n x \ \rho(x) \mathcal{O}(x)}{\int_{\mathbb{R}^n} \mathrm{d}^n x \ \rho(x)}. \tag{1}$$

Positive representations

Old idea: find $P \geq 0$ such that

$$\int_{\mathbb{R}^n} \mathrm{d}^n x \; \rho(x)\mathcal{O}(x) = \mathcal{N} \int_{\mathbb{C}^n} \mathrm{d}^n x \mathrm{d}^n y \; P(x,y)\mathcal{O}(x+iy). \qquad (2)$$

- Complex Langevin[1].
- Thimbles[2] (up to residual sign problem).
- Matching conditions.
- Beyond Complex Langevin[3] (BCL).

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[1] Parisi (83), Klauder (84)
[2] Pham (83), Christofretti, Di Renzo, Scorzato (12)
[3] Wosiek (15)

Solving the matching conditions[4]

- Variant of the moment problem.
- Space of solutions $\mathcal{P}$ infinite dimensional.
- $\mathcal{P}$ convex and invariant under smearing.
- Very regular solutions exist.
- To solve the problem you need the solution?

[4] Salcedo (97), Weingarten (02), Salcedo (07), Seiler, Wosiek (17), Ruba, Wyrzykowski (17)

BCL approach

Question

How to satisfy matching conditions without solving the theory?

Idea

Obtain ρ by integrating out auxillary variables.

BCL approach
Generalities

$$z = x + iy, \tag{3a}$$
$$\bar{z} = x - iy. \tag{3b}$$

- Allow $\bar{z} \neq z^*$.
- x, y become complex.
- $\bar{z}$ will be integrated out.

Step 1 find $P(z, \bar{z})$ positive for $\bar{z} = z^*$ and holomorphic such that

$$\rho(z) = \int_{\bar{\Gamma}} \mathrm{d}^n \bar{z}\ P(z, \bar{z}). \tag{4}$$

Step 2 using generalized Cauchy show that

$$\begin{aligned} \int_{\mathbb{R}^n} \mathrm{d}^n z\ \mathcal{O}(z)\rho(z) &= \overbrace{\int_{\Gamma} \mathrm{d}^n z\ \mathcal{O}(z) \int_{\bar{\Gamma}} \mathrm{d}^n \bar{z}\ P(z, \bar{z})}^{z,\bar{z}\text{ independent}} \\ &= \underbrace{\int_{\mathbb{C}^n} \mathrm{d}^n x \mathrm{d}^n y\ \mathcal{O}(x + iy) P(x, y)}_{\bar{z} = z^*}. \end{aligned} \tag{5}$$

BCL approach
Generalities

Crucial equation

$$\underbrace{\int_{\mathbb{C}^n} \mathrm{d}^n x \mathrm{d}^n y \; P(z,\bar{z}) \mathcal{O}(x+iy)}_{\bar{z}=z^*} = \underbrace{\int_{\Gamma} \mathrm{d}^n z \int_{\bar{\Gamma}} \mathrm{d}^n \bar{z} \; P(z,\bar{z}) \mathcal{O}(z)}_{z,\bar{z} \text{ independent}} . \quad (6)$$

holds provided that

$$\mathbb{C}^n \xrightarrow{\text{homotopy}} \Gamma \times \bar{\Gamma} \quad (7)$$

with $\int \neq \infty$ at intermediate stages.

This condition is vacuous **only** for gaussian integrals.

Gaussian theories

Simple way to find positive representations for gaussian ρ:

Gaussian ansatz → Integrate out $\bar{z}$ → Match with ρ

Locality? Symmetries?

Gaussian theories

Free particle

Let's apply this scheme to the quantum free particle model.

$$S[x] = \sum_j \frac{1}{2\epsilon}(x_{j+1} - x_j)^2. \tag{8}$$

Local gaussian ansatz preserving symmetries $\longrightarrow$ New family of solutions

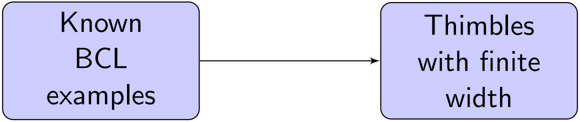
Gaussian theories
Free particle

$$\underbrace{\int Dx\ e^{iS[x]}\mathcal{O}[x]}_{\text{explicitly } \mu \text{ independent}} = \int DxDy\ \underbrace{P[x,y;\mu]\mathcal{O}[x+iy]}_{\text{integrand depends on } \mu}. \qquad (9)$$

Interesting $\mu \to \infty$ limit:

$$P[x,y;\mu] \sim \delta[y - y_{\rm thimble}[x]]e^{iS[x+iy]} \qquad (\text{for } \mu \to \infty) \qquad (10)$$

BCL and thimbles



What about the oscillator?

$$S[x] = \sum_j \left(\frac{1}{2\epsilon}(x_{j+1} - x_j)^2 - \frac{\epsilon}{2}x_j^2 \right). \tag{11}$$

Local gaussian ansatz → Matching condition → e^{x^2} type solutions

This can be understood by looking at thimbles.

Step 1 Fourier transformation

$$S[x] = \sum_k \lambda_k \tilde{x}_k \tilde{x}_{-k}, \tag{12a}$$

$$\lambda_k = \frac{1}{2\epsilon} \sin^2\left(\frac{k\pi}{n} - \frac{\epsilon}{2}\right) \tag{12b}$$

Step 2 Contour rotations such that $iS < 0$.

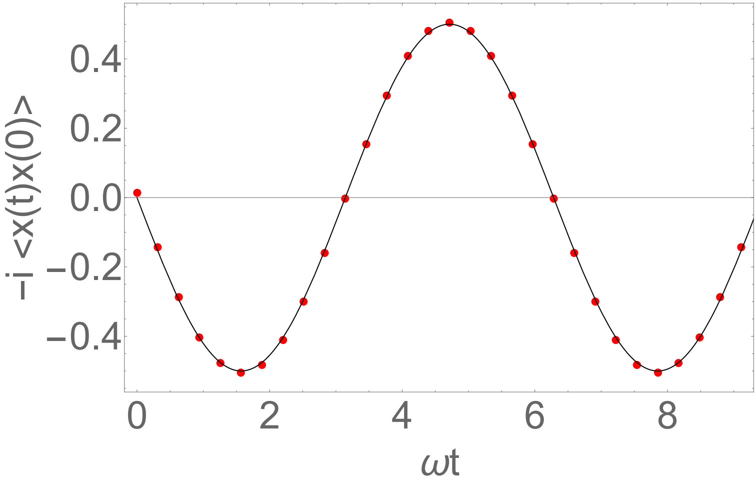
$$\tilde{x}_k = \begin{cases} e^{-\frac{i\pi}{4}}\ \tilde{q}_k & \text{if } \lambda_k < 0, \\ e^{\frac{i\pi}{4}}\ \tilde{q}_k & \text{if } \lambda_k > 0, \end{cases} \tag{13}$$

Step 3 Inverse Fourier $\longrightarrow$ nonlocal $S[q] \leq 0$.

$$iS[q] = \sum_j \left(-\frac{1}{2\epsilon}(q_{j+1} - q_j)^2 + \frac{\epsilon}{2} q_j^2 \right) - 2 \sum_{\lambda_k < 0} |\lambda_k| |\tilde{q}_k|^2 \quad (14)$$

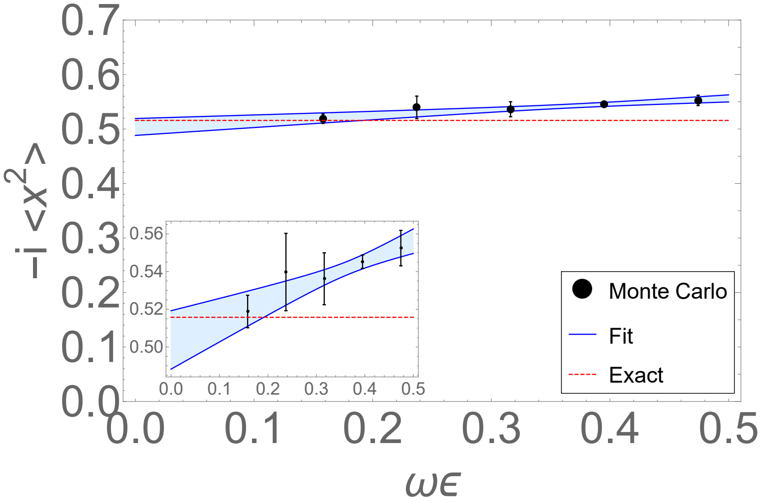
Gaussian theories

Harmonic oscillator - Monte Carlo result for the propagator



Gaussian theories

Harmonic oscillator - continuum limit of $\langle x^2 \rangle$



Gaussian theories

Lessons from Morse theorem

Caustics $\rightarrow$ Eigenvalues of $\delta^2 S\big|_{x_{\mathrm{cl}}}$ change sign $\rightarrow$ Singularities of thimbles

- Caustics $\longrightarrow$ obstructions to locality of BCL actions.
- Generalization to field theories self-evident.
- Next slides: free complex scalars in $d = 1 + 1$.

Action

$$S[\phi] = \frac{a^2}{2} \sum_{x,\mu} \left(\partial_\mu \phi_x \partial^\mu \phi_x - m^2 \phi_x^2 \right) . \tag{15}$$

Repeat the trick that worked for the oscillator:

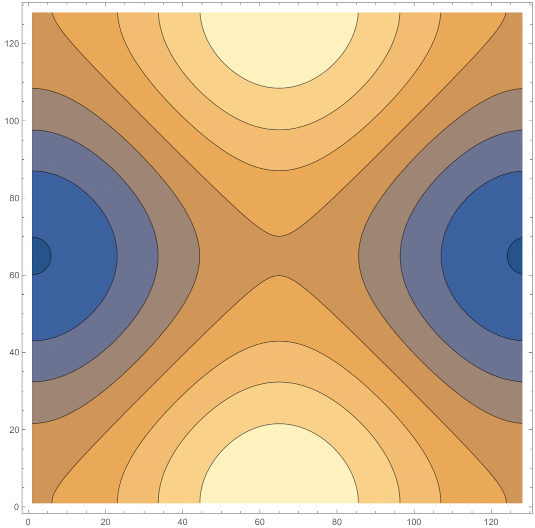
$$\phi_x = K_{xy} \psi_y , \tag{16a}$$

$$\bar{\phi}_x = \bar{K}_{xy} \psi_y^* , \tag{16b}$$

$$iS[\psi] = -\frac{1}{2V} \sum_p \left| \hat{p^\mu} \hat{p_\mu} - m^2 \right| \left| \tilde{\psi}(p) \right|^2 . \tag{16c}$$

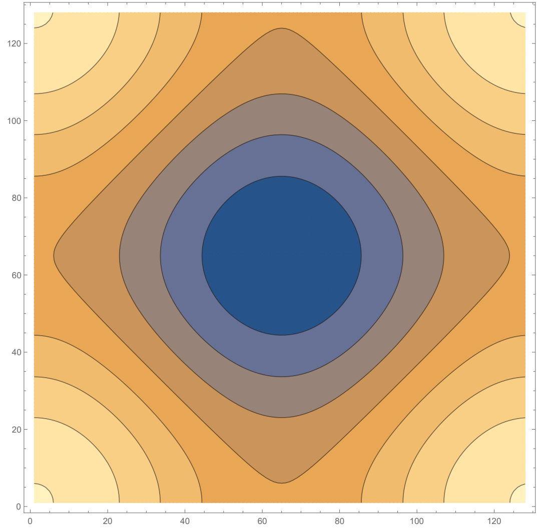
Gaussian theories

Complex scalar - real parts of eigenvalues

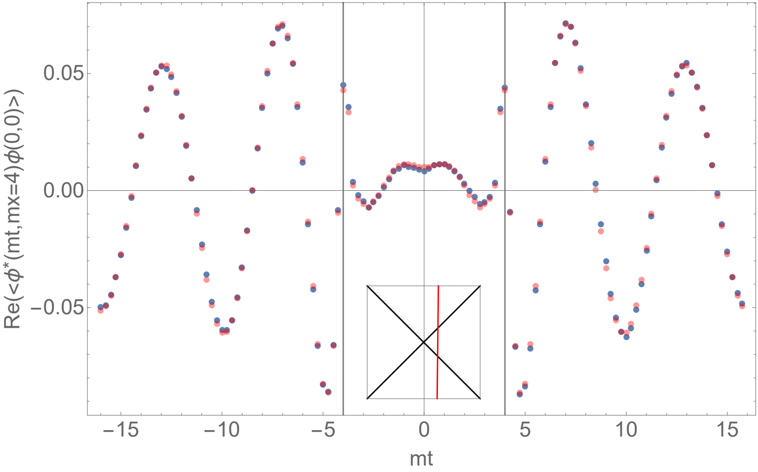


Gaussian theories

Complex scalar - imaginary parts of eigenvalues

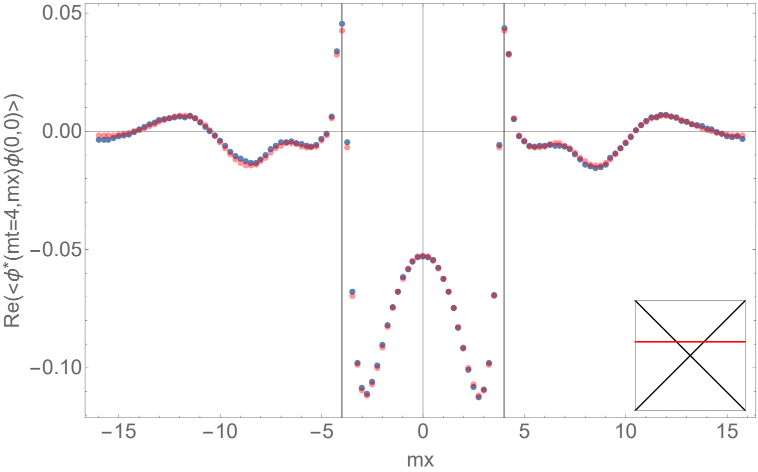


Gaussian theories
Complex scalar - timelike cut of the propagator



Gaussian theories

Complex scalar - spacelike cut of the propagator



Gaussian theories

Free complex scalar - some observations

- Infinitesimal Wick's rotation is crucial.
- Number of negative eigenvalues $\sim \left(\frac{\Lambda_{\mathrm{UV}}}{\Lambda_{\mathrm{IR}}}\right)^{d-1}$.
- $S[\phi]$ strongly nonlocal and UV-singular.
- Light-cone and UV structure nicely reproduced up to cutoff.
- Finite volume effects much larger than in Euclidean space.

Summary

- We simulated gaussian field theory in Minkowski time.
- Nonlocal, but positive path integral measure was used.
- Connection between our approach and thimbles was seen.
- Including interactions is an open problem.

Generalized Cauchy theorem

$$\mathrm{d}\left(\overbrace{P(z,\bar{z})\ \mathrm{d}^n z \wedge \mathrm{d}^n \bar{z}}^{\text{denote } \omega}\right) = 0 \iff \overbrace{\frac{\partial P}{\partial z^*} = \frac{\partial P}{\partial \bar{z}^*} = 0}^{\text{Cauchy-Riemann}} \tag{17}$$

Stokes' theorem:

$$\int_{\mathrm{boundary}^{2n}} \omega = \int_{\mathrm{bulk}^{2n+1}} \mathrm{d}\omega = 0. \tag{18}$$

Free particle - positive representation

$$S = \sum_j \left[\frac{1}{2\epsilon} (\bar{z}_{j+1} - \bar{z}_j)(z_{j+1} - z_j) + \sigma (x_j - y_j)^2 \right]. \tag{19}$$

Continuum limit:

$$\epsilon \to 0, \tag{20a}$$

$$\frac{\sigma}{\epsilon} \to \infty. \tag{20b}$$

Thimble transformation in the continuum

$$\phi(x) = \int \mathrm{d}^d y \left(e^{\frac{i\pi}{4}} \delta_+(x-y;m) + e^{-\frac{i\pi}{4}} \delta_-(x-y;m) \right) \psi(y), \tag{21a}$$

$$\delta_\pm(x;m) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \theta(\pm(p^2 - m^2)) e^{-ipx}, \tag{21b}$$

$$\delta_+(x;m) + \delta_-(x;m) = \delta(x). \tag{21c}$$

Thimble transformation in the continuum

Distributions $\delta_\pm$ = Bessel functions. For $d = 2$:

$$\delta_+(x; m)\big|_{d=2} = \frac{m}{2\pi^2\sqrt{-(x-i0)^2}} K_1\left(m\sqrt{-(x-i0)^2}\right) + \mathrm{c.c.} \quad (22)$$

UV singularity:

$$\delta_+(x; m) \sim \frac{1}{2\pi^2}\left(-\frac{1}{(x-i0)^2} + \frac{m^2}{2}\log\left(m\sqrt{-(x-i0)^2}\right)\right) + \mathrm{c.c.} \quad (23)$$