Towards a Dual Representation of Lattice QCD

G. Gagliardi, W. Unger

Universität Bielefeld

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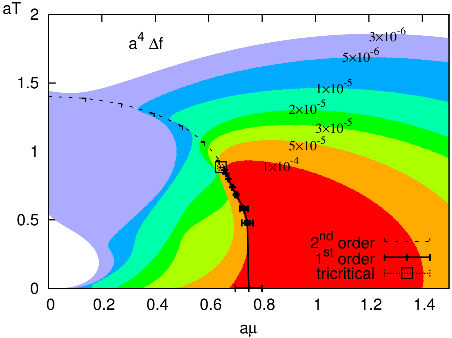
Motivation for Strong Coupling QCD

- Strong coupling expansion is a Taylor expansion in the lattice gauge coupling $\beta = \frac{2N_c}{g^2}$.
- At $\beta = 0$ the gauge d.o.f. can be integrated out analytically giving rise to a dual formulation.
- Average complex phase $\langle e^{i\phi} \rangle_{pq} = e^{-\frac{V}{T}(f_{pq} - f)}$ close to one.

$\Longrightarrow$ **sign problem** is mild and **phase diagram** can be mapped out.



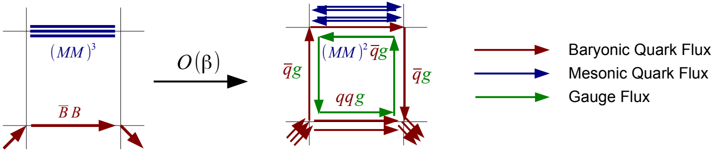
[W. Unger et al, arXiv:1406.4397].

$$Z_{sc} = \int [dU] \int d\chi\bar{\chi}\, e^{-(\cancel{S_g} + S_f)}$$

Sign problem is **representation dependent!**

Strong Coupling QCD and dual representation.

- *Dual representation: rewriting the partition function in terms of new degrees of freedom* (**dual variables**). At $\beta = 0$ just color singlets **[mesons & baryons]** $\Longrightarrow$ **MDP formulation** [Wolff & Rossi, 1984].
- β corrections to strong coupling needed to decrease the lattice spacing. Plaquette excitations produce **world sheets** bounded by **quark fluxes**:



- **Caveat:** Dual representation does not solve, *per sè*, the sign problem.
- **Goal:** Find a representation where the gluon dynamics does not reintroduce a *sign problem*.

First step: Find a dual, sign problem free, formulation for pure Yang-Mills theory valid for all β.

Overview of the available dual formulations

Pure Yang-Mills theory:

- Alternative formulation for $SU(3)$ using Hubbard Stratonovich transformation: [H. Vairinhos & P. De Forcrand '14]
- Plaquette expansion for pure Yang-Mills SU(2) gauge theory: [Leme, Oliveira,Krein '17]

Dual formulations with matter fields:

- Nuclear Physics from lattice QCD at strong coupling: [P. De Forcrand & M. Fromm '09]
- Dual lattice simulation of the U(1) gauge-Higgs model at finite density: [Mercado, Gattringer, Schmidt '13]
- Dual simulation of the massless lattice Schwinger model with topological term and non-zero chemical potential: [Göschl '17]
- Abelian color cycles (**ACC**): [Gattringer & Marchis '17]
- Dual $U(N)$ LGT with staggered fermions: [Borisenko et al '17]

Strong coupling expansion

A new strategy:

$$\mathcal{Z}_{YM} = \int_{SU(N)} [dU] e^{\frac{\beta}{2N_c} \sum_p \left[\mathrm{Tr}\, U_p + \mathrm{Tr}\, U_p^\dagger\right]}$$

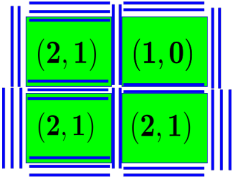
- **Taylor expand** the action in terms of plaquette(anti-plaquette) occupation numbers $\{n_p, \bar{n}_p\}$:

$$\mathcal{Z}_{YM} = \sum_{\{n_p, \bar{n}_p\}} \frac{(\beta/2N_c)^{\sum_p n_p + \bar{n}_p}}{\prod_p n_p! \bar{n}_p!} \underbrace{\prod_\ell \prod_p \int_{SU(N)} dU_\ell \left(\mathrm{Tr}\, U_p\right)^{n_p} \left(\mathrm{Tr}\, U_p^\dagger\right)^{\bar{n}_p}}_{\mathbf{W(\{n_p, \bar{n}_p\})}}$$

- **Plaquette constraint:** For each link $\ell = (x, \mu)$:

$$\sum_{\nu > \mu} \underbrace{\delta n_{x,\mu,\nu} - \delta n_{x-\nu,\mu,\nu}}_{\delta \mathbf{n_p = n_p - \bar{n}_p}} = \begin{cases} 0 & \mathbb{U}(\mathbb{N}) \\ 0 \; mod \; N & \mathbb{SU}(\mathbb{N}) \end{cases}$$

$$\underbrace{d_{\ell=(x,\mu)}}_{\textbf{dimer number}} := \min \left\{ \begin{array}{c} \sum_{\nu>\mu} n_{x,\mu,\nu} + \bar{n}_{x-\nu,\mu,\nu} \\ \sum_{\nu>\mu} \bar{n}_{x,\mu,\nu} + n_{x-\nu,\mu,\nu} \end{array} \right\}$$



Link Integration

How to compute the $W(\{n_p, \bar{n}_p\})$?

- The following quantities are needed:

$$\mathcal{I}^{a,b}_{i^j k^\ell} = \int_{SU(N_c)} dU U^{j_1}_{i_1} \ldots U^{j_a}_{i_a} (U^\dagger)^{\ell_1}_{k_1} \ldots (U^\dagger)^{\ell_b}_{k_b} \qquad | a - b | = q \cdot N_c$$

- Cases $q = 0$ and $q = 1$ addressed: [Collins '03,'06, Zuber '17].
- We extended their results by computing the **generating functional**:

$$Z^{a,b}[K,J] = \int_{SU(N)} dU \left[\mathrm{Tr}(UK)\right]^a \left[\mathrm{Tr}(U^\dagger J)\right]^b$$

$$\overset{n=\min\{a,b\}}{=} (qN_c + n)! \prod_{i=0}^{N_c-1} \frac{i!}{(i+q)!} (\det K)^q \sum_{\rho \vdash n} \tilde{W}^{n,q}_g(\rho, N_c) t_\rho(JK)$$

$$\tilde{W}^{n,q}_g(\rho, N_c) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq N_c}} \frac{1}{(n!)^2} \frac{d^2_\lambda \chi^\lambda(\rho)}{D_{\lambda, N_c+q}}, \qquad t_\rho(A) = \prod_{\rho_i} \mathrm{Tr}(A^{\rho_i})$$

Link Integration

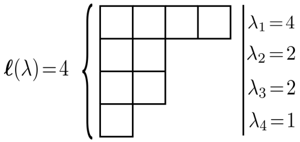
$Z^{a,b}[K,J]$ expressed as a sum over integer partitions weighted by the **modified Weingarten functions** $\tilde{W}_g$.

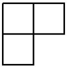
- Group theoretical factors enter the expression for $\tilde{W}_g^{n,q}$: d_λ, $D_{\lambda,N}$ dimensions of irrep λ of S_n and $SU(N_c)$. $\chi^\lambda(\rho)$ are the **irreducible characters** of the symmetric group.

- $\tilde{W}_g^{n,q}$ take as an argument both N_c and ρ. **Conjugacy classes of permutations** can also be represented as integer partitions.

$$\tilde{W}_g^{n,q}(\rho, N_c) = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq N_c}} \frac{1}{(n!)^2} \frac{d_\lambda^2 \chi^\lambda(\rho)}{D_{\lambda, N_c+q}}$$

$\lambda \vdash 9 = (4,2,2,1)$

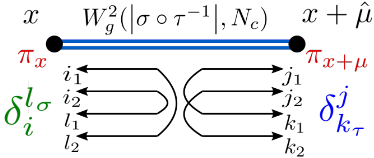


$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \cong (12)(3) \cong$$ 

$\mathcal{I}^{a,b}$ obtained from $Z^{a,b}[K,J]$ by differentiating in J, K.

Dual form of the partition function: U(N) case

$$\mathcal{I}^{n,n}_{i^j k^\ell} = \sum_{\sigma,\tau\in S_n} W^n_g(|\sigma\circ\tau^{-1}|, N_c)\,\delta_i^{\ell_\sigma}\,\delta^j_{k_\tau}$$



For $U(N)$ only the $\mathcal{I}^{a,b}$ with $a = b$ are non-zero. $W(\{n_p, \bar{n}_p\})$ can be evaluated as follows:

- Associate a pair of permutation $(\sigma_\ell, \tau_\ell) \in S_{d_\ell}$ to each **bond**.
- The **delta functions** δ^σ and δ_τ contract on each **vertex**.
- An additional permutation π_x sitting on each vertex tells us how to **re-orient the color flux**:
  - i.e. how to contract the indices between δ's associated to links that join the same vertex.

$$W(\{n_p, \bar{n}_p\}) = \sum_{\{\sigma_\ell,\tau_\ell\in S_{d_\ell}\}} \prod_\ell \underbrace{W^{d_\ell}_g(|\sigma_\ell\circ\tau_\ell^{-1}|, N_c)}_{\gtrless 0} \quad \underbrace{\prod_x N_c^{\ell(\hat{\sigma}\circ\pi_x)}}_{\text{from delta contraction}}$$

Dual form of the partition function: U(N) case

- **Problems:** Analytic resummation too expensive. Half of the W_g's are negative. No possibility of reweighting. E.g. :

$$\tilde{W}_g^{2,0}(2, N_c) = \frac{-1}{N_c(N_c^2 - 1)}$$

- **Idea:** Express the Gauge Integrals in a different basis by introducting a new class of operators $P_\lambda^{a,b}$:

$$\mathcal{I}^{n,n}_{i^j k^\ell} = \sum_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq N_c}} \frac{1}{D_{\lambda, N_c}} \cdot \left(P_\lambda^{a,b}\right)_i^\ell \left(P_\lambda^{a,b}\right)_k^j \qquad P_\lambda^{a,b} = \frac{1}{d_\lambda} \sum_{\pi \in S_n} M_\lambda^{a,b}(\pi) \delta_\pi$$

- $M_\lambda^{a,b}(\pi)$ are the matrix elements of the **irrep.** λ of S_n in the **Young-Yamanouchi** basis.
- $M_\lambda(\pi)$ is an orthogonal matrix for all $\pi \in S_n$.
- Matrix elements computed using computer algebra.

Dual form of the partition function: U(N) case

Properties of the operators $P_\lambda^{a,b}$

- Define **hermitian product** between operators in $(\mathbb{C}^{N_c})^{\otimes n}$:
  $\langle A, B\rangle := \mathrm{Tr}(A^\dagger B) \implies \langle \delta_\pi, \delta_\sigma\rangle = N_c^{\ell(\sigma\circ\pi^{-1})}$
- $P_\lambda^{a,b}$ is a complete, **orthogonal set**, with respect to $\langle\cdot\rangle$. We obtain:

$$W(\{n_p, \bar{n}_p\}) = \sum_{\substack{\{\lambda_\ell \vdash d_\ell\} \\ \ell(\lambda_\ell)\leq N_c}} \overbrace{\left[\sum_{a_\ell, b_\ell} \prod_\ell \frac{1}{D_{\lambda_\ell, N_c}} \prod_x w(x)\right]}^{W(\{n_p,\bar{n}_p\},\{\lambda_\ell\})\geq 0}$$

$$w(x) = \langle \bigotimes_{\ell\in nb(x)} P_{\lambda_\ell}^{a_\ell, b_\ell}, \delta_{\pi_x}\rangle$$

Advantages: Quantity in brackets is positive and much faster to compute $\implies$ **allows for importance sampling**. Orthogonality helps us understanding which configurations have non zero weight. Extension to $SU(N)$ easier in this orthogonal basis.

Our new degrees of freedom are $\{n_p, \bar{n}_p\}$ and integer partitions $\lambda_\ell \vdash d_\ell$.

- Different types of updates to ensure **ergodicity** without violating the **plaquette constraint**:

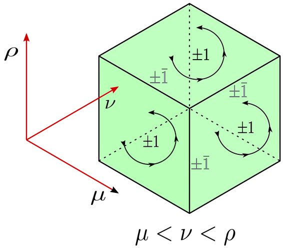
1) Select a plaquette p' and propose $(n_{p'}, \bar{n}_{p'}) \to (n_{p'} \pm 1, \bar{n}_{p'} \pm 1)$. Randomly choose new partitions $\lambda'_{\ell'}$ on links $\ell' \in p'$. Accept new configuration with probability:

$$P_{acc} = \min\left\{1, \frac{(\beta/2N_c)^{\pm 2}}{\left[(n_{p'} \pm 1)\cdot(\bar{n}_{p'} \pm 1)\right]^{\pm 1}} \cdot \frac{W\left(\{n_p \pm \delta_{p,p'}, \bar{n}_p \pm \delta_{p,p'}\}, \{\lambda'_\ell\}\right)}{W(\{n_p, \bar{n}_p\}, \{\lambda_\ell\})}\right\}$$

2) Select a plaquette **p** and propose $n_p \to n_p \pm N_c$ or $\bar{n}_p \to \bar{n}_p \pm N_c$.

3) **At fixed** $\{n_p, \bar{n}_p\}$ select a random link ℓ' and accept $\lambda_{\ell'} \to \lambda'_{\ell'}$ with probability:

$$P_{acc} = \min\left\{1, \frac{W\left(\{n_p, \bar{n}_p\}, \{\lambda'_\ell\}\right)}{W(\{n_p, \bar{n}_p\}, \{\lambda_\ell\})}\right\}$$



4) Propose **cube updates** to change $n_p - \bar{n}_p \mod N_c$.

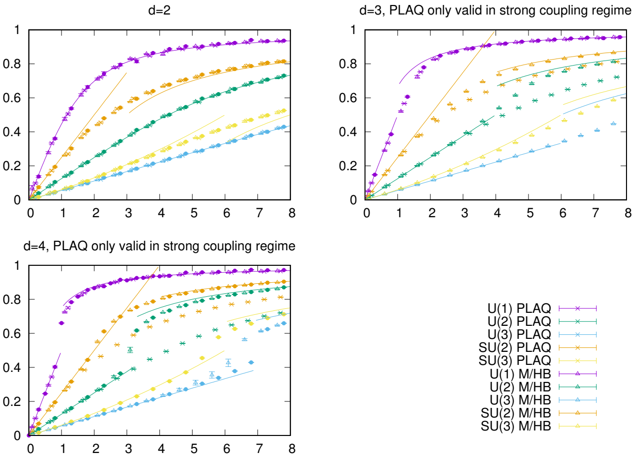
Summary

- **Results:**
  - We obtained a **fully dualized** partition function for *Yang-Mills* theory. We are able to compute each weight and we checked the correctness of our approach comparing with known results.
  - By resumming a subset of weights we obtained **only positive configurations** which are labelled by integer partitions.
- **Outlook:**
  - Implement a Markov Chain Monte Carlo simulation for the dualized partition function. Possible observables: Mean plaquette, glueball and screening masses.
  - Extend our approach to matter fields: scalar QCD, staggered fermions.

Backup slides

Average plaquette for pure Yang-Mills in the dual representation and comparison with standard heat bath algorithm for various dimensions:

- Weights computed using **invariants** (valid for $n_p - \bar{n}_p \bmod N_c = 0$)
- Cube updates **still missing**.



Observables in the dual representation

- Mean plaquette:

$$\left\langle \frac{\mathrm{Re}\,\mathrm{Tr}\, U_{\mu,\nu}(x)}{N_c} \right\rangle \stackrel{\textbf{dual}}{=} \left\langle \frac{2n_{x,\mu,\nu}}{\beta} \right\rangle$$

- Scalar glueball $J^{PC} = 0^{++}$: Extracted from temporal correlator of spatial plaquettes:

$$\begin{aligned} C(t) &= \langle \psi(t)\psi(0)\rangle - \langle \psi(t)\rangle\langle\psi(0)\rangle \\ \psi(t) &= \frac{1}{N_c}\sum_{\vec{x}}\sum_{\mu<\nu}^{\mu\neq 0} \mathrm{Re}\,\mathrm{Tr}\, U_{\mu,\nu}(\vec{x},t) \stackrel{\textbf{dual}}{=} \sum_{\vec{x}}\sum_{\mu<\nu}^{\mu\neq 0} \frac{n_{(\vec{x},t),\mu,\nu} + \bar{n}_{(\vec{x},t),\mu,\nu}}{\beta} \end{aligned}$$