Higher order fluctuations form imaginary chemical potential

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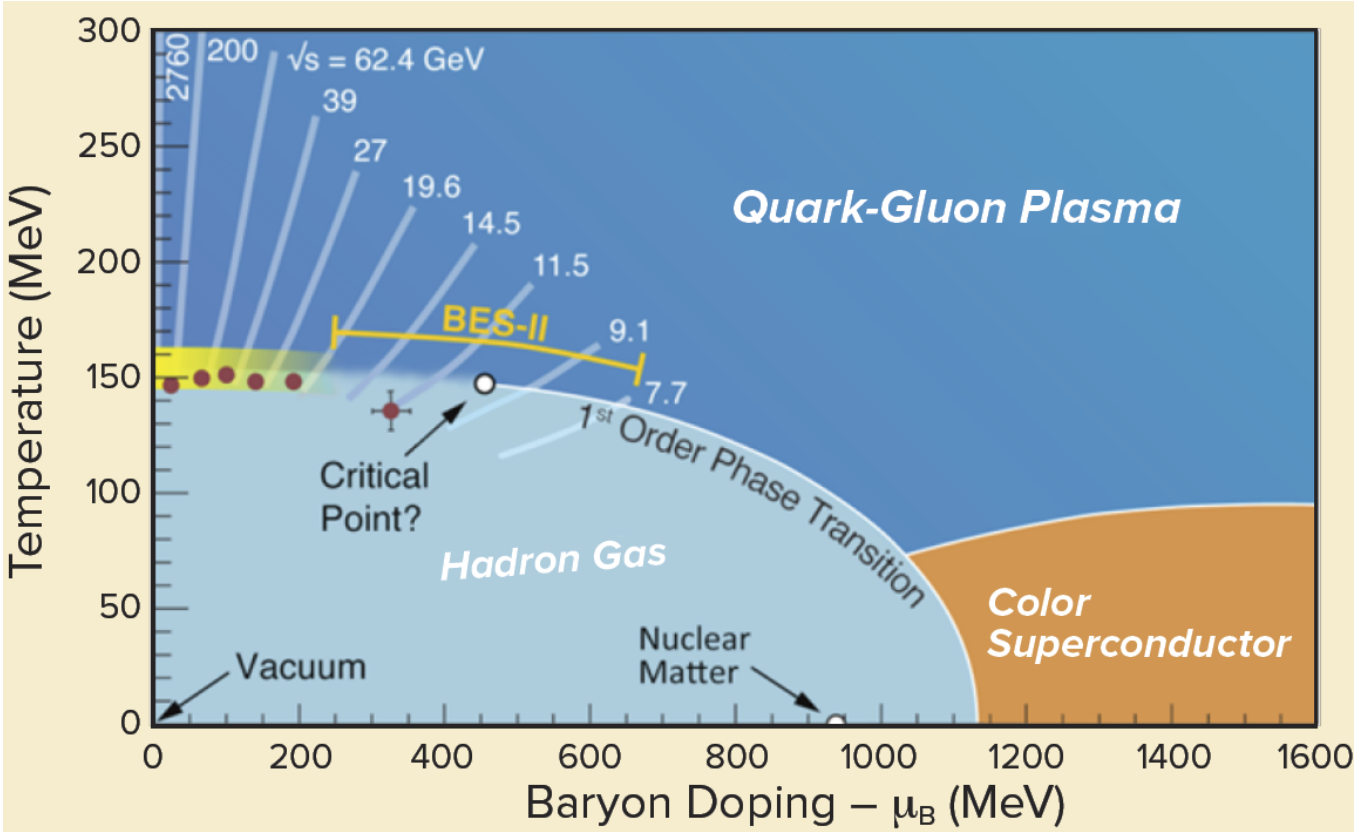
SFB TR55

WB collaboration

The (T, μ_B)-phase diagram of QCD

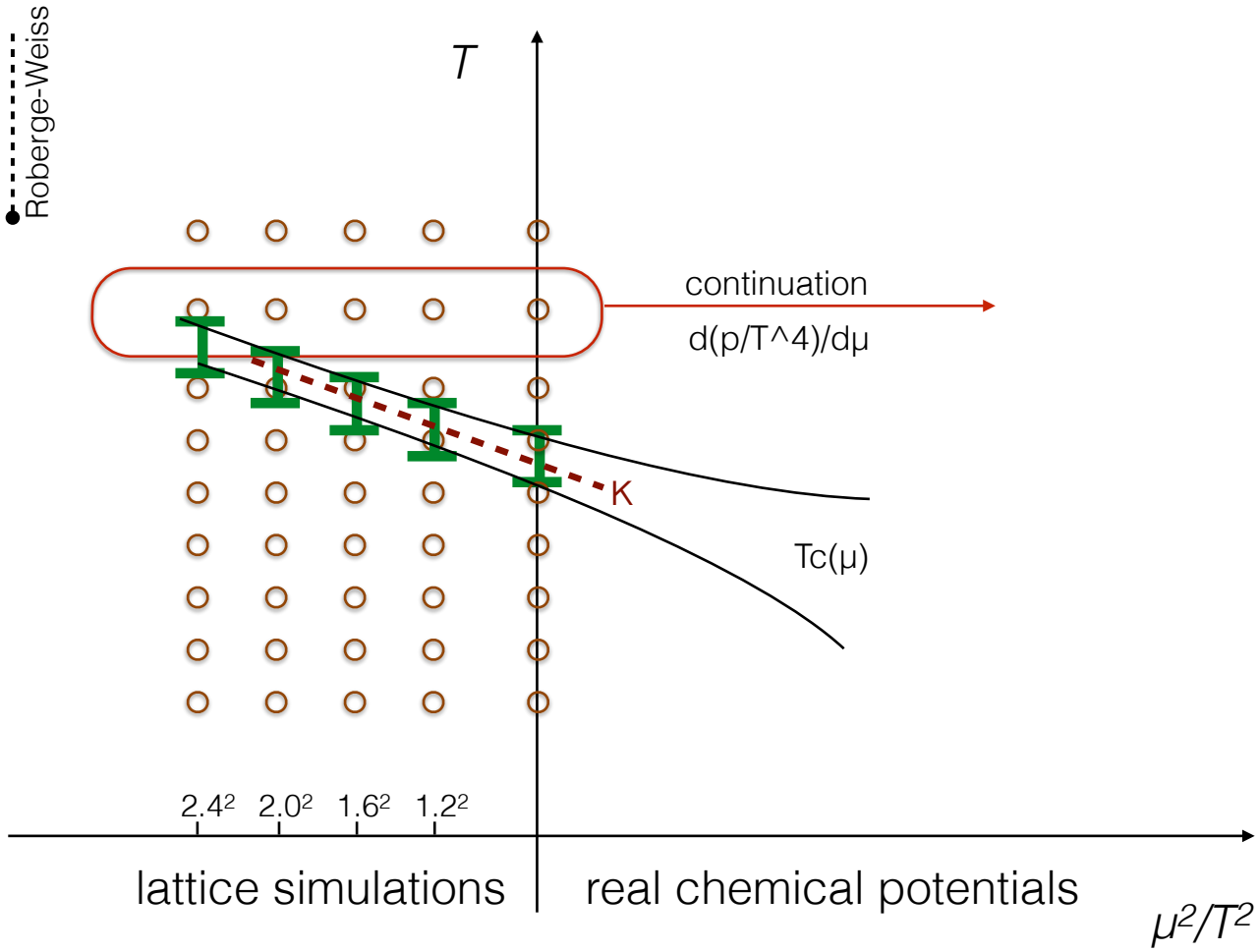


Our observables:
T_c, Equation of state, Fluctuations

Dealing with the sign problem

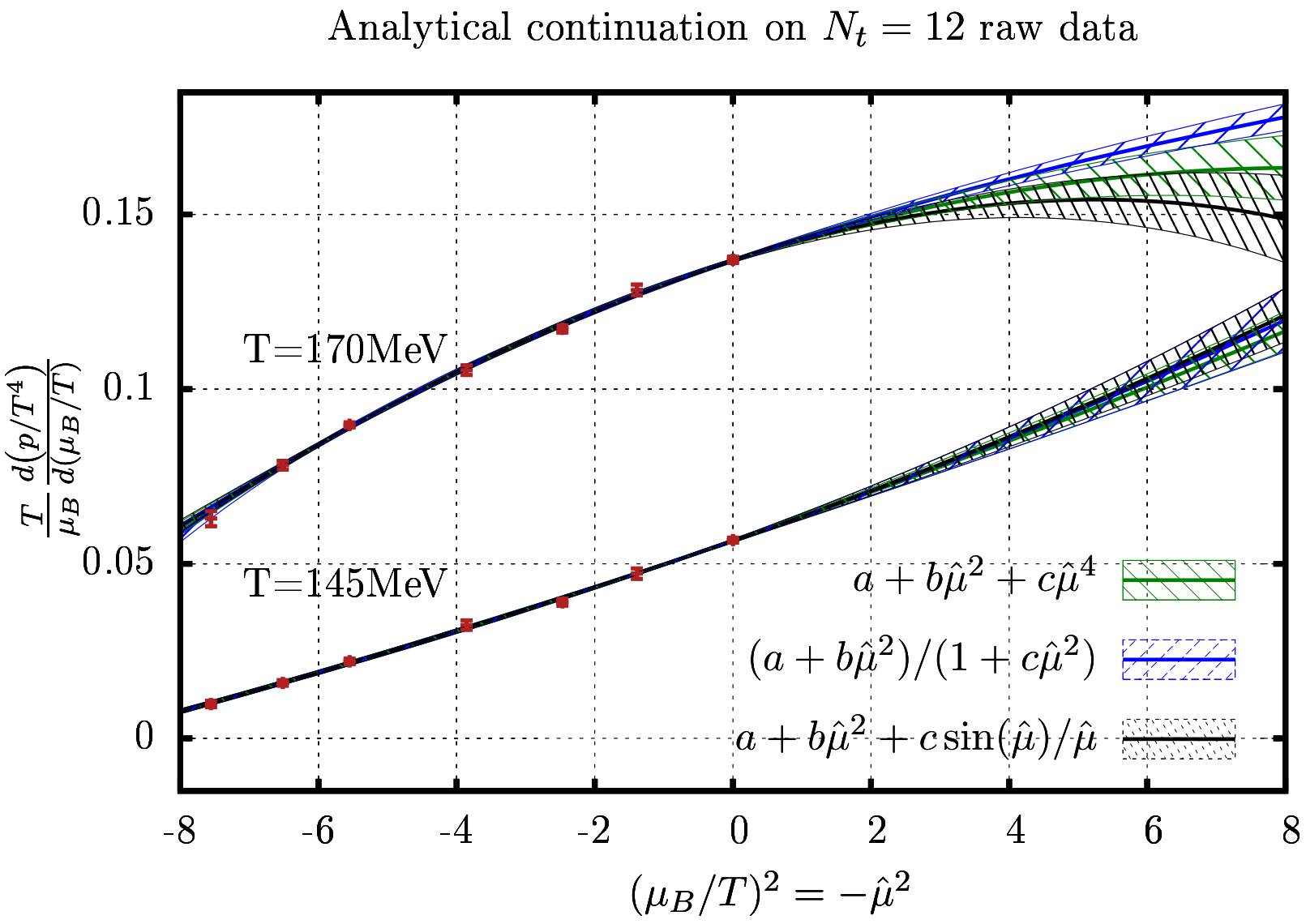
- Reweighting techniques
- Canonical ensemble
- Complex Langevin
- Lefshetz Thimble
- Density of state methods
- Dual variables
- Taylor expansion $\longrightarrow$ previous talk
- Imaginary μ
- ...

Analytic continuation



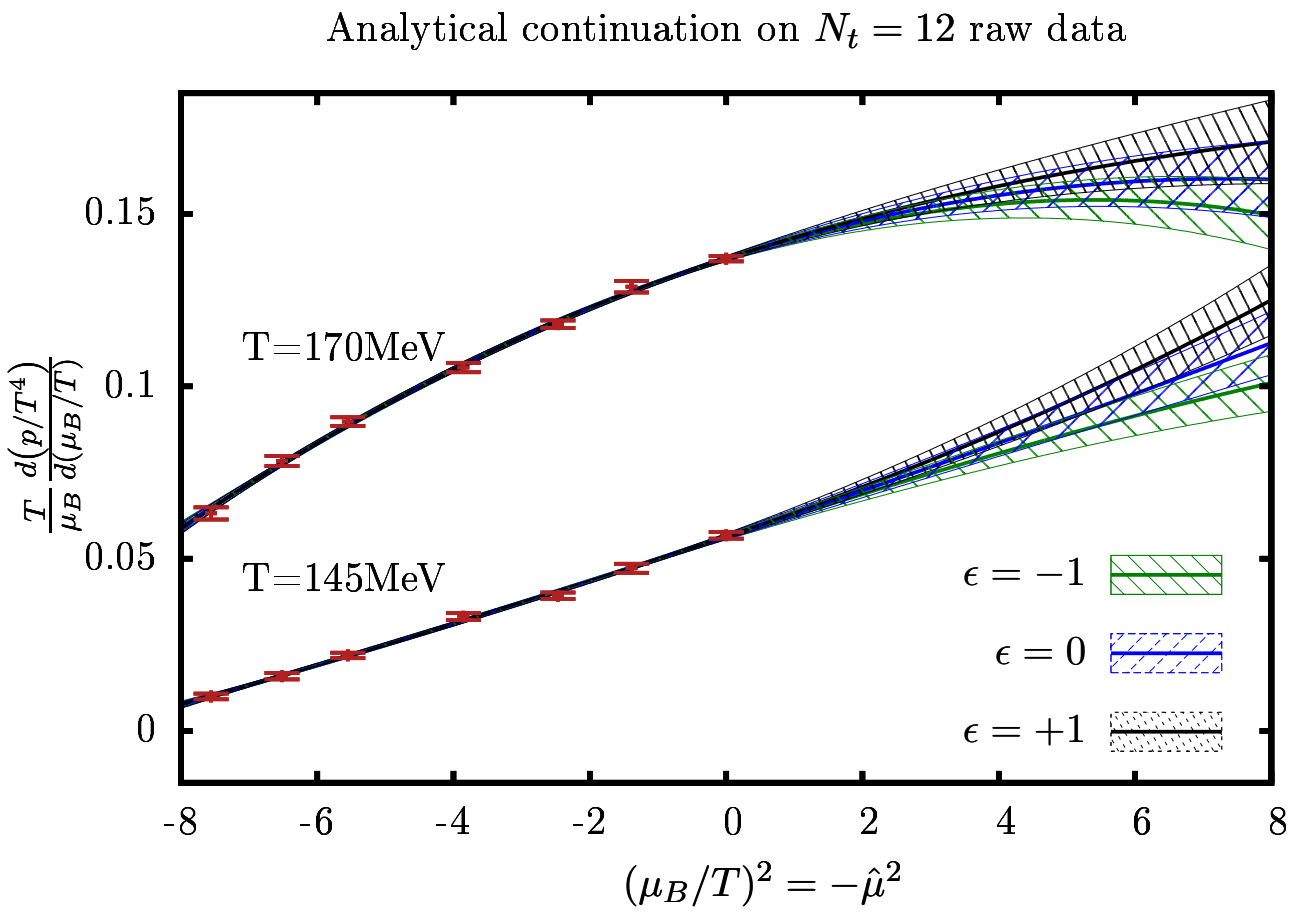
Common technique: [de Forcrand, Philipsen, deForcrand:2002hgr], [Bonati et al., Bonati:2015bha], [Cea et al., Cea:2015cya], [D'Elia et al., DElia:2016jqh], [Bonati et al., Bonati:2018nut] . . .

Different functions

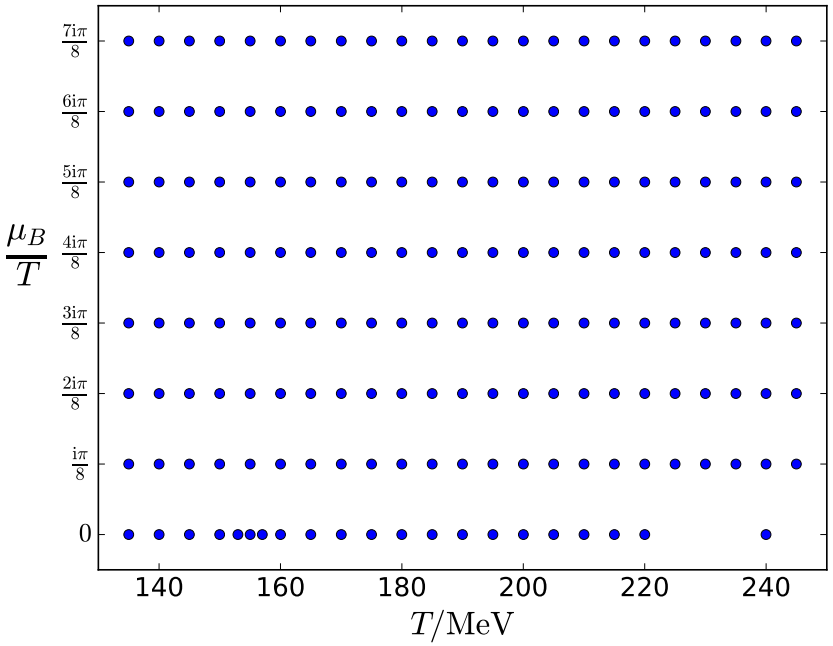


Different functions

Condition: $\chi_8 \lesssim \chi_4 \longrightarrow f(\hat{\mu}_B) = a + b\hat{\mu}_B^2 + c\hat{\mu}_B^4 + \frac{b\epsilon}{840}\hat{\mu}_B^6$



Simulation details



- Borsanyi et al., Borsanyi:2018grb, arXiv:1805.04445
- Action: tree-level Symanzik improved gauge action, with four times stout smeared staggered fermions
- 2+1+1 flavor, on LCP with pion and kaon mass
- Simulation at $\mu_S = \mu_Q = 0$
- Lattice size: $48^3 \times 12$
- $\frac{\mu_B}{T} = \mathrm{i}\frac{j\pi}{8}$ with $j = 0$, 1, 2, 3, 4, 5, 6 and 7

The fit function

$$\chi_{i,j,k}^{B,Q,S} = \frac{\partial^{i+j+k}(p/T^4)}{(\partial\hat{\mu}_B)^i(\partial\hat{\mu}_Q)^j(\partial\hat{\mu}_S)^k}\,, \quad \hat{\mu} = \frac{\mu}{T}$$

$$\frac{p}{T^4} = \chi_0^B + \frac{1}{2!}\chi_2^B\hat{\mu}_B^2 + \frac{1}{4!}\chi_4^B\hat{\mu}_B^4 + \frac{1}{6!}\chi_6^B\hat{\mu}_B^6 + \frac{1}{8!}\chi_8^B\hat{\mu}_B^8 + \frac{1}{10!}\chi_{10}^B\hat{\mu}_B^{10}$$

From this we can calculate the derivatives that we can measure on the lattice:

$$\chi_1^B(\hat{\mu}_B) = \chi_2^B\hat{\mu}_B + \frac{1}{3!}\chi_4^B\hat{\mu}_B^3 + \frac{1}{5!}\chi_6^B\hat{\mu}_B^5 + \frac{1}{7!}\chi_8^B\hat{\mu}_B^7 + \frac{1}{9!}\chi_{10}^B\hat{\mu}_B^9$$

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$$\chi_3^B(\hat{\mu}_B) = \chi_4^B\hat{\mu}_B + \frac{1}{3!}\chi_6^B\hat{\mu}_B^3 + \frac{1}{5!}\chi_8^B\hat{\mu}_B^5 + \frac{1}{7!}\chi_{10}^B\hat{\mu}_B^7$$

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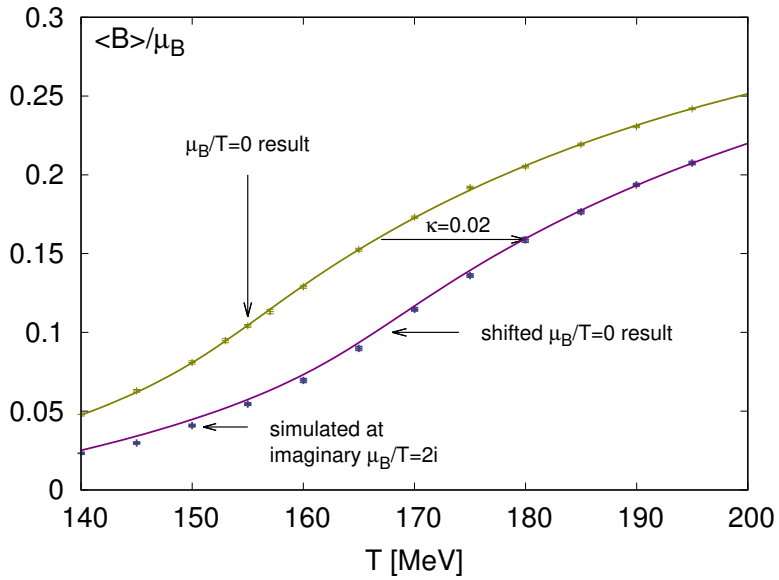
where ϵ_1 and ϵ_2 are drawn randomly from a normal with $\mu = -1.25$ and $\sigma = 2.75$ distribution.

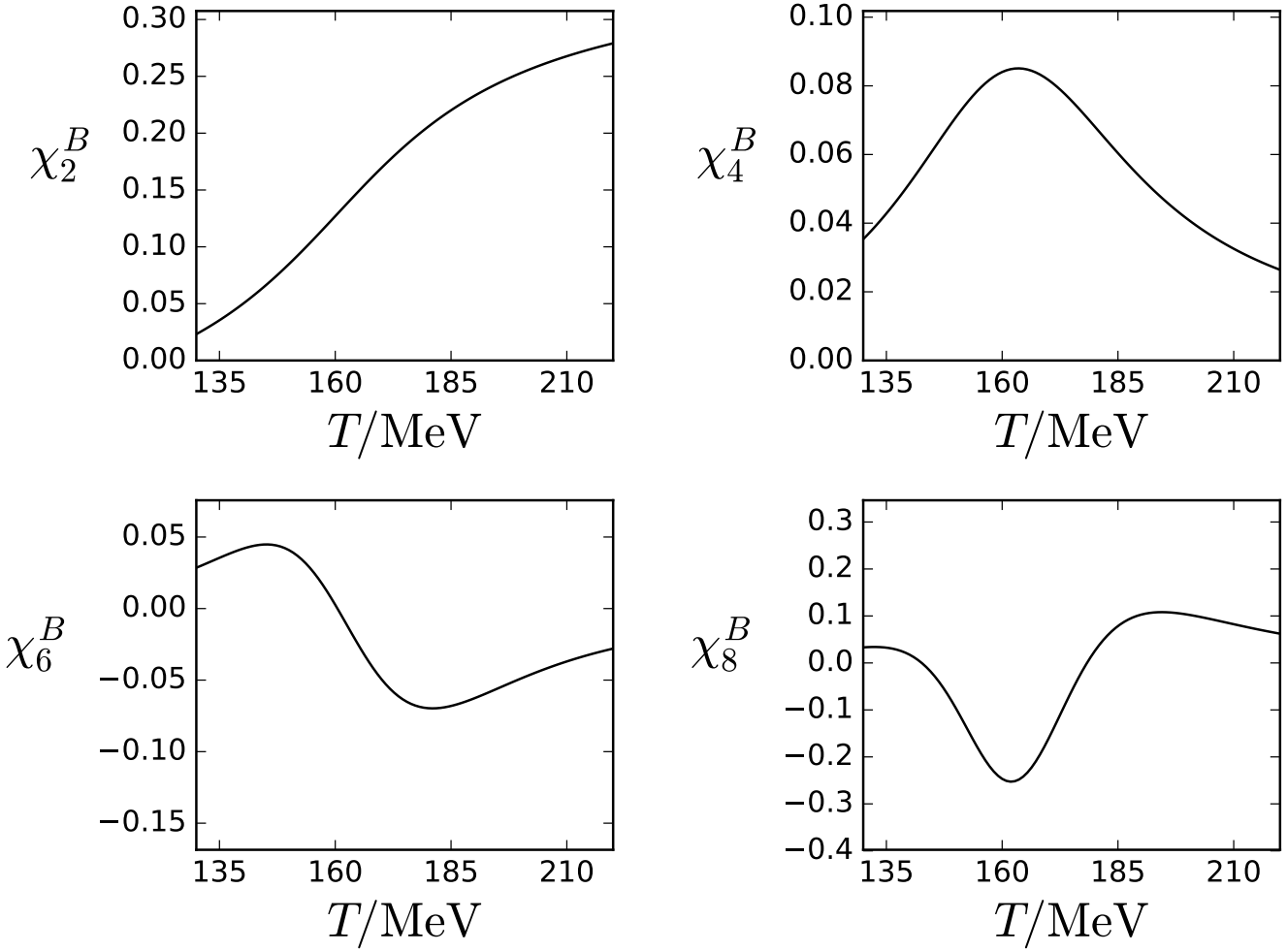
Motivating the prior - Toy model without critical endpoint



- Start with some parametrization of the curve χ_1^B/μ_B at $\mu = 0$
- Assume that the only difference in the physics at finite μ is a shift in this curve
- The inflection point of this curve is one possible definition of T_c, so shift the curve by using the κ values found in the literature
- You now have a model prediction of χ_1^B for any finite μ, differentiate it a few times at $\mu = 0$ to get estimates of χ_4^B, χ_6^B and χ_8^B

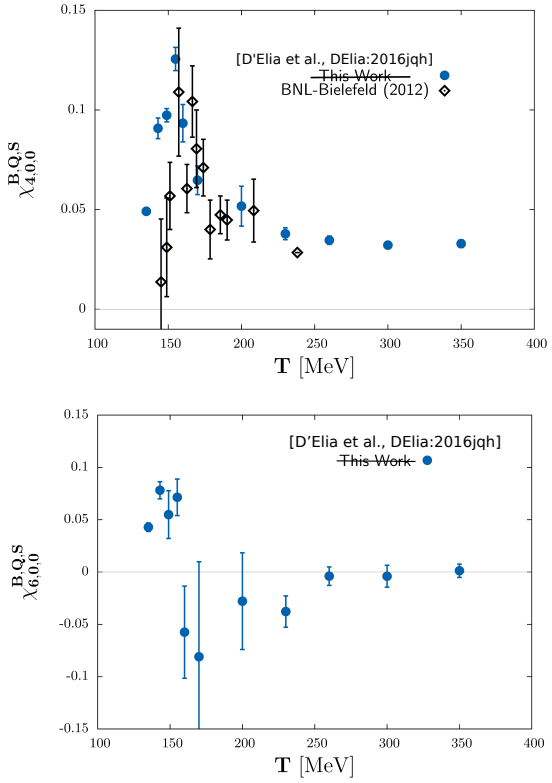
NOTE: The model assumes no criticality

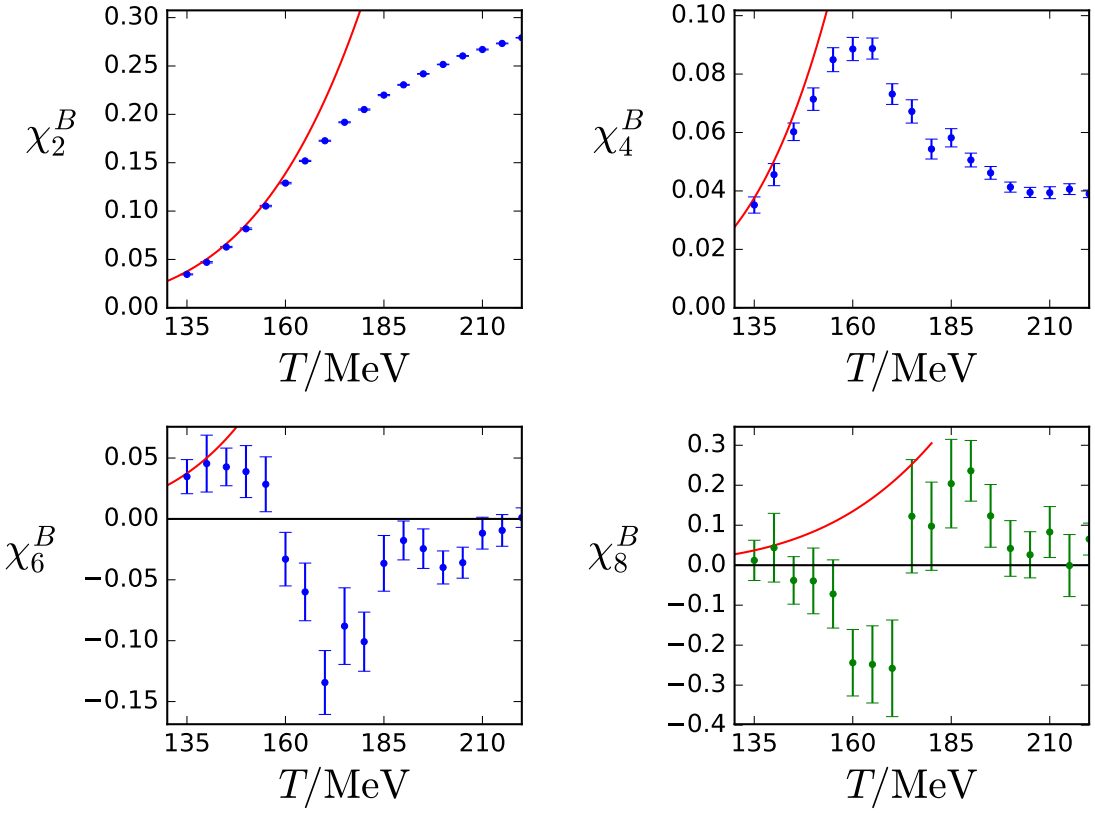
Fluctuations in the toy model



The simple model described in the previous slide without criticality

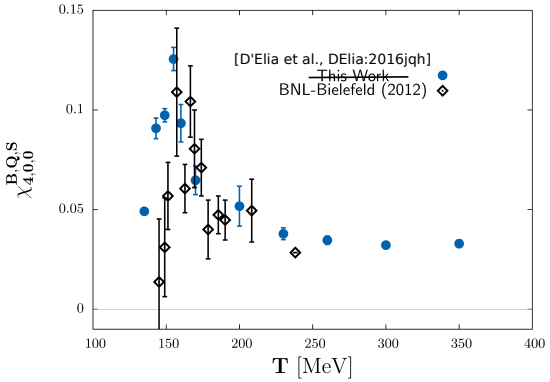
χ_2^B, χ_4^B, χ_6^B and χ_8^B

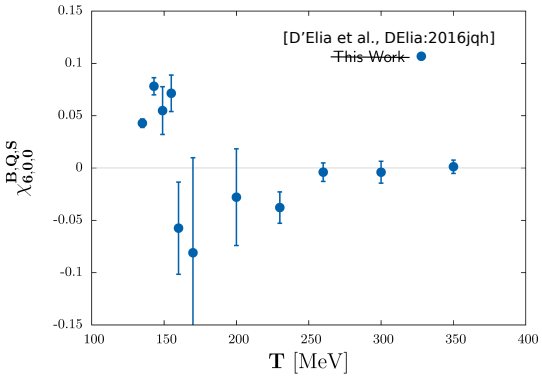


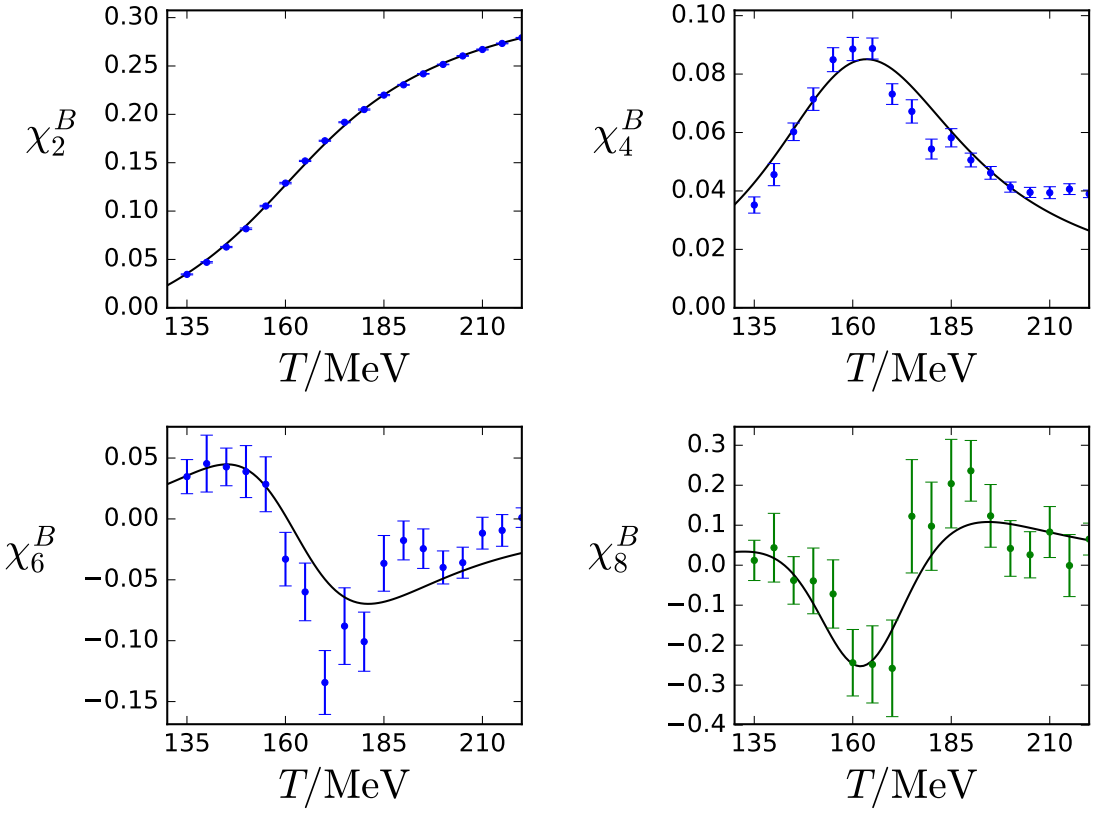


[D'Elia et al., DElia:2016jqh], see also [Bazavov et al., Bazavov:2017dus]

χ_2^B, χ_4^B, χ_6^B and χ_8^B





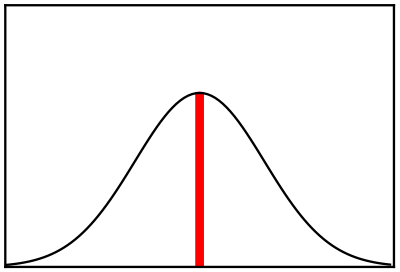


[D'Elia et al., DElia:2016jqh], see also [Bazavov et al., Bazavov:2017dus]

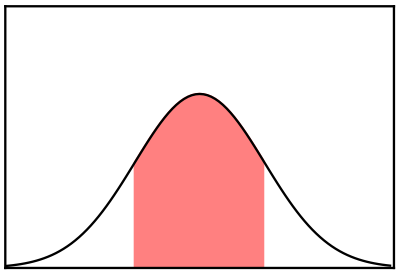
Observables

Cumulants of the net baryon number distributions:

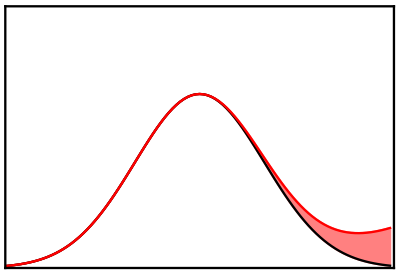
- mean M_B



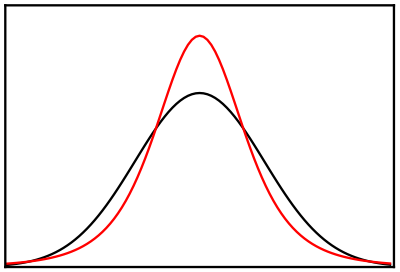
- variance σ_B^2



- skewness S_B: asymmetry of the distribution

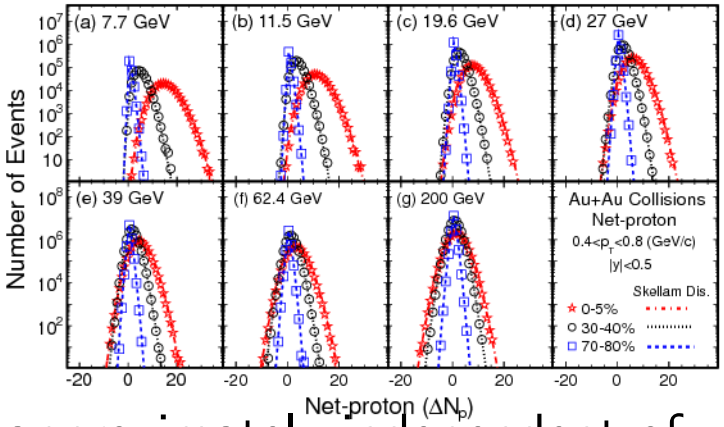


- kurtosis κ_B: "tailedness" of the distribution



Calculating observables

We have derivatives with respect to $\hat{\mu}_B$, $\hat{\mu}_Q$ and $\hat{\mu}_S$ of the pressure at $\mu_S = \mu_Q = 0$. Notation:



$$\chi^{B,Q,S}_{i,j,k} = \frac{\partial^{i+j+k}(p/T^4)}{(\partial\hat{\mu}_B)^i(\partial\hat{\mu}_Q)^j(\partial\hat{\mu}_S)^k},$$

with $\hat{\mu}_i = \mu/T$.
We want ratios of the cumulants that are approximately independent of the volume at $\mu_B > 0$, $\langle n_S\rangle = 0$ and $\langle n_Q\rangle = 0.4\langle n_B\rangle$:

$$\frac{M_B}{\sigma_B^2} = \frac{\chi_1^B(T,\hat{\mu}_B)}{\chi_2^B(T,\hat{\mu}_B)} = \hat{\mu}_B r_{12}^{B,1} + \hat{\mu}_B^3 r_{12}^{B,3} + \dots$$

$$\frac{S_B\sigma_B^3}{M_B} = \frac{\chi_3^B(T,\hat{\mu}_B)}{\chi_1^B(T,\hat{\mu}_B)} = r_{31}^{B,0} + \hat{\mu}_B^2 r_{31}^{B,2} + \dots$$

$$\kappa_B\sigma_B^2 = \frac{\chi_4^B(T,\hat{\mu}_B)}{\chi_2^B(T,\hat{\mu}_B)} = r_{42}^{B,0} + \hat{\mu}_B^2 r_{42}^{B,2} + \hat{\mu}_B^4 r_{42}^{B,4} + \dots$$

[Bazavov et al., Bazavov:2017dus], [Karsch, Karsch:2017zzw]
plot: [STAR, Adamczyk:2013dal]

Calculating observables II

$$\kappa_B \sigma_B^2 = \frac{\chi_4^B(T, \hat{\mu}_B)}{\chi_2^B(T, \hat{\mu}_B)} = r_{42}^{B,0} + \hat{\mu}_B^2 r_{42}^{B,2} + \hat{\mu}_B^4 r_{42}^{B,4} + \dots$$

The $r_{42}^{B,x}$ can be expressed in terms of the χ_{ijk}^{BQS}

$$r_{42}^{B,0} = \frac{\chi_4^B}{\chi_2^B}$$

$\longrightarrow$ up to 4th derivative

Calculating observables II

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The $r_{42}^{B,x}$ can be expressed in terms of the χ_{ijk}^{BQS}

$$r_{42}^{B,0} = \frac{\chi_4^B}{\chi_2^B}$$

$\longrightarrow$ up to 4th derivative

$$\begin{aligned}
r_{42}^{B,2} &= \frac{1}{\chi_2^B}\left(\frac{1}{2}\chi_{42}^{BS} s_1^2 + \chi_{411}^{BQS} q_1 s_1 + \frac{1}{2}\chi_{42}^{BQ} q_1^2 + \chi_{51}^{BS} s_1 + \chi_{51}^{BQ} q_1 + \frac{1}{2}\chi_6^B\right) \\
&- \frac{\chi_4^B}{(\chi_2^B)^2}\left(\frac{1}{2}\chi_{22}^{BS} s_1^2 + \chi_{211}^{BQS} q_1 s_1 + \frac{1}{2}\chi_{22}^{BQ} q_1^2 + \chi_{31}^{BS} s_1 + \chi_{31}^{BQ} q_1 + \frac{1}{2}\chi_4^B\right) \\
q_1 &= \frac{(0.4\chi_{11}^{BS} - \chi_{11}^{QS})\chi_{11}^{BS} - (0.4\chi_2^B - \chi_{11}^{BQ})\chi_2^S}{(0.4\chi_{11}^{BQ} - \chi_2^Q)\chi_2^S - (0.4\chi_{11}^{BS} - \chi_{11}^{QS})\chi_{11}^{QS}} \\
s_1 &= -\left(\frac{\chi_{11}^{QS}}{\chi_2^S}\,\frac{(0.4\chi_{11}^{BS} - \chi_{11}^{QS})\chi_{11}^{BS} - (0.4\chi_2^B - \chi_{11}^{BQ})\chi_2^S}{(0.4\chi_{11}^{BQ} - \chi_2^Q)\chi_2^S - (0.4\chi_{11}^{BS} - \chi_{11}^{QS})\chi_{11}^{QS}} + \frac{\chi_{11}^{BS}}{\chi_2^S}\right)
\end{aligned}$$

$\longrightarrow$ up to 6th derivative

Calculating observables II

$$\kappa_B \sigma_B^2 = \frac{\chi_4^B(T,\hat{\mu}_B)}{\chi_2^B(T,\hat{\mu}_B)} = r_{42}^{B,0} + \hat{\mu}_B^2 r_{42}^{B,2} + \hat{\mu}_B^4 r_{42}^{B,4} + \dots$$

The $r_{42}^{B,x}$ can be expressed in terms of the χ_{ijk}^{BQS}

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r_{42}^{B,2} &= \frac{1}{\chi_2^B}\left(\frac{1}{2}\chi_{42}^{BS} s_1^2 + \chi_{411}^{BQS} q_1 s_1 + \frac{1}{2}\chi_{42}^{BQ} q_1^2 + \chi_{51}^{BS} s_1 + \chi_{51}^{BQ} q_1 + \frac{1}{2}\chi_6^B\right) \\
&- \frac{\chi_4^B}{(\chi_2^B)^2}\left(\frac{1}{2}\chi_{22}^{BS} s_1^2 + \chi_{211}^{BQS} q_1 s_1 + \frac{1}{2}\chi_{22}^{BQ} q_1^2 + \chi_{31}^{BS} s_1 + \chi_{31}^{BQ} q_1 + \frac{1}{2}\chi_4^B\right) \\
q_1 &= \frac{(0.4\chi_{11}^{BS} - \chi_{11}^{QS})\chi_{11}^{BS} - (0.4\chi_2^B - \chi_{11}^{BQ})\chi_2^S}{(0.4\chi_{11}^{BQ} - \chi_2^Q)\chi_2^S - (0.4\chi_{11}^{BS} - \chi_{11}^{QS})\chi_{11}^{QS}} \\
s_1 &= -\left(\frac{\chi_{11}^{QS}}{\chi_2^S}\,\frac{(0.4\chi_{11}^{BS} - \chi_{11}^{QS})\chi_{11}^{BS} - (0.4\chi_2^B - \chi_{11}^{BQ})\chi_2^S}{(0.4\chi_{11}^{BQ} - \chi_2^Q)\chi_2^S - (0.4\chi_{11}^{BS} - \chi_{11}^{QS})\chi_{11}^{QS}} + \frac{\chi_{11}^{BS}}{\chi_2^S}\right)
\end{aligned}$$

$\longrightarrow$ up to 6th derivative

$$r_{42}^{B,4} = \dots$$

$\longrightarrow$ up to 8th derivative

Measured observables

On each ensemble we measure the $\chi^{BQS}_{i,j,k}$ up to the fourth derivative:

- χ^{B}_{1}
- χ^{B}_{2}
- χ^{B}_{3}
- χ^{B}_{4}

- χ^{Q}_{1}
- $\chi^{BQ}_{1,1}$
- $\chi^{BQ}_{1,2}$
- $\chi^{BQ}_{1,3}$

- χ^{S}_{1}
- $\chi^{BS}_{1,1}$
- $\chi^{BS}_{1,2}$
- $\chi^{BS}_{1,3}$

- χ^{Q}_{2}
- $\chi^{BQ}_{2,1}$
- $\chi^{BQ}_{2,2}$

- χ^{S}_{2}
- $\chi^{BS}_{2,1}$
- $\chi^{BS}_{2,2}$

- $\chi^{QS}_{1,1}$
- $\chi^{BQS}_{1,1,1}$
- $\chi^{BQS}_{2,1,1}$

- χ^{Q}_{3}
- $\chi^{BQ}_{3,1}$

- χ^{S}_{3}
- $\chi^{BS}_{3,1}$

- $\chi^{QS}_{2,1}$
- $\chi^{BQS}_{1,2,1}$

- $\chi^{QS}_{1,2}$
- $\chi^{BQS}_{1,1,2}$

- χ^{Q}_{4}

- χ^{S}_{4}

- $\chi^{QS}_{3,1}$

- $\chi^{QS}_{2,2}$

- $\chi^{QS}_{1,3}$

Measured observables

On each ensemble we measure the $\chi^{BQS}_{i,j,k}$ up to the fourth derivative:

- χ^{B}_{1}
- χ^{B}_{2}
- χ^{B}_{3}
- χ^{B}_{4}

- χ^{Q}_{1}
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- $\chi^{BQ}_{1,2}$
- $\chi^{BQ}_{1,3}$

- χ^{S}_{1}
- $\chi^{BS}_{1,1}$
- $\chi^{BS}_{1,2}$
- $\chi^{BS}_{1,3}$

- χ^{Q}_{2}
- $\chi^{BQ}_{2,1}$
- $\chi^{BQ}_{2,2}$

- χ^{S}_{2}
- $\chi^{BS}_{2,1}$
- $\chi^{BS}_{2,2}$

- $\chi^{QS}_{1,1}$
- $\chi^{BQS}_{1,1,1}$
- $\chi^{BQS}_{2,1,1}$

- χ^{Q}_{3}
- $\chi^{BQ}_{3,1}$

- χ^{S}_{3}
- $\chi^{BS}_{3,1}$

- $\chi^{QS}_{2,1}$
- $\chi^{BQS}_{1,2,1}$

- $\chi^{QS}_{1,2}$
- $\chi^{BQS}_{1,1,2}$

- χ^{Q}_{4}

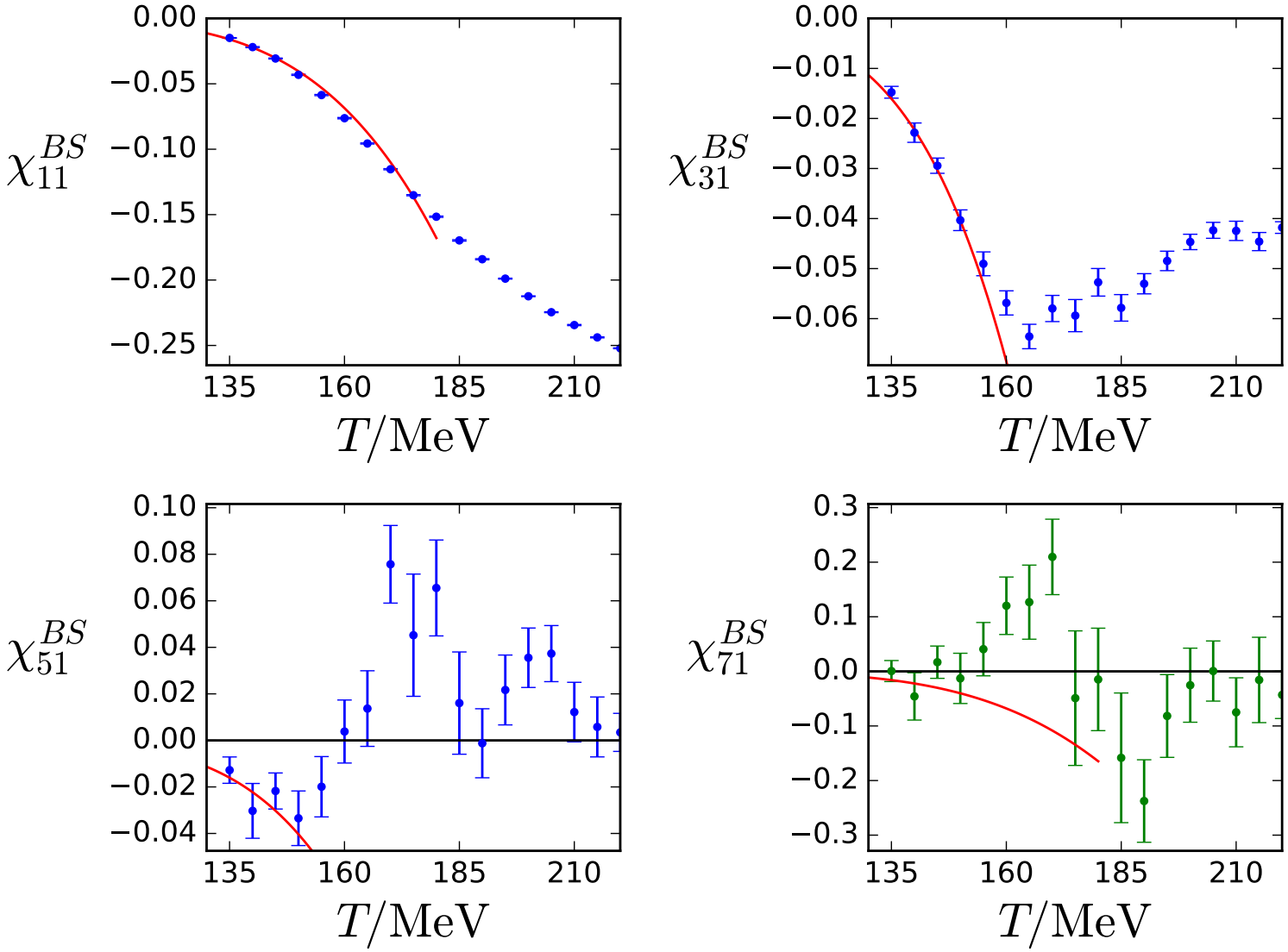
- χ^{S}_{4}

- $\chi^{QS}_{3,1}$

- $\chi^{QS}_{2,2}$

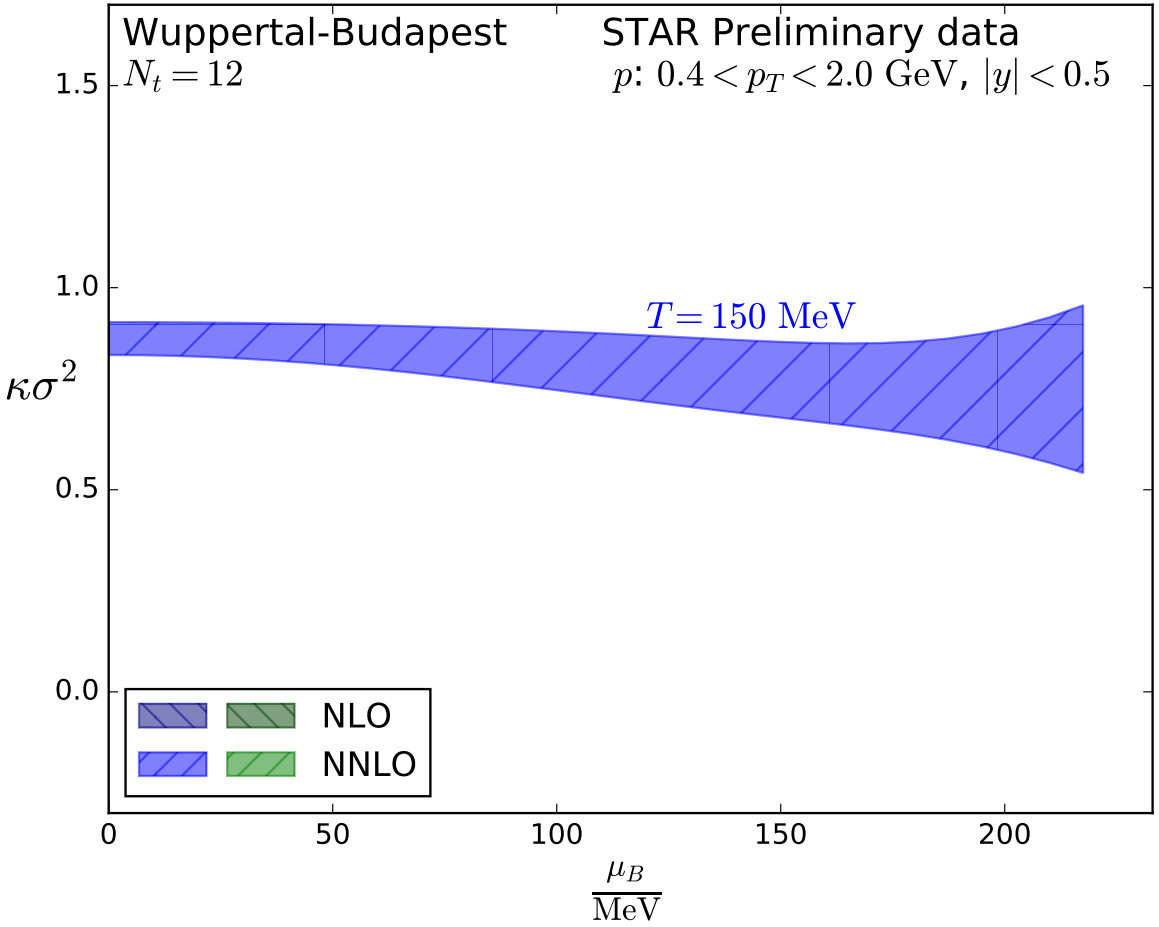
- $\chi^{QS}_{1,3}$

χ_{11}^{BS}, χ_{31}^{BS}, χ_{51}^{BS} and χ_{71}^{BS}



Extrapolation $\kappa\sigma^2$

$$\kappa_B\sigma_B^2 = \frac{\chi_4^B(T,\hat{\mu}_B)}{\chi_2^B(T,\hat{\mu}_B)} = r_{42}^{B,0} + \hat{\mu}_B^2 r_{42}^{B,2} + \hat{\mu}_B^4 r_{42}^{B,4} + \dots$$



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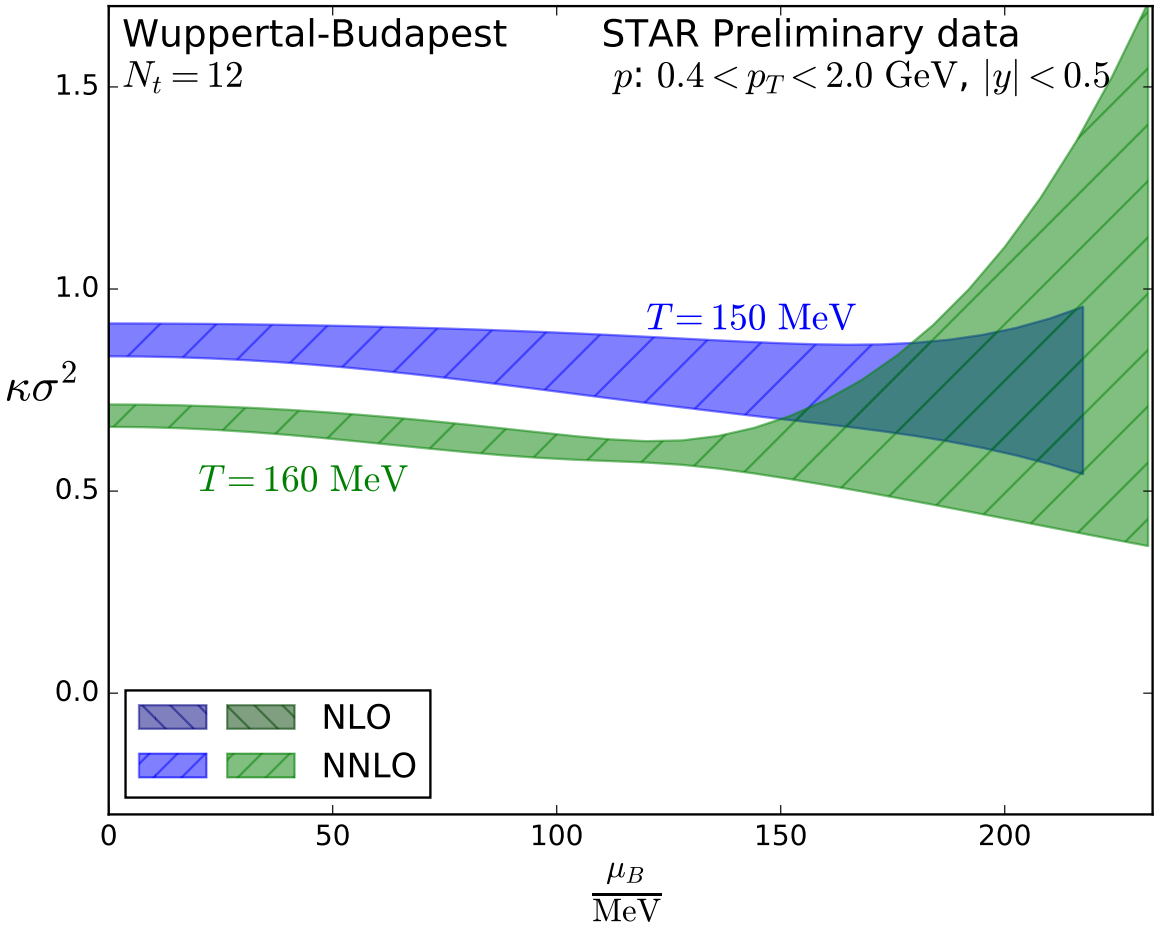

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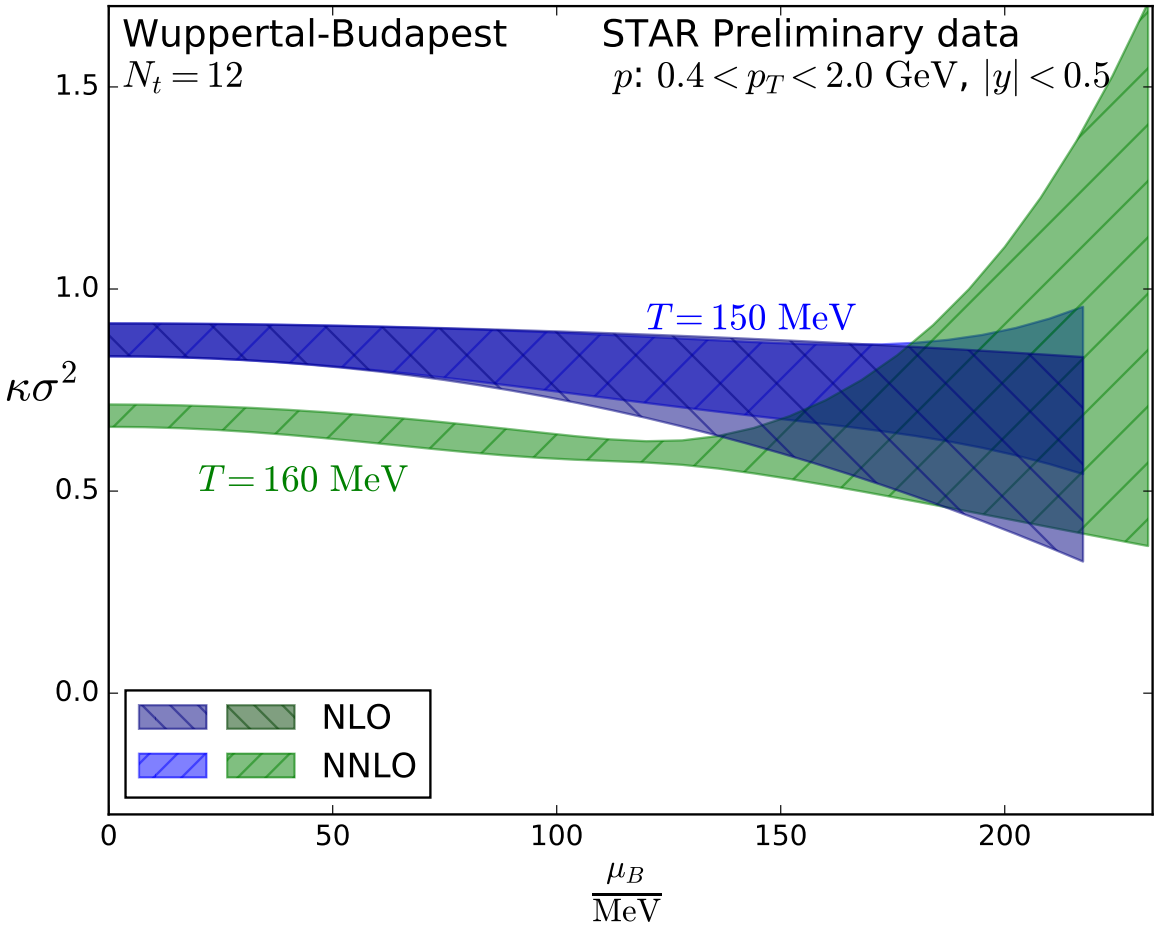
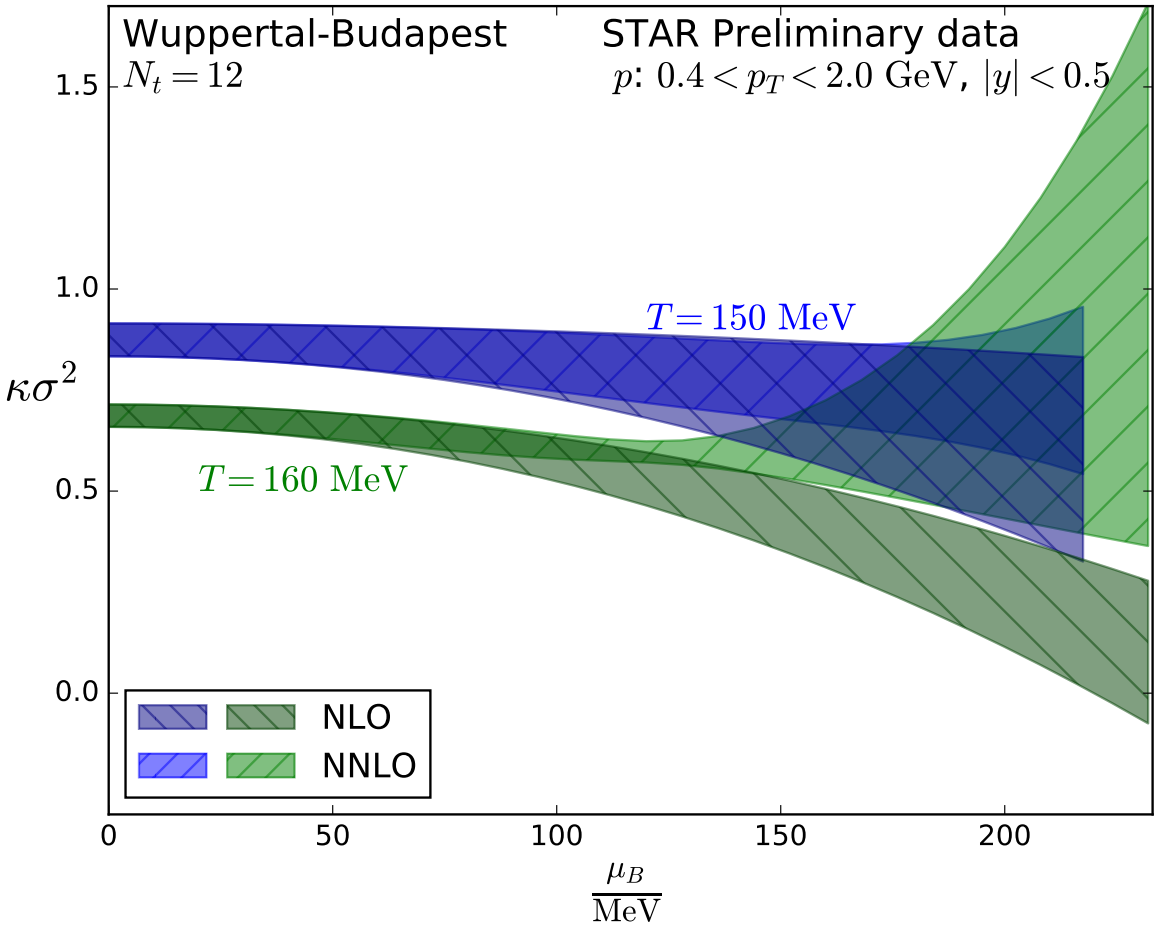


Extrapolation $\kappa\sigma^2$

$$\kappa_B\sigma_B^2 = \frac{\chi_4^B(T,\hat{\mu}_B)}{\chi_2^B(T,\hat{\mu}_B)} = r_{42}^{B,0} + \hat{\mu}_B^2 r_{42}^{B,2} + \hat{\mu}_B^4 r_{42}^{B,4} + \dots$$



Extrapolation $\kappa\sigma^2$

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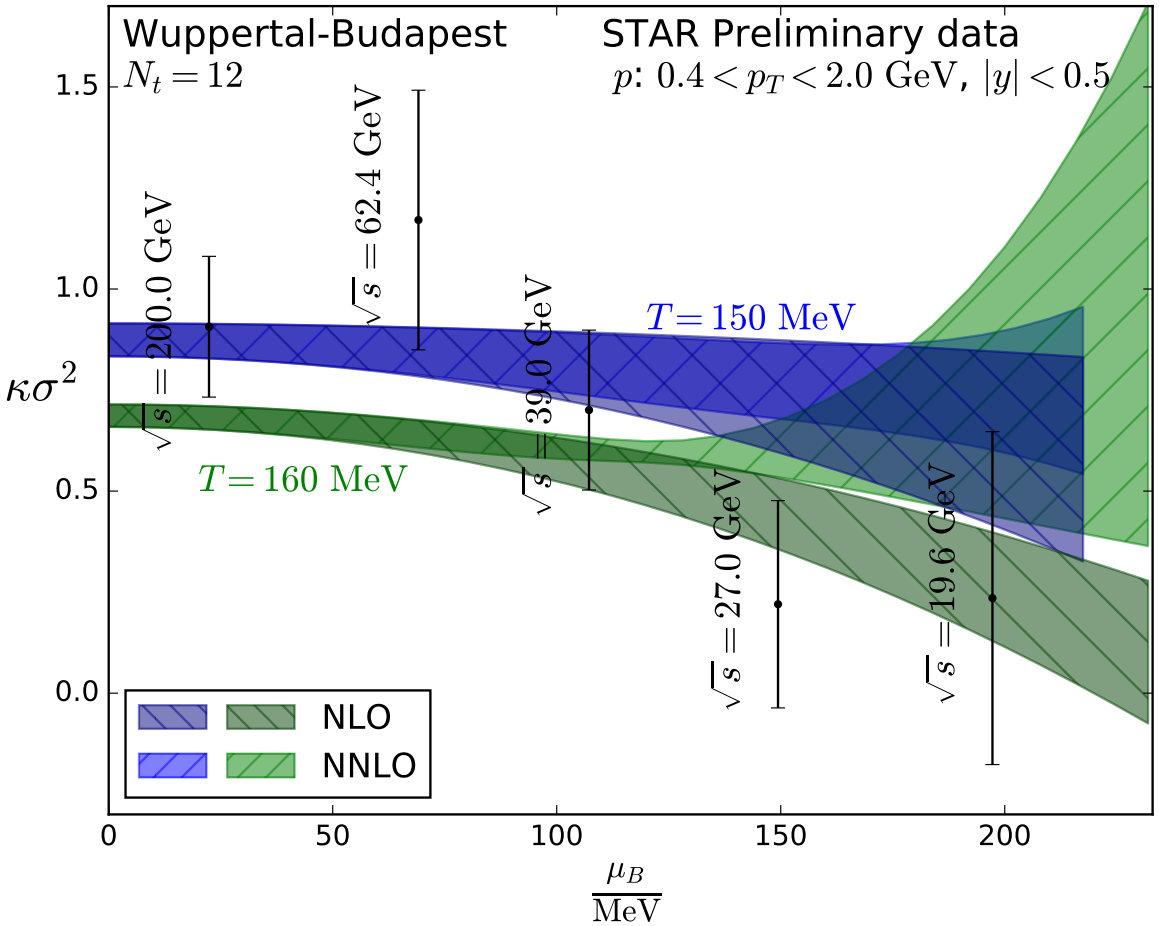
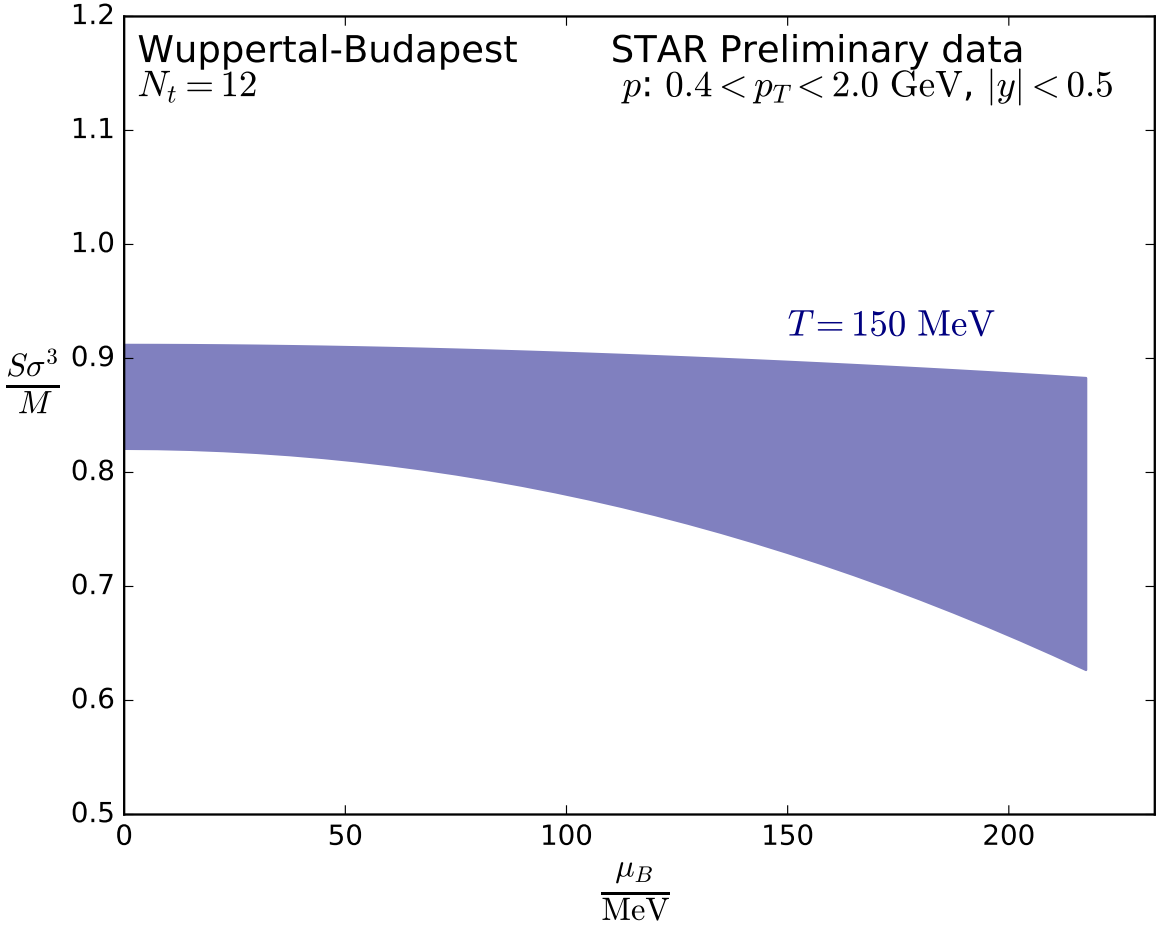


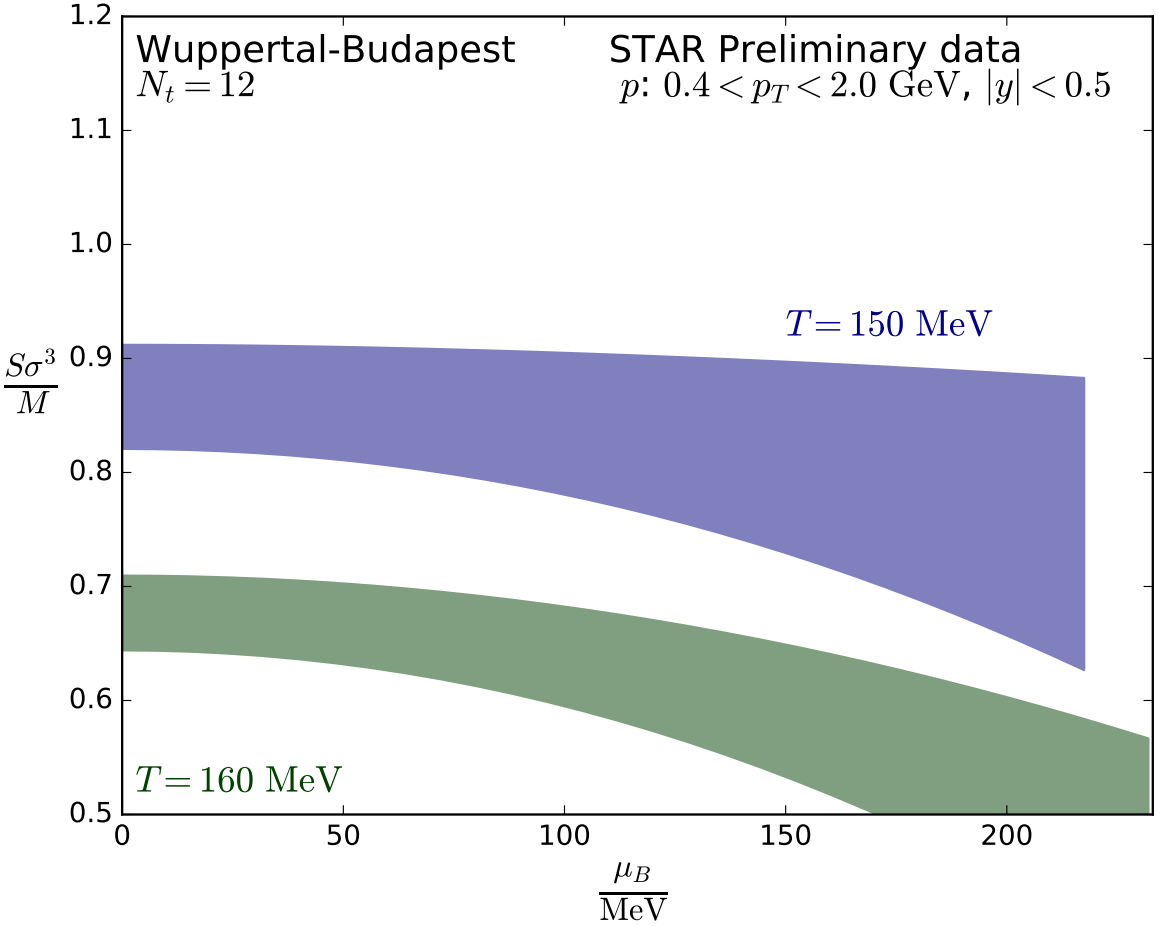
Extrapolation $\kappa\sigma^2$

$$\frac{S_B\sigma_B^3}{M_B} = \frac{\chi_3^B(T,\hat{\mu}_B)}{\chi_1^B(T,\hat{\mu}_B)} = r_{31}^{B,0} + \hat{\mu}_B^2 r_{31}^{B,2} + \dots$$



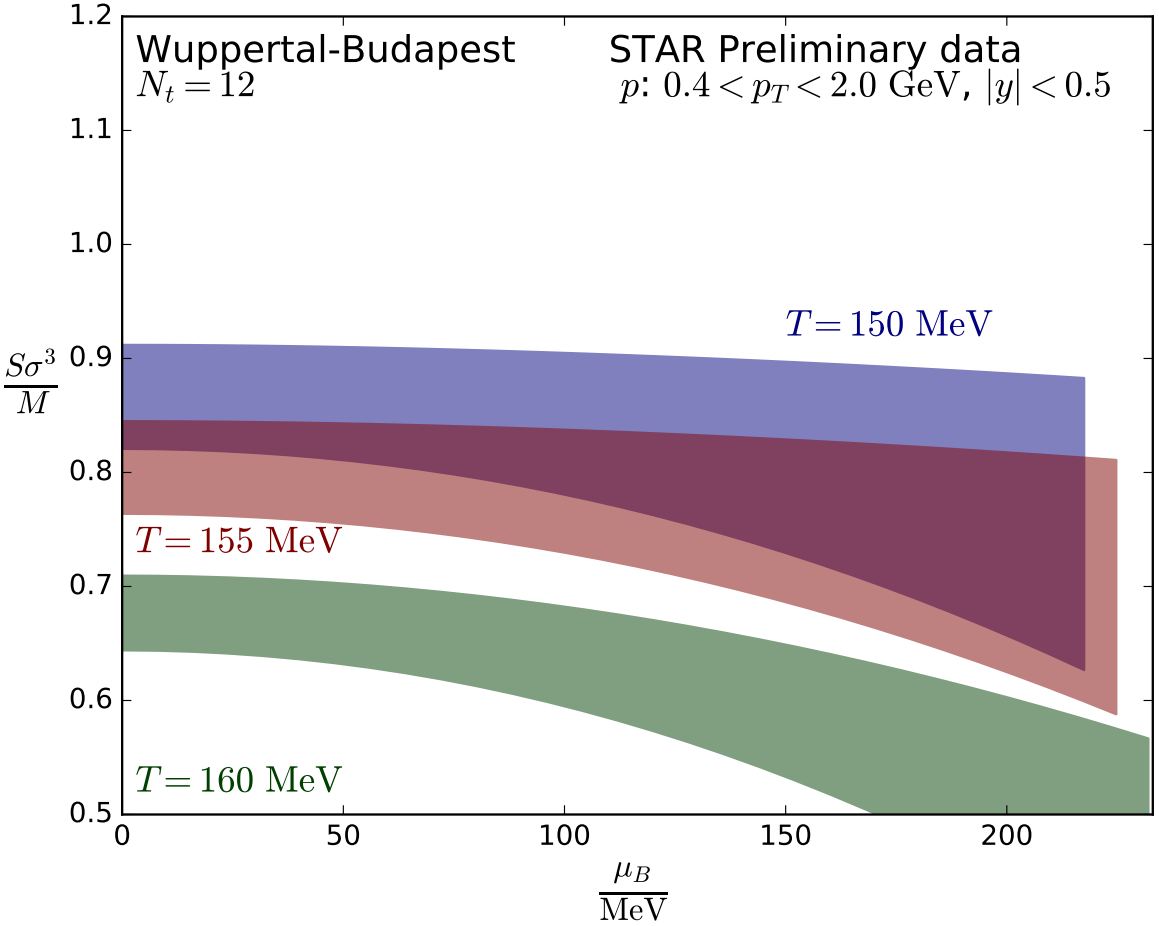
Extrapolation $\kappa\sigma^2$

$$\frac{S_B\sigma_B^3}{M_B} = \frac{\chi_3^B(T,\hat{\mu}_B)}{\chi_1^B(T,\hat{\mu}_B)} = r_{31}^{B,0} + \hat{\mu}_B^2 r_{31}^{B,2} + \dots$$



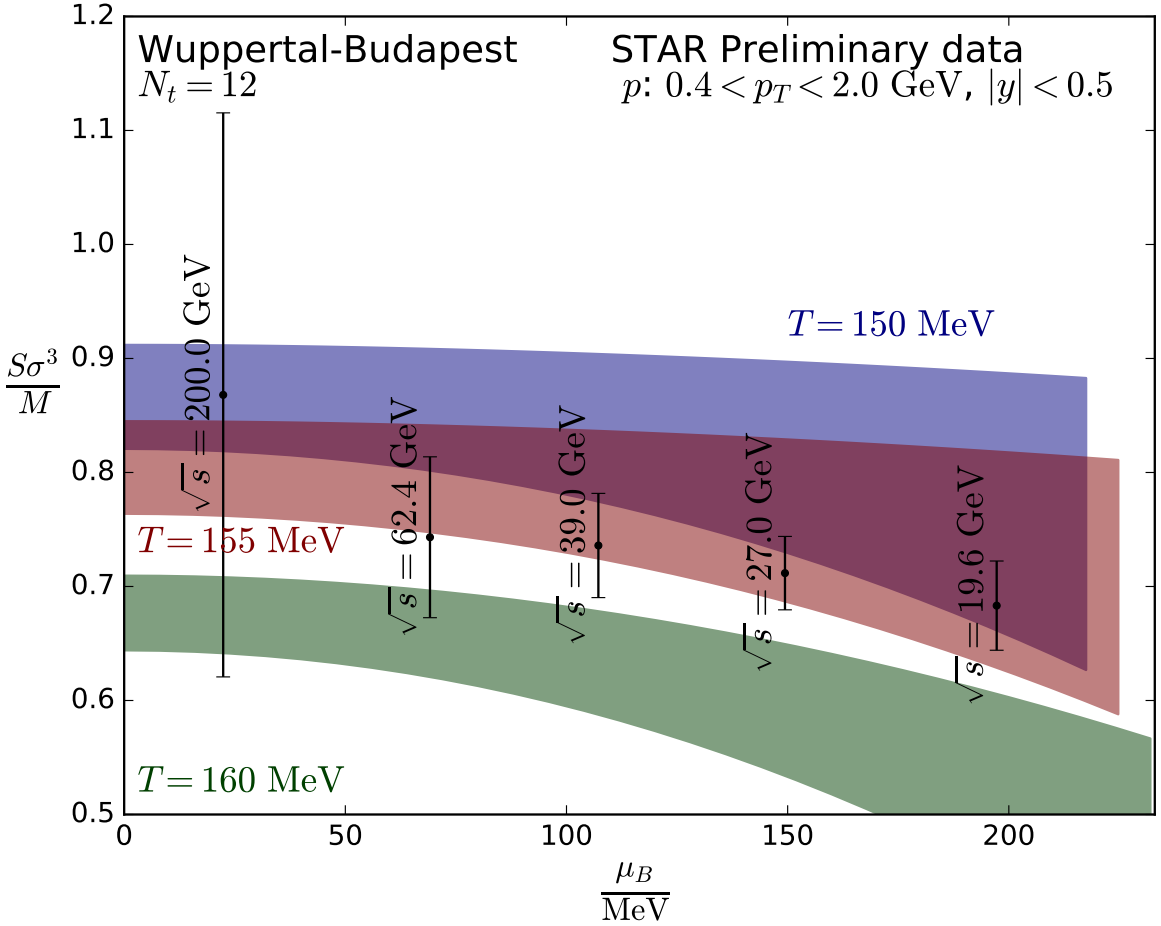
Extrapolation $\kappa\sigma^2$

$$\frac{S_B\sigma_B^3}{M_B} = \frac{\chi_3^B(T,\hat{\mu}_B)}{\chi_1^B(T,\hat{\mu}_B)} = r_{31}^{B,0} + \hat{\mu}_B^2 r_{31}^{B,2} + \dots$$

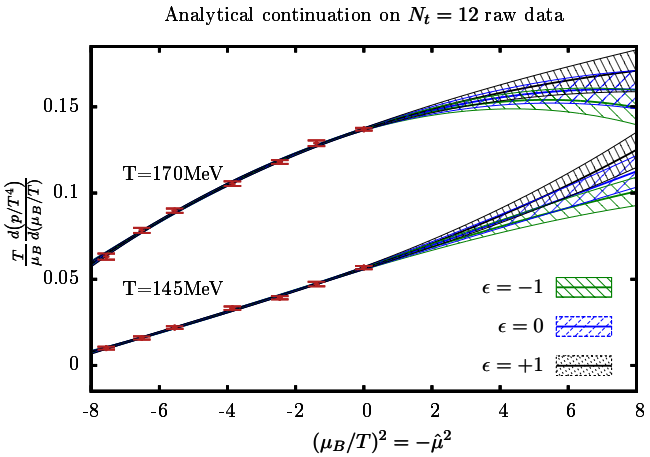


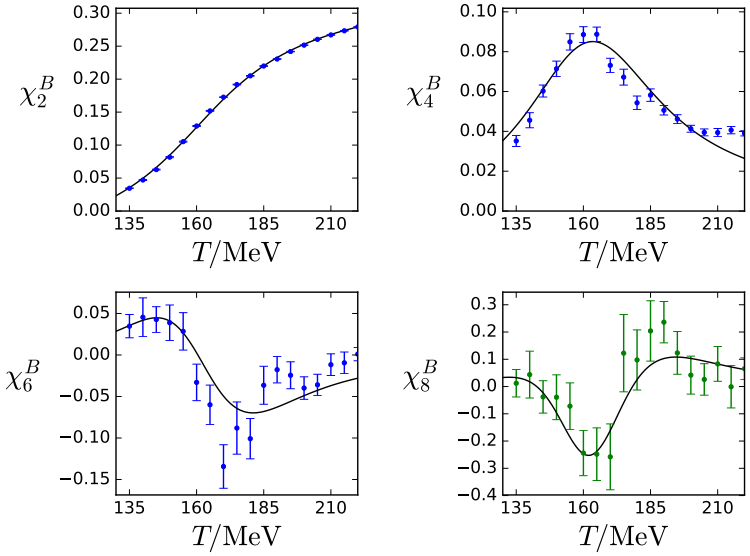
Extrapolation $\kappa\sigma^2$

$$\frac{S_B\sigma_B^3}{M_B} = \frac{\chi_3^B(T,\hat{\mu}_B)}{\chi_1^B(T,\hat{\mu}_B)} = r_{31}^{B,0} + \hat{\mu}_B^2 r_{31}^{B,2} + \dots$$



Summary





The μ_B dependence can be written in terms of the Taylor expansion:

$$\chi^{BQS}_{i,j,k}(\hat{\mu}_B) = \chi^{BQS}_{i,j,k}(0) + \hat{\mu}_B \left[\chi^{BQS}_{i+1,j,k}(0) + q_1 \chi^{BQS}_{i,j+1,k}(0) + s_1 \chi^{BQS}_{i,j,k+1}(0)\right] + \frac{1}{2}\hat{\mu}_B^2 \left[\chi^{BQS}_{i+2,j,k}(0) + s_1^2 \chi^{BQS}_{i,j+2,k}(0) + q_1^2 \chi^{BQS}_{i,j,k+2} + 2q_1 s_1 \chi^{BQS}_{i,j+1,k+1}(0) + 2s_1 \chi^{BQS}_{i+1,j+1,k}(0) + 2q_1 \chi^{BQS}_{i+1,j,k+1}(0)\right] + \ldots$$