Thermodynamics at Strong Coupling on Anisotropic Lattices

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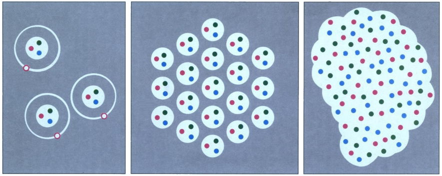
Collaborators: Dennis Bollweg and Marc Klegrewe

Lattice 2018

East Lansing, MI, 25.07.2018



Universität Bielefeld

CRC-TR 211
Strong-interaction matter under extreme conditions

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Overview

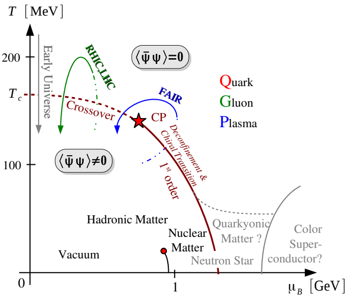
Motivation:

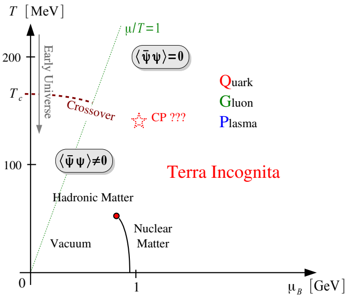
1. **Anisotropic lattices:** necessary to study thermodynamics in strong coupling regime (β fixed)
2. This has been studied via a dual formulation in the **chiral limit**
3. We extend these results to **finite quark mass**

Content:

1. Lattice QCD at Strong Coupling, Role of Anisotropy
2. Thermodynamic Observables
3. Results on Anisotropy and Continuous Time Limit
4. Some results on the phase diagram

QCD Phase Diagram and Sign Problem





- **Sign problem:** no direct RHMC simulations at finite μ due to complex weights
- Complexifying parameter space (**Complex Langevin, Lefshetz Thimbles**: promising, but not (yet) applicable to full QCD

Sign problem is **representation dependent!**

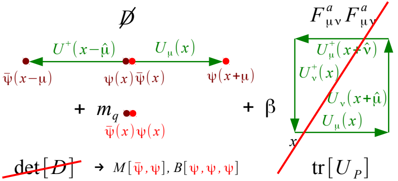
- **dual representations** oftentimes solve or milden sign problems
- for lattice QCD, a dual representation is well known in the **strong coupling limit**

Lattice QCD at Strong Coupling

Alternative approach:

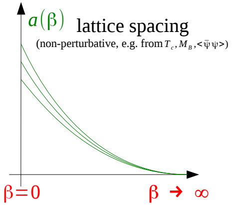
> Limit of strong coupling: $\beta = \frac{6}{g^2} \to 0$

- gauge fields $U_\mu(x)$ can be integrated out
- **"dual" representation:** via color singlets on the links!
- at strong coupling: **mesons** and **baryons**



Advantage:

- (almost) no sign problem
- fast simulations (no supercomputers necessary)
  $\Rightarrow$ **complete phase diagram** can be calculated



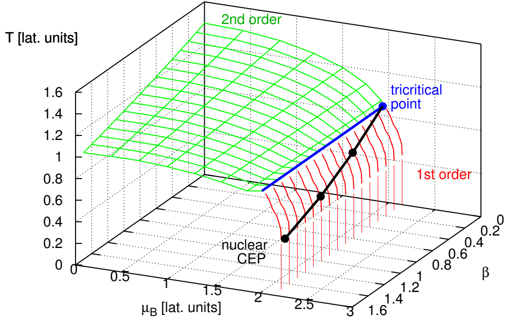
Strong Coupling Expansion:

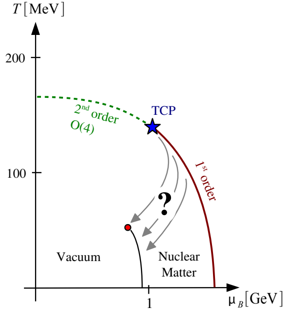
- difficult to include gauge corrections $\beta > 0$ to make lattice finer
  $\to$ talk by **Giuseppe Gagliardi** "Towards a Dual Representation of Lattice QCD"

Status of the SC-QCD Phase Diagram in the Chiral Limit

Via reweighting in β from $\beta = 0$: $\mathcal{O}(\beta)$ **corrections for SU(3)**

[Langelage, de Forcrand, Philipsen & U., *PRL* **113** (2014)]





Immediate Question:

- Do the **nuclear and chiral transition split** at sufficiently large β?
- New simulations obtained by sampling plaquette contributions via **world sheets** give similar results [G. Gagliardi, Kim & U. arXiv:1710.07564]
- Is a **finite quark mass necessary** for splitting?

Status of SC-QCD on Anisotropic Lattices

Anisotropic Lattices, $\xi = \frac{a_s}{a_t} > 1$:

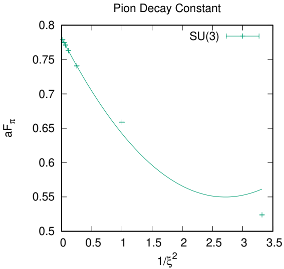
- needed in strong coupling regime to modify the temperature (details below)
- anisotropy dependence has been studied for $m_q = 0$
- defines unambigously the **continuous time limit:**
  $a_t \to 0 \qquad (N_t \to \infty,\ \xi \to \infty$ at fixed $aT = \frac{\xi}{N_t})$

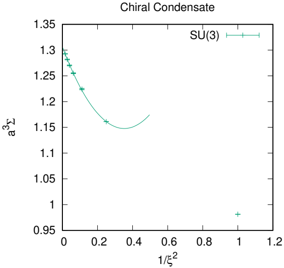
To be addressed here:

- extend **anisotropy callibration** to finite quark mass
- apply results to thermodynamic observables and the phase diagram

Not yet addressed:

- how to obtain gauge corrections on anisotropic lattices at finite quark mass
  (so far only in chiral limit)





[de Forcrand, U., Vairinhos *Phys. Rev. D* **97** (2018)]

Strong Coupling Partition Function

Exact rewriting after Grassmann integration: Mapping onto **discrete system:**

[Rossi & Wolff (1984)], [Karsch & Mütter (1989)]

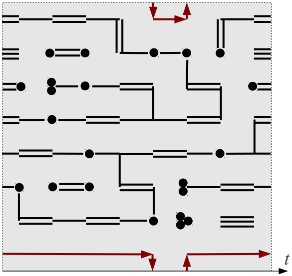
$$Z_F(m_q,\mu) = \sum_{\{k,n,\ell\}} \underbrace{\prod_{b=(x,\mu)} \frac{(N_c-k_b)!}{N_c!k_b!}}_{\text{meson hoppings } M_x M_y} \underbrace{\prod_x \frac{N_c!}{n_x!}(2am_q)^{n_x}}_{\text{chiral cond. operator } \bar\psi\psi} \underbrace{\prod_\ell w(\ell,\mu)}_{\text{baryon hoppings } \bar{B}_x B_y}$$

$k_b \in \{0,\dots N_c\}$, $n_x \in \{0,\dots N_c\}$, $\ell_b \in \{0,\pm 1\}$, QCD: $N_c = 3$

- Grassmann constraint:

$$n_x + \sum_{\hat\mu=\pm\hat0,\dots\pm\hat d} \left(k_{\hat\mu}(x) + \frac{N_c}{2}|\ell_{\hat\mu}(x)| \right) = N_c$$

- weight $w(\ell,\mu)$ and sign $\sigma(\ell)\in\{-1,+1\}$ for oriented baryonic loop ℓ depends on loop geometry



finite quark mass

Strong Coupling LQCD at Finite Temperature

How to vary the temperature?

- $aT = 1/N_t$ is discrete with N_t even
- $aT_c \simeq 1.5 \quad \Rightarrow \quad$ we cannot address the phase transition!

Solution: introduce an **anisotropy** γ in the Dirac couplings such that $a_t \neq a_s = a$:

$$\mathcal{L}_{\mathrm{F}} = \sum_{\mu} \frac{\gamma^{\delta_{\mu 0}}}{2} \eta_{\nu}(x) \left(e^{\mu \delta_{\mu 0}} \bar{\chi}(x) U_{\nu}(x) \chi(x+\hat{\mu}) - e^{-\mu \delta_{\mu 0}} \bar{\chi}(x+\hat{\mu}) U_{\mu}^{\dagger}(x) \chi(x) \right)$$

$$Z_F(m_q, \mu, \gamma) = \sum_{\{k,n,\ell\}} \prod_{b=(x,\mu)} \frac{(N_{\mathrm{c}} - k_b)!}{N_{\mathrm{c}}! k_b!} \gamma^{2 k_b \delta_{\mu 0}} \prod_{x} \frac{N_{\mathrm{c}}!}{n_x!} (2am_q)^{n_x} \prod_{\ell} w(\ell, \mu)$$

- Meanfield at strong coupling: $\frac{a_s}{a_t} \equiv \xi(\gamma) = \gamma^2$, since $\gamma_c^2 = N_t \frac{(d-1)(N_{\mathrm{c}}+1)(N_{\mathrm{c}}+2)}{6(N_{\mathrm{c}}+3)}$

$\Rightarrow$ definition of the temperature: $\boxed{aT = \frac{\xi(\gamma)}{N_t}}$

However: Need to know the **precise correspondence between** $\xi \equiv a_s/a_t$ **and** γ

- Nonperturbative result: $\boxed{\xi(\gamma) \approx \kappa \gamma^2 + \frac{\gamma^2}{1+\lambda\gamma^4}, \quad \kappa = 0.781(1)}$

γ-Dependence of Thermodynamic Observables

Consider U(3) in the chiral limit first:

$$Z(\gamma) = \sum_{\{k\}} \left(\prod_{b=(x,\mu)} \frac{(3-k_b)!}{3!k_b!} \right) \gamma^{2N_{Dt}}$$

with N_{Dt} the total number of **timelike dimers**, $n_{Dt} = \frac{1}{N_s^3 N_t} N_{Dt}$

The dimensionless thermodynamic observables are:

- energy density: $a_s^3 a_t \epsilon = - \frac{a^3 a_t}{V} \frac{\partial \log Z}{\partial T^{-1}}\bigg|_V = \frac{\xi}{\gamma} \frac{d\gamma}{d\xi} \langle 2n_{Dt} \rangle$
- pressure: $a_s^3 a_t p = - a_s^3 a_t T \frac{\partial \log Z}{\partial V}\bigg|_T = \frac{\xi}{3\gamma} \frac{d\gamma}{d\xi} \langle 2n_{Dt} \rangle$
- interaction measure: $\epsilon - 3p = 0$
- entropy density: $s = \frac{4\epsilon}{3T}$

γ-Dependence of Thermodynamic Observables

Now consider SU(3) at finite quark mass:

$$Z(m_q, \mu_B, \gamma) = \sum_{\{k,n,\ell\}} \sigma(\ell) \left(\prod_{b=(x,\mu)} \frac{(3-k_b)!}{3!k_b!} \right) \left(\prod_x \frac{3!}{n_x!} \right) (2a_t m_q)^{N_M} \gamma^{N_q} e^{N_t a_t \mu_B \Omega(\ell)}$$

with $N_q = 2N_{Dt} + 3N_{Bt}$, and $N_{Bt} = \sum_x |b_{x,0}|$ the number of timelike baryon segments.

The dimensionless thermodynamic observables are:

- baryon density: $\boxed{a_s^3 \rho_B = a_s^3 \frac{T}{V} \frac{\partial \log Z}{\partial \mu_B}\Big|_{V,T} = \frac{\langle \Omega \rangle}{N_\sigma^3} = \langle \omega \rangle}$
- chiral condensate: $\boxed{a_s^3 \langle \bar{\chi}\chi \rangle = a_s^3 \frac{\langle N_M \rangle}{N_\sigma^3 N_t a_s^3 a_t} = \frac{1}{a_t m_q} \langle n_M \rangle}$
- energy density: $\boxed{a^3 a_t \epsilon = \mu_B \rho_B - \frac{a^3 a_t}{V} \frac{\partial \log Z}{\partial T^{-1}}\Big|_{V,\mu_B} = \frac{\xi}{\gamma} \frac{d\gamma}{d\xi} \langle n_q \rangle - \langle n_M \rangle}$
- pressure: $\boxed{a_s^3 a_t p = -\, a_s^3 a_t T \frac{\partial \log Z}{\partial V}\Big|_{T,\mu_B} = \frac{\xi}{3\gamma} \frac{d\gamma}{d\xi} \langle n_q \rangle}$
- interaction measure: $\boxed{\epsilon - 3p = -\frac{\langle n_M \rangle}{a_s^3 a_t} = -m_q \langle \bar{\chi}\chi \rangle}$
- entropy density: $\boxed{s = \frac{1}{T} \left(\frac{4\epsilon}{3} - \mu_B \rho_B \right)}$

How to determine $\xi(\gamma)$

Idea: anisotropy callibration

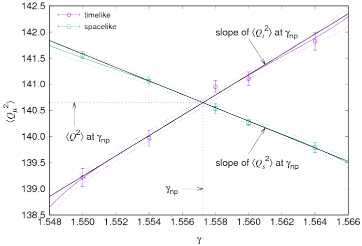
- Grassmann constraint implies **locally conserved current:**

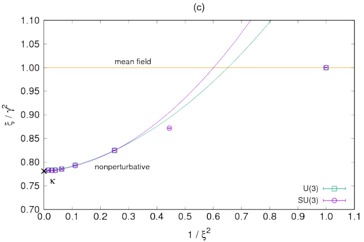
[Chandrasekharan & Jiang '03]

$$j_\mu(x) = \sigma(x)\left(k_\mu(x) - \frac{3}{2}|b_{x,\mu}| - \frac{3}{2d}\right) \quad \rightarrow \quad \sum_{\pm\hat{\mu}}(j_\mu(x) - j_\mu(x-\hat{\mu})) = 0$$

- conserved charge $Q_\mu = \sum_{x\perp\mu} j_\mu(x)$ has $\langle Q_\mu\rangle = 0$, but non-zero variance: $\left\langle Q_\mu^2\right\rangle \neq 0$
- calibration of $\xi(\gamma)$ via renormalization condition on **physically isotropic box:**

$$a_t N_t = a_s N_s \quad \Leftrightarrow \quad \left\langle Q_t^2\right\rangle(\gamma_0) \stackrel{!}{=} \left\langle Q_s^2\right\rangle(\gamma_0), \qquad \frac{a_s}{a_t} = \frac{N_t}{N_s} = \xi(\gamma_0)$$





[de Forcrand, U., Vairinhos *Phys. Rev. D* **97** (2018)]

How to determine $\xi(\gamma, \hat{m})$ at finite quark mass?

Problem:

- $j_\mu(x)$ is **no longer a conserved current**
- on a given configuration, e.g. $Q_t(t_1) \neq Q_t(t_2)$

Solution:

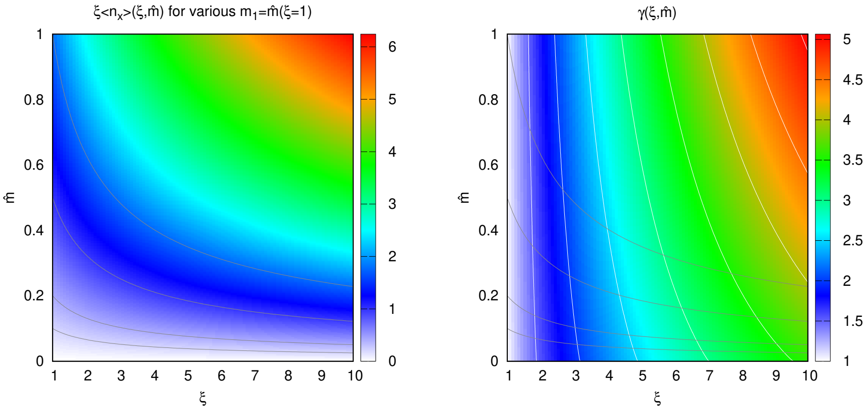
- after averaging over parallel hypersurfaces: $Q_t = \frac{1}{Nt}\sum_t (Q_t(t))$
  the **charge has same distribution** as for $m_q = 0$
  (same number of monomers on even and odd sites)
- again, we demand the fluctuations to be equal $\left\langle Q_t^2 \right\rangle (\gamma_0) \stackrel{!}{=} \left\langle Q_s^2 \right\rangle (\gamma_0)$
- however: at finite bare quark mass $\hat{m} \equiv am_q$, we need to **keep the physics constant** e.g. $M_\pi L$ =const or $[\hat{m}\langle\bar{\chi}\chi\rangle]_L$ =const
  $\rightarrow$ we need to determine $\hat{m}(\xi)$ as well! (see also [Levkova et al., PRD 72 (2006)])

Our strategy: require the monomer density to be constant:

$$\boxed{a_s^4 \hat{m} \langle\bar{\chi}\chi\rangle = a_s^3 a_t\, \xi \langle n_x\rangle}$$

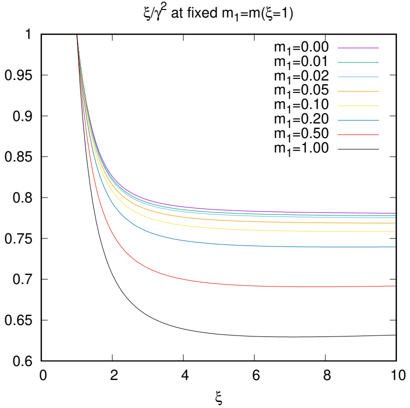
(not the case that lattice spacings factorize $(a_t m_q)(a^3 \langle\bar{\chi}\chi\rangle)$

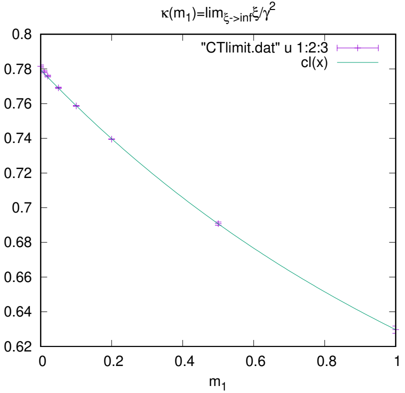
Results from the anisotropy callibration



- the condition $\xi \langle n_x \rangle$ =const defines **lines of constant physics** $\hat{m}(\xi)$ (with $m_1 = \hat{m}(\xi = 1) = am_q$ the isotropic bare mass)
- determine the bare anisotropy γ for various ξ, i.e. for temperatures $aT = \frac{\xi}{N_t}$
- **continuous time limit** $N_t \to \infty$ well defined also for finite am_q with m_q/T fixed.

Continuous Time Limit at finite quark mass





- continuous time limit well defined
- **non-perturbative correction factor** has simple quark mass dependence:

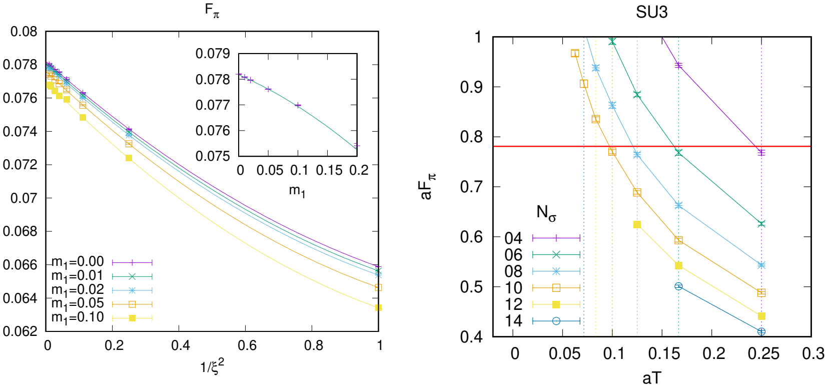
$$\kappa(m_1) = \frac{\kappa_0}{1 + c_1 m_1 + c_2 m_1^2}$$

- temperature and chemical potential: $aT = \kappa(m_1)[aT]_{\mathrm{mf}}$, $a\mu_B = \kappa(\hat{m})[a\mu_B]_{\mathrm{mf}}$

Results from $\xi(\gamma, \hat{m})$ for Zero Temperature

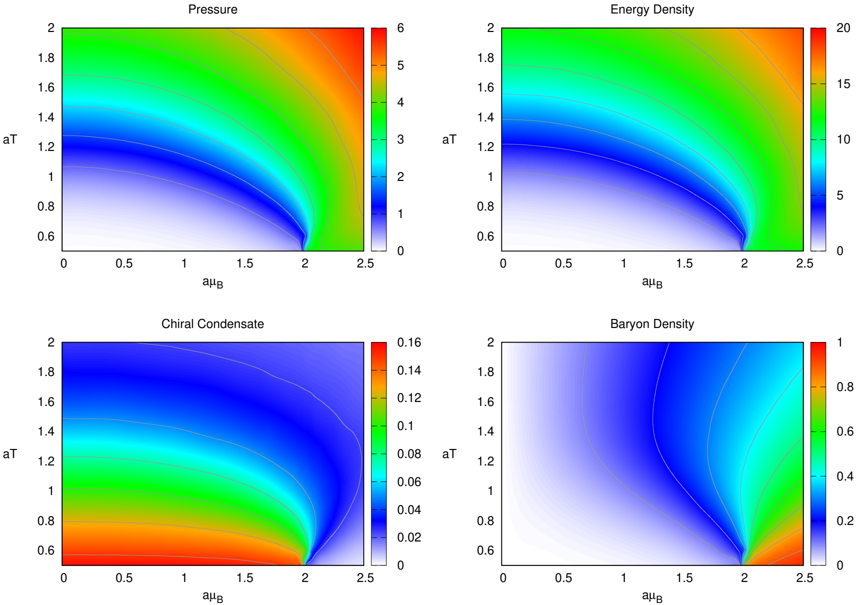
The Pion Decay Constant F_π is obtained from the charge fluctuation:

$$aF_\pi = \lim_{N_s \to \infty} a^2 \Upsilon, \qquad a^2\Upsilon = \frac{1}{N_s^2} \left\langle Q^2 \right\rangle_{\gamma_{np}}$$

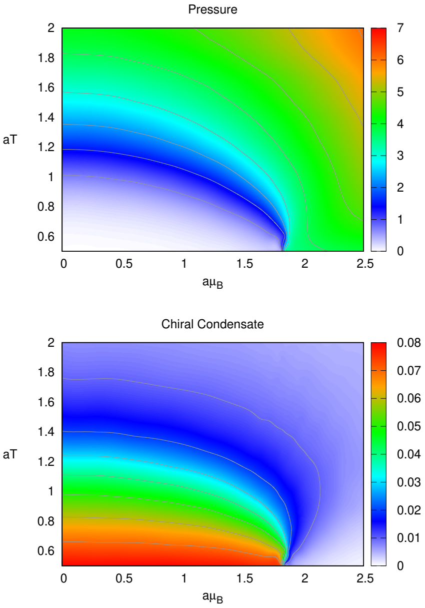


- *Left:* determination of F_π for various quark masses, extrapolated in CT limit
- *Right:* determination of F_π via continuous time Monte Carlo

Thermodynamic Observables at $m_q = 0.1$



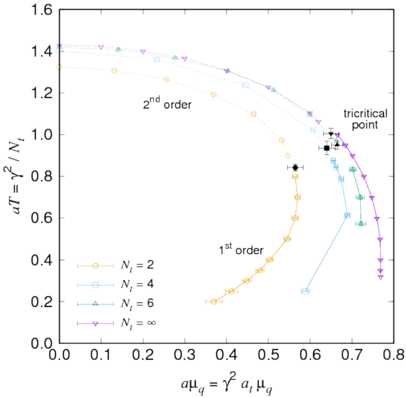
Thermodynamic Observables at $m_q = 0.05$

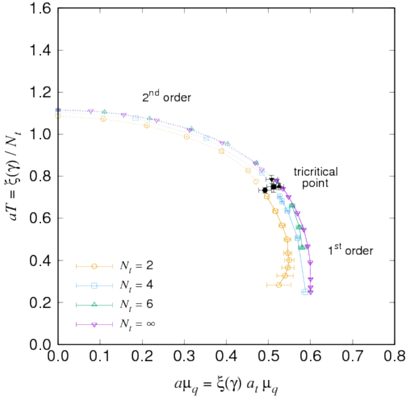


The phase diagram in the strong coupling limit

Well established is the chiral limit:

- corrected phase boundaries with $\xi(\gamma)$ rescales all $N_t > 2$ on universal curve
- matches on continuous time phase boundary ($N_t = \infty$ [de Forcrand & U. '11])





Phase diagram for finite quark mass:

- has been studied by Jangho Kim via mean field definitions $[aT]_{\rm mf}$, $[a\mu_B]_{\rm mf}$ [arXiv:1611.09120]
- non-perturbative result is under preparation

Conclusions

Results:

- anisotropy $\frac{a}{a_t} \equiv \xi(\gamma, \hat{m})$ determined non-perturbatively also for $m_q > 0$
- allows to **measure thermodynamical observables correctly**
- allows unambigous identification of phase boundary
- simulations in continuous time limit $\xi \to \infty$ confirm extrapolated results

$\rightarrow$ next talk by **Marc Klegrewe** on
"Temporal Correlators in the Continuous Time Limit of Lattice QCD"

Goals: address **anisotropy for** $\beta > 0$:

- non-perturbative determination of $a/a_t \equiv \xi(\gamma_F, \gamma_G, \beta)$
- simulations to all orders of β
- determine interaction measure for $\beta > 0$, $\mu > 0$

Anouncement

We would like ot invite you to the ...

Sign Workshop:

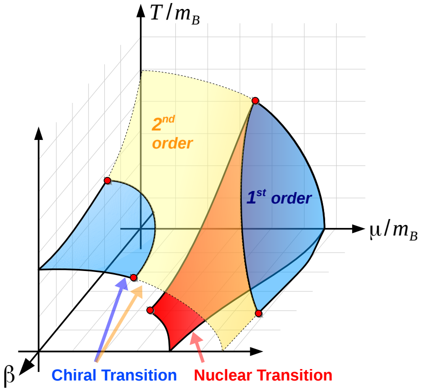
- Date: 10-14 Sep 2018
- Location: Bielefeld University
- Topics: QCD and related theories, Nuclear Many Body, Dual Formulations, Langevin and Lefschetz Approaches, Algorithmic Developments
- Website: www.physik.uni-bielefeld.de/sign18
- Stipends: available for PhD students
- Deadline: Augst 5th 2018
- Contact: Christian Schmidt, Wolfgang Unger

sign18@physik.uni-bielefeld.de

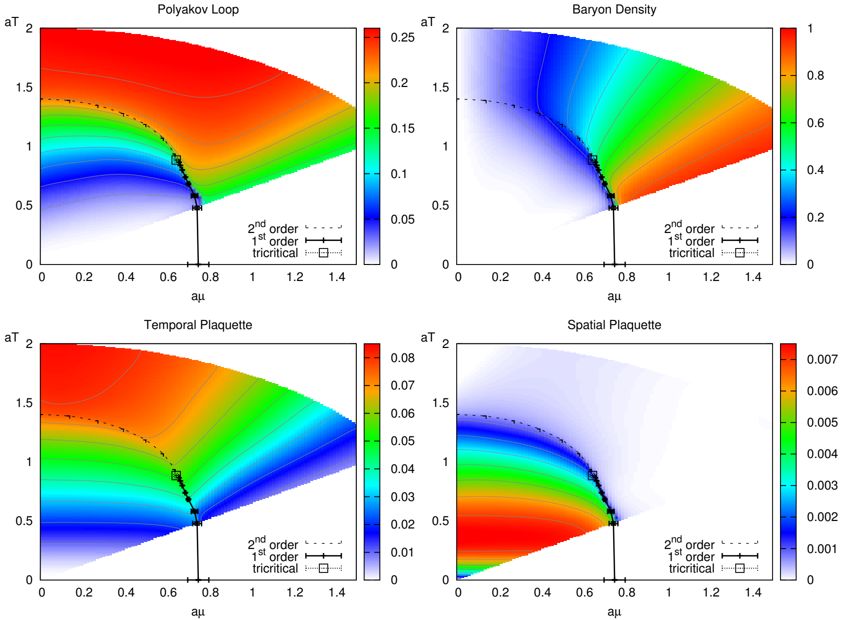
Backup: Connection Between SC and Continuum Limit?

One of several **possible scenarios** for the extension to the continuum:

- back plane: strong coupling phase diagram ($\beta = 0$), $N_\mathrm{f} = 1$
- front plane: continuum phase diagram ($\beta = \infty$, $a = 0$)
- due to fermion doubling, corresponds to $N_\mathrm{f} = 4$ in continuum (no rooting)



Backup: Polyakov Loop and Plaquettes in the Chiral Limit



Backup: Severity of Sign Problem at finite quark mass