Examples of renormalization group transformations for image sets

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Introduction

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- Preprint available at:
  - https://arxiv.org/abs/1807.10250

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Introduction

- Show how configurations of the 2D Ising model generated using the worm algorithm can be represented as 2D images.
- Illustrate how principal component analysis can be used to describe the critical behavior of the model near the phase transition.
- Propose a renormalization group transformation for arbitrary images.
- Discuss how quantities of interest behave under successive applications of these blocking (coarse-graining) transformations.
- Suggest ways to obtain data collapse and compare results with the two state tensor RG approximation near the fixed point.

Character Expansion

- The 2D Ising model has the usual partition function

$$Z = \sum_{\{\sigma_i\}} e^{\beta \sum_{\langle i,j \rangle} \sigma_i \sigma_j} \tag{1}$$

- Difficult to construct explicitly the RG transformation mapping couplings among spins.
  - **Use TRG!**
- Character expansion:

$$\exp(\beta\sigma) = \cosh(\beta) + \sigma \sinh(\beta) \tag{2}$$

$$= \cosh(\beta)\left(1 + \sigma \tanh(\beta)\right) \tag{3}$$

- This leaves us with integer variables on the links of the lattice, and requires the sum of the link variables associated with each site to be even (closed paths).

Character expansion

- Rewrite the partition function

$$Z = 2^V \left(\cosh(\beta)\right)^{2V} \sum_{N_b} \left(\tanh(\beta)\right)^{N_b} \mathcal{N}(N_b) \tag{4}$$

- Use $\frac{\partial}{\partial \beta} \ln Z$ to relate $\langle N_b \rangle \propto \langle E \rangle$ and the bond number fluctuations

$$\langle \Delta^2_{N_b} \rangle \equiv \langle \left(N_b - \langle N_b \rangle\right)^2 \rangle \tag{5}$$

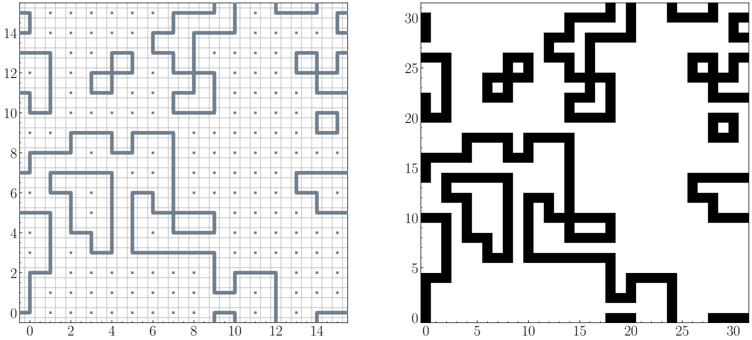
to the specific heat per site.

- From the logarithmic singularity of the specific heat we find

$$\frac{\langle \Delta^2_{N_b} \rangle}{V} = -\frac{2}{\pi} \ln(|T - T_c|) + \cdots \tag{6}$$

From Loops to Images

- Contributions to Z are sampled using the Worm Algorithm which constructs closed-loop paths that live on the links between sites.
- Represent these configurations as images of $2L \times 2L$ pixels with size equal to one half the lattice spacing.
- These images can then be flattened into a vector $\vec{v} \in \mathbb{R}^{4V}$ with $v_i \in \{0, 1\}$.



Principal Component Analysis and Criticality

- Used to isolate the "most relevant" directions of a dataset by calculating the eigenvalues λ_α and eigenvectors u_α of its corresponding covariance matrix.
- Useful for reducing the dimensionality at a minimal cost.
- For our example, at a given T, our data matrix consists of N_{configs} images $\{\vec{v}^{\,n}\}_{n=1}^{N_{\text{configs}}}$, each of which is a vector $\vec{v}^{\,n} \in \mathbb{R}^{4V}$.
- Construct the covariance matrix,

$$S_{ij} = \frac{1}{N_{\text{configs}}} \sum_{n=1}^{N_{\text{configs}}} \left(v_i^n - \langle v_i \rangle\right) \left(v_j^n - \langle v_j \rangle\right) \tag{7}$$

where indices $i, j = 1, \ldots, 4V$ label the pixels of the image $\vec{v}^{\,n}$.

PCA and Criticality

- PCA extracts the solutions to

$$S u_\alpha = \lambda_\alpha u_\alpha \tag{8}$$

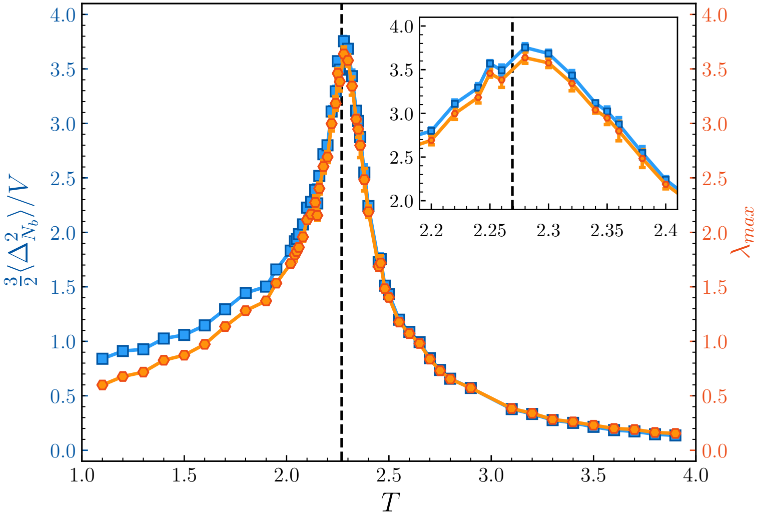
and orders them in decreasing magnitude of λ_α (≥ 0).

- Not immediately obvious what λ_α, u_α represent physically, or how they might be useful for describing our model.
- Under mild assumptions and an easily justified approximation, we were able to find a relation between the largest PCA eigenvalue λ_{max} and the logarithmic divergence of the specific heat, namely

$$\lambda_{\text{max}} \approx \frac{3}{2} \frac{\langle \Delta_{N_b}^2 \rangle}{V} \tag{9}$$

$$\approx -\frac{3}{\pi} \ln \left(|T - T_c| \right) \tag{10}$$

PCA and Criticality: Results



TRG Coarse-graining

- We can use the constraints imposed by the character expansion to build a tensor

$$T^{(i)}_{xx'yy'}(\beta) = [\tanh(\beta)]^{(n_x+n_{x'}+n_y+n_{y'})/2} \times \delta_{n_x+n_{x'}+n_y+n_{y'},0 \text{ mod } 2}.$$

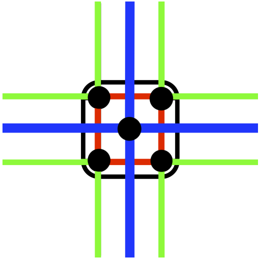
  located at the i^{th} site of the lattice, with integer variables $n_{\hat{\mu}} \in \{0,1\}$ on each of the links x, x', y, y' and the $\delta_{i,j}$ is satisfied if the sum of the $n_{\hat{\mu}}$'s are even.

- By contracting these tensors together in the pattern of the lattice, we recreate the closed loop paths sampled by the worm algorithm.

TRG Coarse-graining

- This formulation can be coarse-grained efficiently!

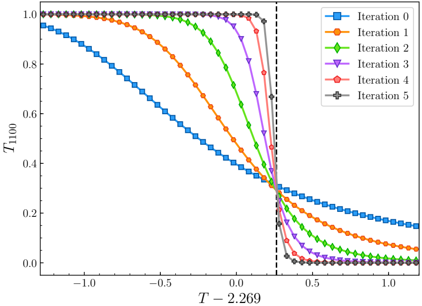
$$Z = 2^V \left(\cosh(\beta)\right)^{2V} \mathrm{Tr} \prod_i T^{(i)}_{xx'yy'}$$
$$\longrightarrow 2^V \left(\cosh(\beta)\right)^{2V} \mathrm{Tr} \prod_{2i} T'^{(2i)}_{xx'yy'}$$



where $2i$ denotes the sites of the coarser lattice with twice the original lattice spacing.

TRG Coarse-graining

- This can't be repeated indefinitely, truncate by projecting the product states onto two-state approximation (arXiv:1211.3675 [hep-lat]).
- Using this blocking procedure, the scale factor is $b = 2$, with eigenvalue in the relevant direction $\lambda = b^{1/\nu} = 2$ since $\nu = 1$.
- Begin with initial lattice of size $L = 64$, so after ℓ iterations we are left with an effective size of the coarse-grained lattice $L_{eff} = L/b^{\ell}$



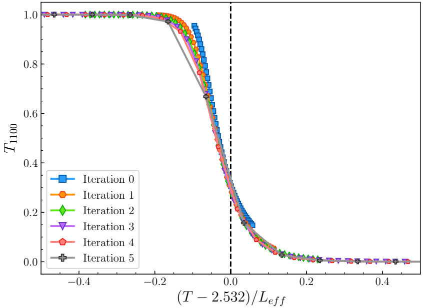


Image Coarse-graining

- As in the TRG coarse-graining procedure, the image is first divided up into blocks of 2×2 squares, each of which is then replaced by a single site with new link variables (blue) determined by the sum of the external link variables in a given direction (green).
- Explicitly, if a given block has exactly one external link in a given direction, the blocked site retains this link in the blocked configuration, otherwise it is neglected $(1 + 1 \to 0)$.

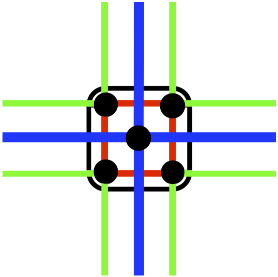


Image Coarse-graining: Example

- Illustration of the $1 + 1 \to 0$ blocking procedure used for coarse-graining sample images.
- Can also be iterated efficiently!

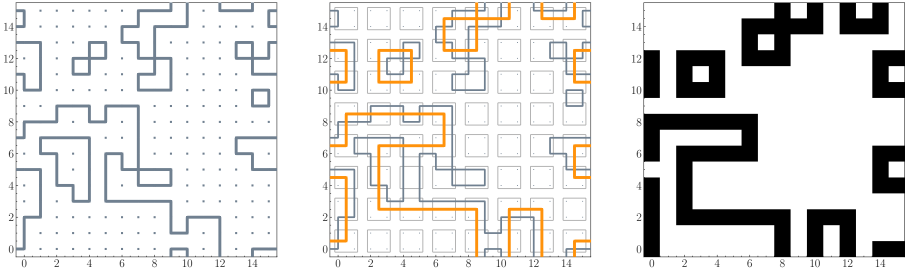


Image Coarse-graining: $\langle \Delta^2_{N_b} \rangle$

- First observation of this iterated blocking procedure is that it preserves the location of the peak of the fluctuations $\langle \Delta^2_{N_b} \rangle$, which can be stabilized for the first few iterations by dividing by $V_{eff} \ln(L_{eff})$.

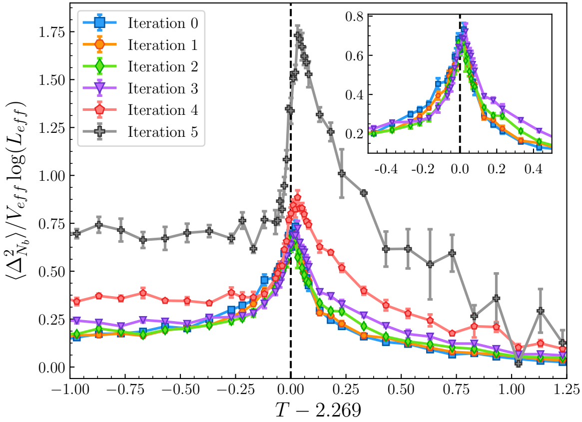


Image Coarse-graining: $\langle N_b \rangle$

- What about $\langle N_b \rangle$?
- At high T, we have sparse configurations with only an occasional plaquette.
- There are four possible plaquettes:
  1. **inside the blocks** (disappear after blocking)
  2. **between two neighboring blocks** (double links, disappear after blocking)
  3. **share a corner with four blocks** (generate a larger plaquette)
- Only $1/4$ survive the blocking transformation!

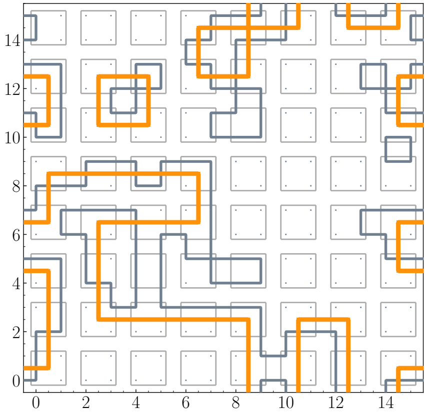
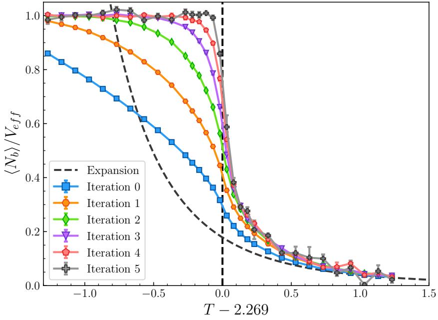
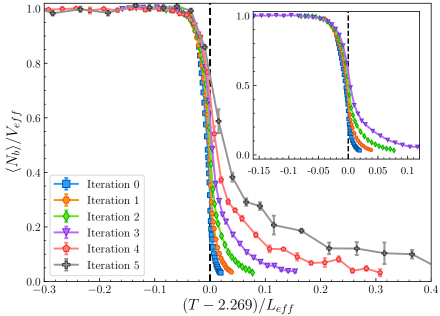


Image Coarse-graining: $\langle N_b \rangle$ results

- Note that on the high temperature side we observe a merging of the data instead of a crossing like in the TRG result.
- This is because with the $1 + 1 \to 0$, only one of the four plaquettes becomes a larger plaquette, which exactly compensates the change in V_{eff} which is also reduced by a factor of four.





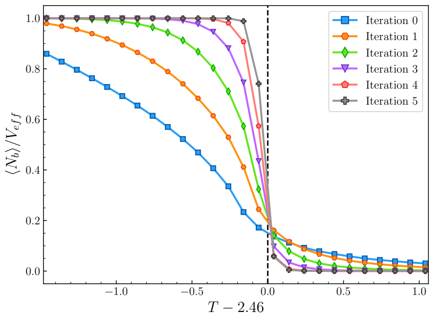
TRG calculation of $\langle N_b \rangle$

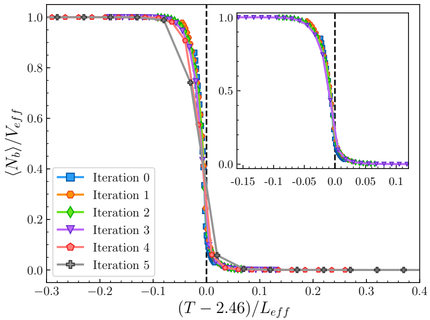
- We can compute $\langle N_b \rangle$ using the tensor method and compare it to our result from the worm algorithm.
- Using $N_b = \sum_\ell n_\ell$ for the sum of bond numbers at every link, we have for $\langle N_b \rangle$

$$
\begin{aligned}
\langle N_b \rangle &= \frac{1}{Z} \sum_{\{n\}} \left(\sum_\ell n_\ell \right) \left(\prod_i \tanh^{n_\ell}(\beta) \right) \left(\prod_i \delta_{n_x + n_{x'} + n_y + n_{y'},\, 0 \,\mathrm{mod}\, 2} \right) \\
&= \sum_\ell \langle n_\ell \rangle
\end{aligned}
$$

- Because of translation and 90° rotational invariance, all $\langle n_\ell \rangle$ are equal.
- It's enough to calculate $\langle n_\ell \rangle$ for one particular link (call it $\langle n \rangle$)and multiply it by $2V$: $\langle N_b \rangle = 2V \langle n \rangle$.

TRG calculation of $\langle N_b \rangle$: Results





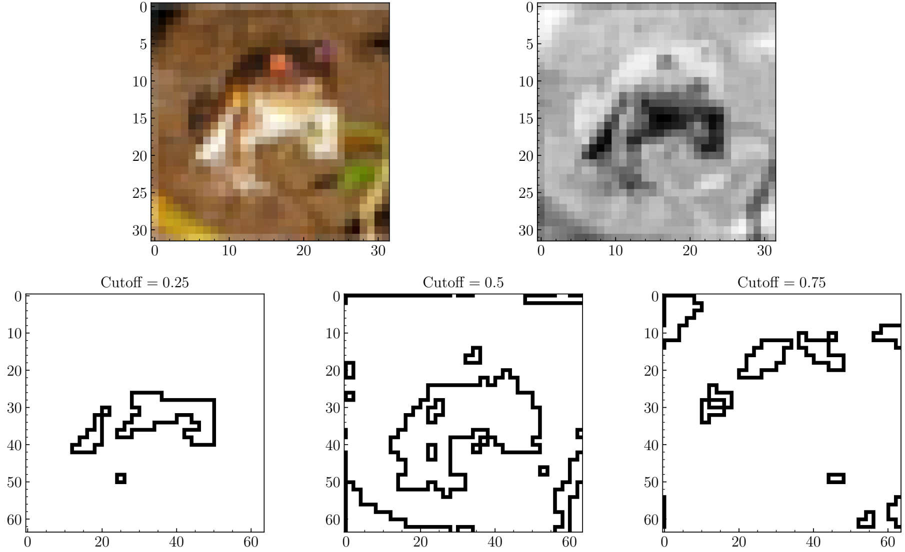
Conclusions

- Used image representation of 2D Ising model configurations using $\langle N_b \rangle$ and $\langle \Delta^2_{N_b} \rangle$ to describe the system.
- Constructed relationship between the logarithmic divergence of the principal component and the critical behavior of the model.
- Proposed RG transformation that can be applied iteratively to an arbitrary 2D image.
- Looked at the behavior of $\langle N_b \rangle$ and $\langle \Delta^2_{N_b} \rangle$ near fixed points of these transformations and observed partial data collapse.
- Relevance to generic 2D image sets?

CIFAR-10

- For a possible application of these ideas, we tried carrying out a similar analysis using the CIFAR-10 dataset.
- **Preprocessing steps:**
  1. Convert images to grayscale with pixel values in the range $[0, 1]$.
  2. Choose grayscale cutoff value so that all pixels with values below the cutoff would become black, and pixels above the cutoff would become white.
  3. Convert to 'worm-like' images by drawing the boundaries separating black and white collections of pixels.

CIFAR-10



CIFAR-10

- No identifiable low temperature phase.
- For cutoff values near both 0 and 1, we obtain images which are mostly empty, similar to the high temperature configurations obtained from the worm algorithm.
- This suggests that there is no such notion of criticality like we found for the two-dimensional Ising model.