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Beyond Complex Langevin: a Progress Report

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I. The sign problem – Complex Langevin approach

Complex case, e.g. $\rho(x) \equiv e^{-S(x)} = e^{-\sigma x^2/2}, \sigma \in \mathcal{C}$

$$S(x) \quad \xrightarrow{\dot{z}(\tau)=-\partial_z S+\eta(\tau)} \quad z(\tau) \quad \longrightarrow \quad P(x,y,\tau)$$

But in general $P(x,y,\tau)$ does not $\xrightarrow{\tau\to\infty} \quad e^{-S}$

Nevertheless in some cases $P(x,y,\infty)$ exists, and indeed

$$\frac{\int f(x)e^{-S(x)}dx}{\int e^{-S(x)}dx} = \frac{\int\int f(x+iy)P(x,y)dxdy}{\int\int P(x,y)dxdy}.$$

Example (Ambjorn and Yang, 1985)

$$\rho(x) = \exp\left(-\frac{\sigma}{2}x^2\right), \quad \sigma = \sigma_R + i\sigma_I, \quad \sigma_R > 0$$

$$P(x,y) = \exp\left(-\sigma_R(x^2 + 2rxy + (1+2r^2)y^2)\right), \quad r = \frac{\sigma_R}{\sigma_I},$$

II. Beyond Complex Langevin

- **Construct ρ, P without any reference to the stochastic process.**

The only requirement (matching conditions) (L. Salcedo, E. Seiler, JW)

$$\int_R f(x)\rho(x)dx \Big/ \int_R \rho(x)dx = \int\int f(x+iy)P(x,y)dxdy \Big/ \int\int P(x,y)dxdy \,.$$

- - **Introduce two, independent complex variables (JW)**

$$z = x + iy, \quad \bar{z} = x - iy$$

then

$$\int_{\Gamma_z} f(z)\rho(z)dz \Big/ \int_{\Gamma_z} \rho(z)dz = \int_{\Gamma_z}\int_{\Gamma_{\bar{z}}} f(z)P(z,\bar{z})dzd\bar{z} \Big/ \int_{\Gamma_z}\int_{\Gamma_{\bar{z}}} P(z,\bar{z})dzd\bar{z}$$

sufficient condition

$$\rho(z) = \int_{\Gamma_{\bar{z}}} P(z,\bar{z})d\bar{z}$$

additionally require

$$P(z,\bar{z})|_{x+iy,x-iy} \quad \text{positive and normalizable}$$

The construction

1. Find $P(z,\bar{z})$ which satisfies

$$\rho(z) = \int_{\Gamma_{\bar{z}}} P(z,\bar{z}) d\bar{z}$$

and

2. Is positive and normalizable at

$$P(z,\bar{z})|_{x+iy,x-iy} = P(x,y)$$

- Then

$$\int f(z)\rho(z)dz \Big/ \int \rho(z)dz = \int\int f(x+iy)P(x,y)dxdy \Big/ \int\int P(x,y)dxdy$$

Example 1 - generalized gaussian model

$$S(z,\bar z) = a^* z^2 + 2bz\bar z + a\bar z^2 \; |_{x+iy,x-iy} = \; 2(b+\alpha)x^2 + 4\beta xy + 2(b-\alpha)y^2,$$

$$a = \alpha + i\beta, \quad b = b^*, \qquad \lambda_\pm = 2(b \pm |a|)$$

$$P(x,y) = e^{-S(x,y)}$$ is positive and normalizable for $b > |a|$

At the same time

$$\rho(z) = \int_{\Gamma_{\bar z}} P(z,\bar z)d\bar z = \frac{1}{2}\sqrt{\frac{\pi}{-a}}\exp\left(-sz^2\right), \quad s = \frac{|a|^2 - b^2}{a}.$$

This reduces to $\exp\left(-\frac{\sigma}{2}z^2\right)$ if

$$b = \frac{\sigma_R}{2}(1+r^2), \quad \alpha = -\frac{\sigma_R}{2}r^2, \quad \beta = \frac{\sigma_R}{2}r, \quad r = \frac{\sigma_R}{\sigma_I}, \quad \sigma_R > 0 \qquad (1)$$

- but is more general.

•• Provides positive representation for any complex a.

$$\langle x^n \rangle_{\rho(x)} \longleftarrow \langle z^n \rangle_{\rho(z)} = \langle (x+iy)^n \rangle_{P(x,y)}$$

For example:

$\alpha = 0, \beta \neq 0$ – $e^{i|s|x^2}$ – "Minkowski" integrals

$\alpha > 0, \beta = 0$ – $e^{+|s|x^2}$ – "a striking example"

• • • $\int f(x+iy)P(x,y)dxdy$ = analytic continuation of $\int \rho(x)f(x)dx$

(1) $\Longrightarrow$ one parameter family of solutions

Błażej Ruba:

$$a = -\mu\sigma^*, \quad b = |\sigma|\sqrt{\mu(\mu+1)}$$

- **interesting limts**

$$\mu \longrightarrow \begin{cases} \left(\frac{Re\sigma}{Im\sigma}\right)^2, & P(x,y) \Rightarrow AY \\ \\ \infty, & P(x,y) \rightarrow \delta(Im\sqrt{\sigma}z)\exp\left(-\sigma z^2\right) \end{cases} \qquad (2)$$

$\Longrightarrow$ **a thimble !**

●● **multi-dimensional Cauchy equation**

$$\int_{\Gamma_z}\int_{\Gamma_{\bar{z}}} f(z)P(z,\bar{z})dzd\bar{z} = \int_{\mathcal{R}^2} f(x+iy)P(x,y)dxdy$$

Example 2 - quartic action

$$S_4(z, \bar{z}) = (a^* z^2 + 2bz\bar{z} + a\bar{z}^2)^2,$$

$$\rho_4(z) \ = \ \frac{i}{2}\int_{\Gamma_{\bar{z}}} d\bar{z} e^{-S_4(z,\bar{z})} = \mathcal{N}\sqrt{\sigma z^2}\exp\left(-\frac{\sigma^2 z^4}{2}\right) K_{\frac{1}{4}}\left(\frac{\sigma^2 z^4}{2}\right),$$

with an arbitrary complex

$$\sigma = -\frac{b^2 - |a|^2}{a}.$$

again

$$\langle z^{2k}\rangle_{\rho_4(z)} = \frac{1}{\pi}\frac{\Gamma\left(\frac{2k+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)}\left(\frac{1}{2\sigma}\right)^k = \langle (x+iy)^n\rangle_{P(x,y)}$$

BR: infinite family of positive representations, as in the gussian case, and

$$\mu \to \infty \quad P(x, y) \to \delta(Im\sqrt{\sigma} z)\rho_4(z)$$

(linear) thimbles again !

III. Path integrals in Minkowski time – a free particle

2N variables $z_i, \bar{z}_i$ $(\equiv z, \bar{z})$ with periodic boundary conditions

$$S_N(z,\bar{z}) = \sum_{i=1}^{N} a\bar{z}_i^2 + 2b\bar{z}_i z_i + 2c\bar{z}_i z_{i+1} + 2c^* z_i \bar{z}_{i+1} + a^* z_i^2, \quad a, c \in C, b \in R.$$

$$\rho(z) \sim \int \prod_{i=1}^{N} d\bar{z}_i \exp\left(-S_N(z,\bar{z})\right) \sim \exp\left(\mathcal{A}\sum_{i=1}^{N}\left(z_{i+1}-z_i\right)^2 - r\left(z_{i+1}-z_{i-1}\right)^2\right)$$

$$2c \equiv 2\gamma = -b + |a|, \qquad \mathcal{A} = \frac{b(b-|a|)}{a}, \qquad r = \frac{b-|a|}{4b}.$$

to be compared with $$S_N^{free} = \frac{im}{2\hbar\epsilon}\sum_{i=1}^{N}(z_{i+1}-z_i)^2$$

- define the new limit (lim_1)

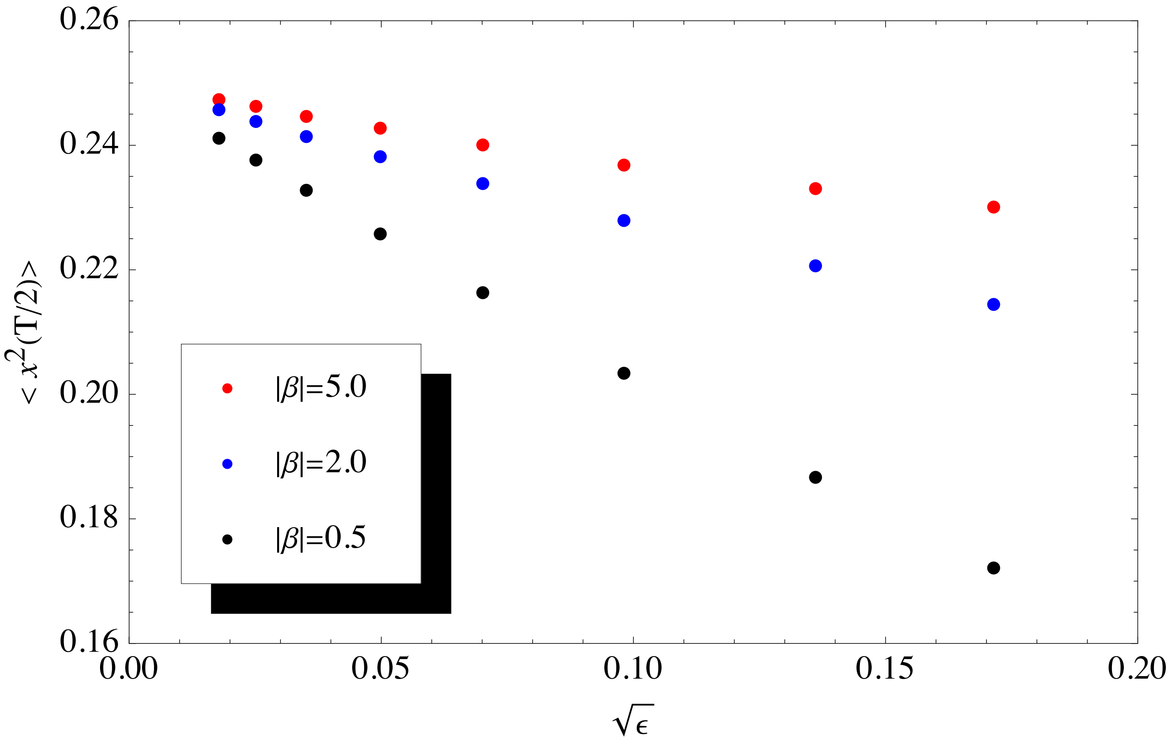
$$|a|, b \to \infty, b - |a| = \frac{m}{2\hbar\epsilon} = const., \quad a = -i|a| \equiv i\beta.$$

- than physics $= lim_{N\to\infty}\ lim_1\ \langle O\rangle_{P_N(x,y)}$

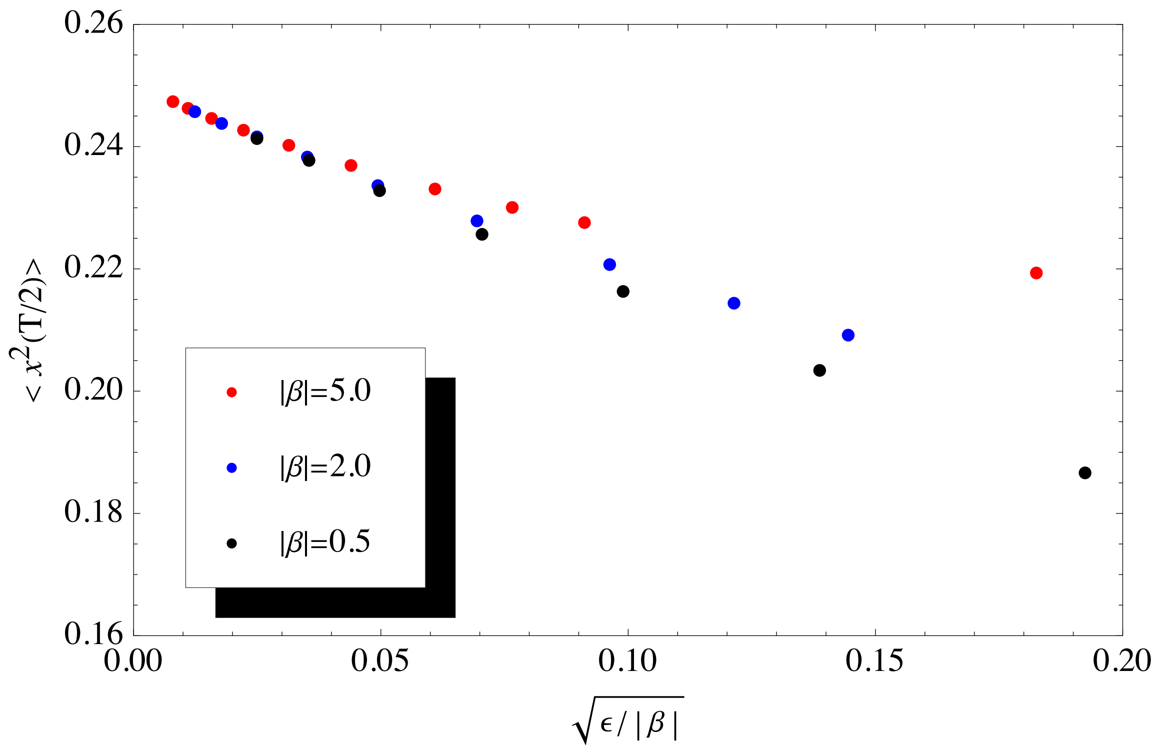
BR : one may also keep the r-term
The only condition:

$$\frac{|\beta|}{\epsilon} >> 1 \qquad (3)$$

can take the continuum limit at fixed, finite β as well



Moreover: $\frac{|\beta|}{\epsilon}$ **is also a natural, i.e. scaling, variable**



Again Błażej finds thimbles (in functional space)

$$P \stackrel{\beta\to-\infty}{\longrightarrow} \mathcal{N} \prod_{j=1}^{N} \delta(x_j - y_j) e^{\frac{im}{2\epsilon}(z_{j+1}-z_j)} \tag{4}$$

IV. Harmonic oscillator in Minkowski time

The continuum limit for HO

$$\rho = \frac{\omega^2 T^2}{2(N-1)^2}, \quad \mu = \frac{m(N-1)}{2\hbar T},$$

$$a = -i|a|, \quad b = \frac{\mu}{\nu}, \quad |a| = \frac{\mu}{\nu}\zeta(\nu,\rho), \quad 2\gamma = -\mu\zeta(\nu,\rho)$$

where

$$\zeta(\nu,\rho) = \frac{\sqrt{1-2\nu^2\rho+\nu^2\rho^2} - \nu(1-\rho)}{1-\nu^2}$$

$$\langle O \rangle = \overbrace{\lim_{N\to\infty}}^{lim2} \quad \overbrace{\lim_{\nu\to 0}}^{lim1} \quad \langle O \rangle_{P_N(x,y)}$$

Harmonic oscillator - example 1

$$\langle x^2(T)\rangle = \langle x^2(0)\rangle \; = \; \frac{\int dx x^2 K(x,x;T)}{\int dx K(x,x;T)}.$$

Classics

$$K(x_b,x_a;T) \; \sim \; \exp\left\{\frac{i}{\hbar}\frac{m\omega}{2\sin\omega T}\Big((x_a^2+x_b^2)\cos\omega T - 2x_a x_b\Big)\right\},$$

gives

$$\langle x^2(T)\rangle \; = \; -\frac{i\hbar T}{4m}\frac{\cot\frac{\omega T}{2}}{\frac{\omega T}{2}}.$$

The new way

$$\langle z_1^2\rangle = \frac{1}{Z}\int\prod_{j=1}^{N} dx_j dy_j (x_1+iy_1)^2 \exp\left\{-X^T M X\right\}.$$

$$lim_1\ \langle z_1^2\rangle = \lim_{\nu\to 0}\langle z_1^2\rangle = -\frac{i\hbar T}{m}\frac{P_N(\omega T/2)}{Q_N(\omega T/2)}.$$

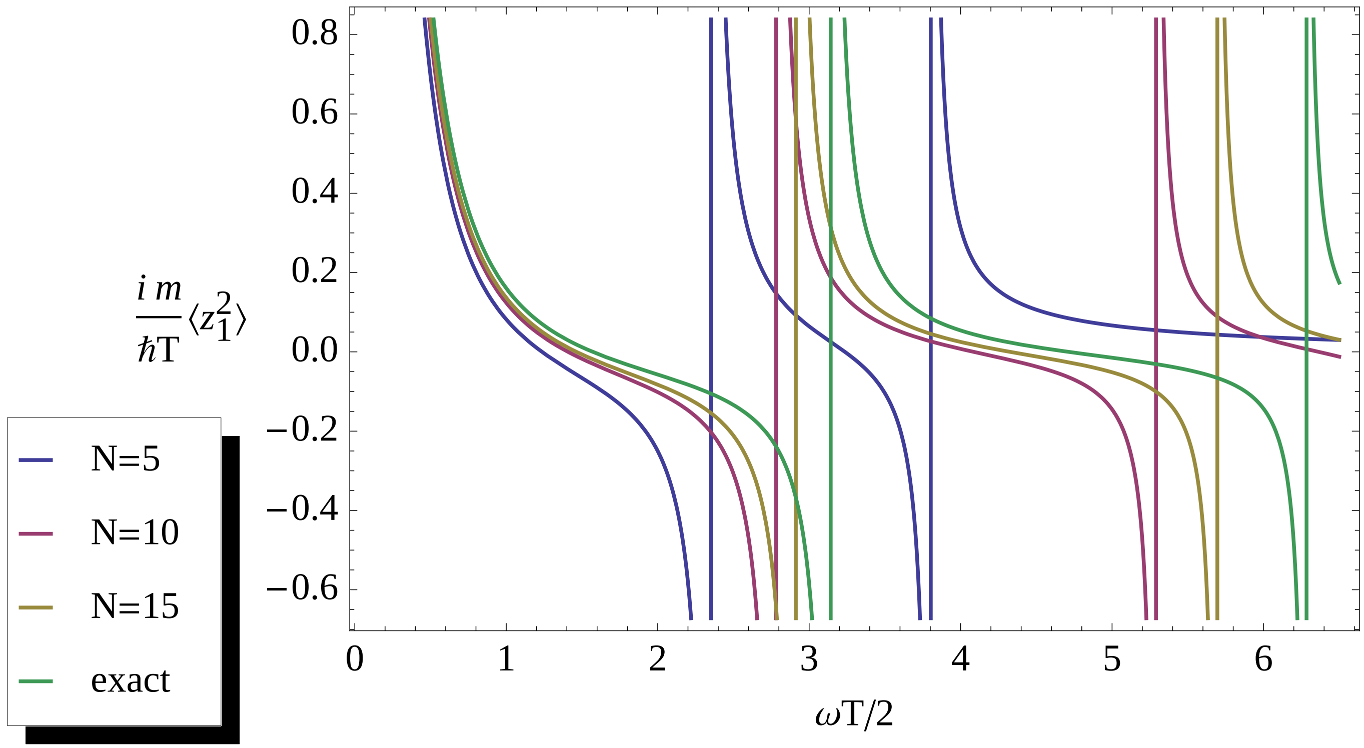
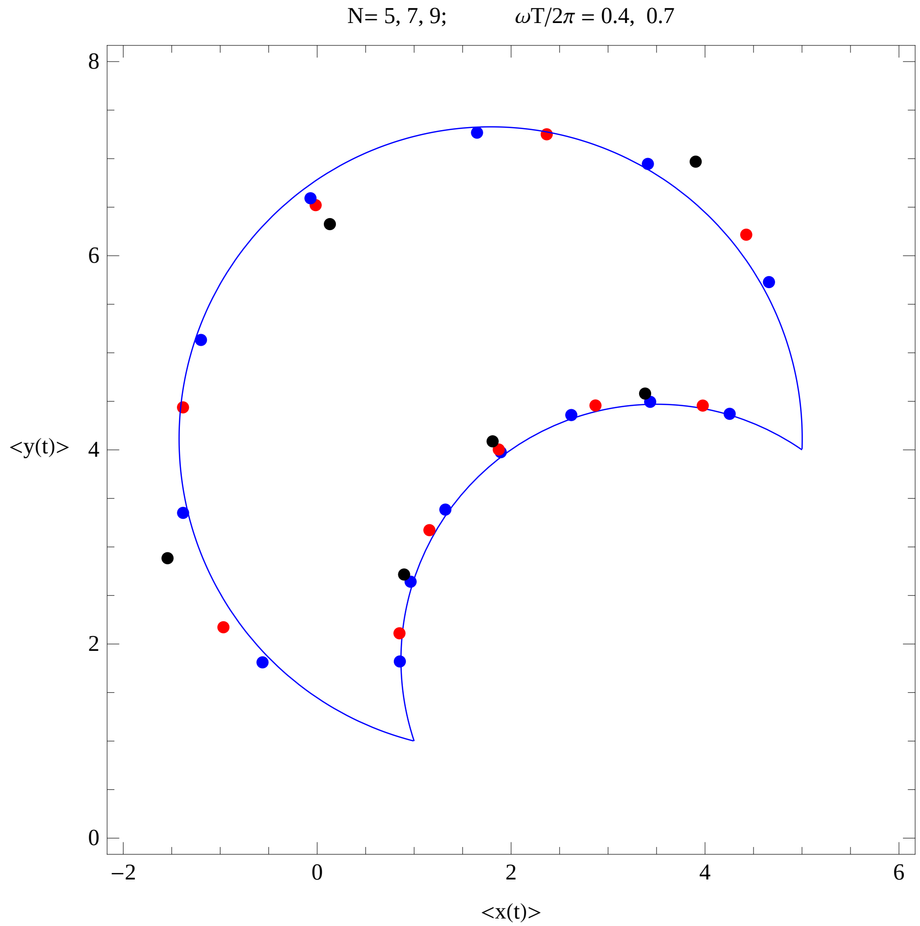


Figure 1:

Example 2. Charged particle in a constant magnetic field



• the average is over positive probability

• lim1 of our average reproduces the standard discretization

• one negative mode

The lowest eigenvalue

$$\lambda_0 = 2(b - |a| + 2\gamma) = \begin{cases} 0 & \text{a free particle} \\ \xrightarrow{lim1} -\frac{m\omega^2 T}{4\hbar(N-1)}, & \text{an harmonic oscillator} \end{cases} \quad (5)$$

One can:

fix it
regularize it – M^{-1} exists
⟶ use our trick again
relax periodic boundary conditions

Ruba's solution

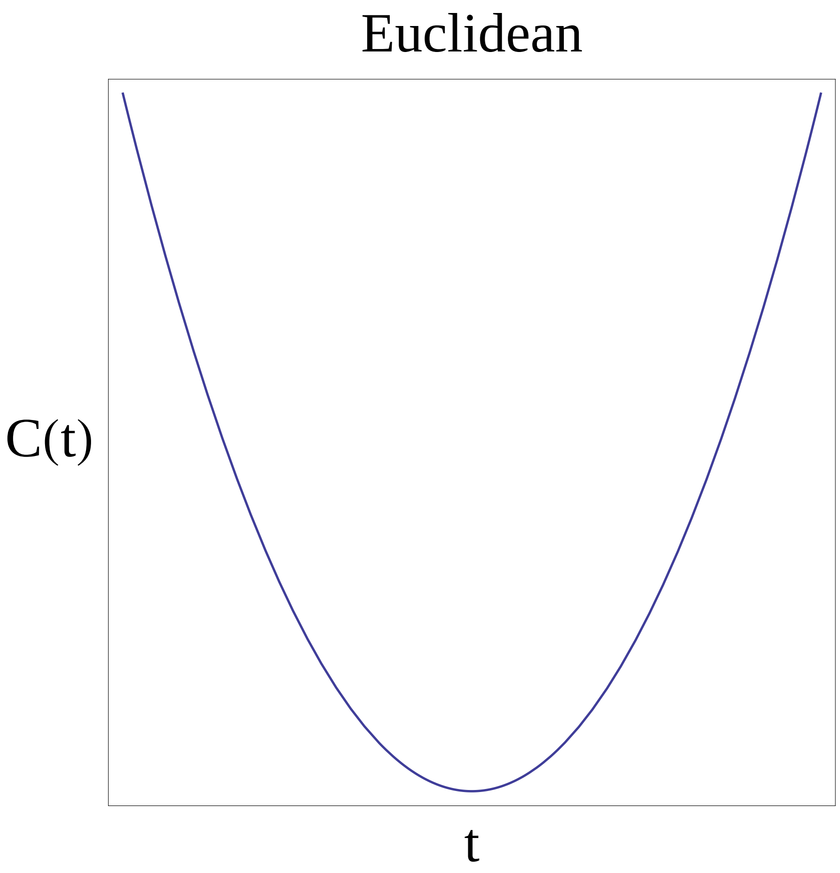
Interpretation

Negative eigenvalues $\Longleftarrow$ Morse Theorem

Classical path passes a focal point $\Longrightarrow$ a negative eigenvalue appears

Procedure

- **Identify negative eigenvalues.**
- **Rotate contours of integrations for negative modes by $\sqrt{-i}$.**
- **Do MC with resulting non-local action, but with strictly positive and normalizable weights.**



Minkowski $\longrightarrow$ **B R**

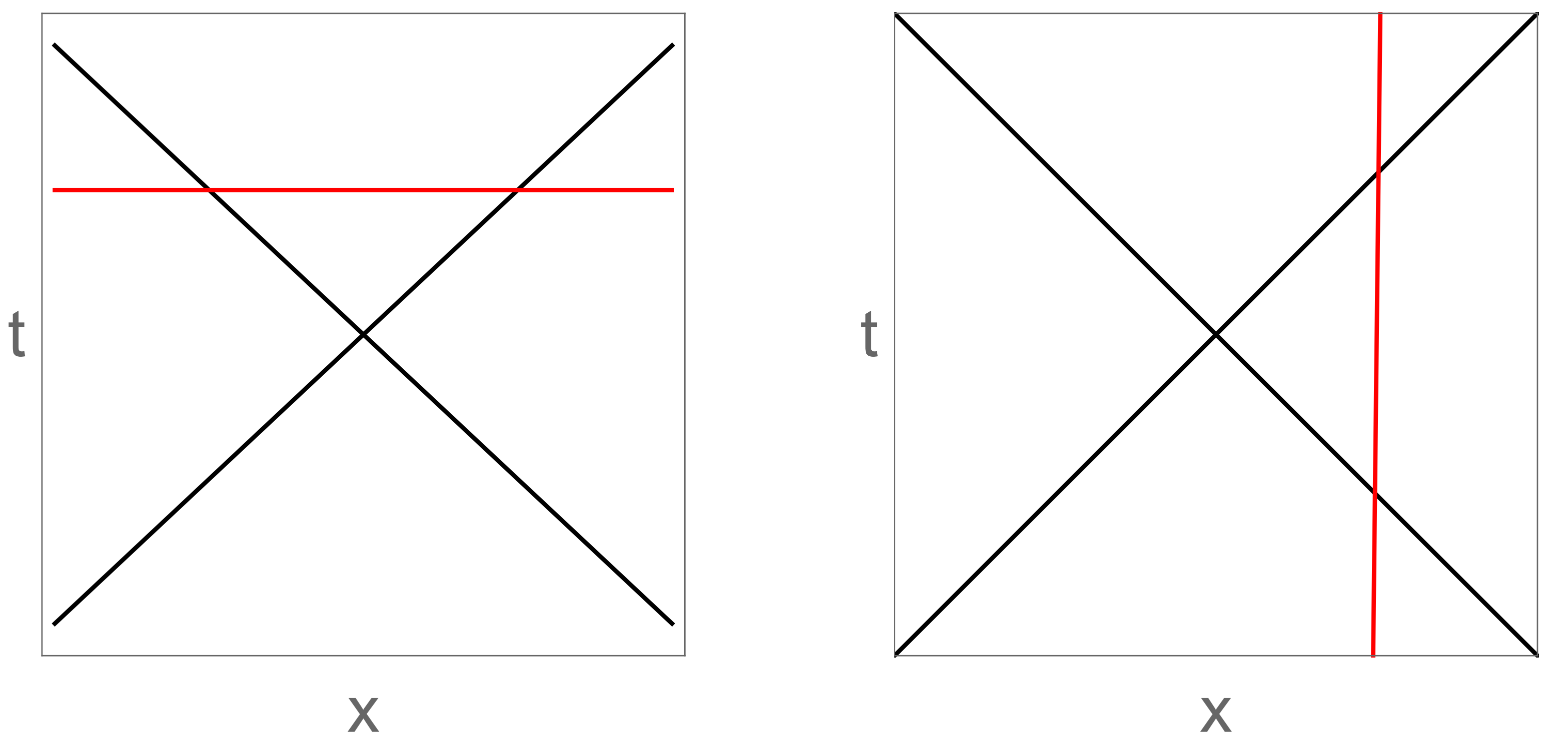
V. Free scalar theory

Focal points for each oscillator

New eigenvalues appear for each oscillator with increasing T

 # negative eigenvalues $\sim$ # of oscillators $= V^{D-1}$

$\Longrightarrow$ **Ruba's procedure works also for gaussian FT's**


t
x
t
x

VI. Summary

Beyond Complex Langevin approach:

Instead of simulating badly converging complex random walks, pairs of corresponding weights were directly constructed.

One degree of freedom:

gaussian model was generalized to arbitrary complex slope
a particular quartic problem was also solved

Quantum mechanical path integrals directly in Minkowski time:

a free particle
an harmonic oscillator
a particle in a constant magnetic field – standard Wick rotation does not give positive representation in this case

New

- **Focal points $\Longrightarrow$ Negative eigenvalues**

• • **Simple algorithm was formulated to avoid the problem**

Minkowski correlation functions were calculated with MC

• • • **Also done for free field theories with Minkowski signature**

• • • **Shopping for interesting applications**

possible example: real time evolutions in external fields