Scattering in a Euclidean formulation of relativistic quantum mechanics

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Motivation and Observations

- **Constructing relativistic quantum mechanical models satisfying cluster properties is complicated.**
- **Locality is logically independent of the rest of the axioms of Euclidean field theory $\rightarrow$ Euclidean formulation of relativistic quantum theory satisfying cluster properties.**
- **Reconstruction theorem: The physical Hilbert space and a unitary representation of the Poincaré group can be directly formulated in the Euclidean representation. Analytic continuation is not necessary.**
- **Given these elements it should be possible to formulate a relativistic treatment of scattering in a Euclidean representation using standard quantum mechanical methods.**

Elements of relativistic quantum mechanics

$$\langle \psi | \phi \rangle \qquad \textbf{Hilbert space}$$

$$U(\Lambda, a) \leftrightarrow \{P^\mu, J^{\mu\nu}\} \qquad \textbf{Relativity}$$

$$P^0 = H \qquad \textbf{Dynamics}$$

$$P^0 = H \geq 0 \qquad \textbf{Spectral condition} \quad \rightarrow \quad \textbf{stability}$$

$$[U(\Lambda, a) - \otimes U_i(\Lambda, a)] |\psi\rangle \rightarrow 0 \qquad (x_i - x_j)^2 \rightarrow \infty$$

Cluster properties: scattering asymptotic conditions

Osterwalder-Schrader (Euclidean) reconstruction

Input: $\{G_{En}(x_1, \cdots, x_n)\}$

Relevant properties

- **Euclidean covariant (invariant)**
- **Cluster property**
- **Reflection positivity**

Construction of the physical Hilbert space: $\mathcal{H}_M$

Vectors (dense set)

$$\psi(x) := (\psi_1(\mathsf{x}_{11}), \psi_2(\mathsf{x}_{21}, \mathsf{x}_{22}), \cdots)$$

$$\psi_n(\mathsf{x}_{n1}, \mathsf{x}_{n2}, \cdots, \mathsf{x}_{nn}) = 0 \qquad \textbf{unless} \qquad 0 < \mathsf{x}^0_{n1} < \mathsf{x}^0_{n2} < \cdots < \mathsf{x}^0_{nn}.$$

$$\theta x := \theta(\tau, \mathbf{x}) = (-\tau, \mathbf{x}) \qquad \textbf{Euclidean time reflection}$$

Physical Hilbert space inner product

$$\langle \psi | \phi \rangle_M = (\theta\psi, G_E \phi)_E = \sum_{kn} \int d^{4k}x d^{4n}y \psi^*_n(\theta \mathsf{x}_{n1}, \theta \mathsf{x}_{n2}, \cdots, \theta \mathsf{x}_{nn}) \times$$
$$G_{E,n+k}(\mathsf{x}_{nn}, \cdots, \mathsf{x}_{1n}; \mathsf{y}_{1k}, \cdots, \mathsf{y}_{kk}) \phi_k(\mathsf{y}_{k1}, \mathsf{y}_{k2}, \cdots, \mathsf{y}_{kk})$$

All variables are Euclidean - no analytic continuation.

Reflection positivity - property of $\{G_{En}\}$

$$\langle \psi | \psi \rangle_M = (\psi, \Pi_{+>} \Theta G_E \Pi_{+>} \psi)_E \geq 0$$

$$\Downarrow$$

Gives the physical Hilbert space and spectral condition.

Illustration

Two-point Green function: Euclidean $\to$ Minkowski

$$\langle\phi|\psi\rangle_M = \int \phi^*(-\tau_x,\mathbf{x})\frac{d^4p\rho(m)dm}{(2\pi)^4}\frac{e^{ip\cdot(x-y)}}{p^2+m^2}\psi(\tau_y,\mathbf{y})d^4xd^4y$$

$$= \int \xi^*_m(\mathbf{p})\frac{d\mathbf{p}\rho(m)dm}{2e_m(\mathbf{p})}\chi_m(\mathbf{p})$$

Euclidean wave function $\to$ Minkowski wave function

$$\chi_m(\mathbf{p}) := \int \frac{d^4y}{(2\pi)^{3/2}}e^{-e_m(\mathbf{p})\tau_y - i\mathbf{p}\cdot\mathbf{y}}\psi(\tau_y,\mathbf{y})$$

$$\xi_m(\mathbf{p}) := \int \frac{d^4x}{(2\pi)^{3/2}}e^{-e_m(\mathbf{p})\tau_x - i\mathbf{p}\cdot\mathbf{x}}\phi(\tau_x,\mathbf{x})$$

$$m^2\psi(\tau_x,\mathbf{x}) = \nabla_4^2\psi(\tau_x,\mathbf{x})$$

Euclidean invariance $\to$ Poincaré invariance

Relativity and $SL(2,\mathbb{C}) \times SL(2,\mathbb{C})$

$$X_m := \begin{pmatrix} t+z & x-iy \\ x+iy & t-z \end{pmatrix} \qquad X_e := \begin{pmatrix} i\tau+z & x-iy \\ x+iy & i\tau-z \end{pmatrix}$$

$$\mathbf{det}(X_M) = t^2 - \mathbf{x}^2 \qquad \mathbf{det}(X_E) = -(\tau^2 + \mathbf{x}^2)$$

$$X \to X' = AXB^t \qquad \mathbf{det}(A) = \mathbf{det}(B) = 1$$

Preserves both $t^2 - \mathbf{x}^2$ and $\tau^2 + \mathbf{x}^2$

Complex Lorentz group = complex orthogonal group

Real orthogonal group = subgroup of complex Lorentz group

Real Lorentz $(A,B) = (A, A^*),\ A \in SL(2,\mathbb{C})$;
Real orthogonal $(A,B) \in SU(2) \times SU(2)$

Relation between Euclidean and Poincaré generators

- **Euclidean time translations $\to$ contractive Hermitian semigroup on $\mathcal{H}_M$:**

$$H_E = P_E^0 = -iH_M = -iP_M^0$$

- **Euclidean space-time rotations $\to$ local symmetric semigroup on $\mathcal{H}_M$:**

$$J_E^{0j} = iJ_M^{0j} = iK^j$$

- **Euclidean space rotations $\to$ unitary one parameter groups on $\mathcal{H}_M$:**

$$J_E^{ij} = J_M^{ij}$$

- **Euclidean space translations $\to$ unitary one parameter groups on $\mathcal{H}_m$:**

$$P_E^i = P_M^i$$

$\{P_M^\mu, J_M^{\mu\nu}\}$ **= 10 self-adjoint generators satisfying the Poincaré commutation relations on $\mathcal{H}_M$**

Spinless case

$$H\psi_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}) = \sum_{k=1}^{n} \frac{\partial}{\partial \mathsf{x}^0_{nk}}\psi_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn})$$

$$\mathbf{P}\psi_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}) = -i\sum_{k=1}^{n} \frac{\partial}{\partial \mathbf{x}_{nk}}\psi_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn})$$

$$\mathbf{J}\psi_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}) = -i\sum_{k=1}^{n} \mathbf{x}_{nk}\times\frac{\partial}{\partial \mathbf{x}_{nk}}\psi_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn})$$

$$\mathbf{K}\psi_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}) = \sum_{k=1}^{n}(\mathbf{x}_{nk}\frac{\partial}{\partial x^0_{nk}} - x^0_{nk}\frac{\partial}{\partial \mathbf{x}_{nk}})\psi_n(\mathsf{x}_{n1},\mathsf{x}_{n2},\cdots,\mathsf{x}_{nn}).$$

All integration variables are Euclidean; Minkowski time is a parameter .

Cluster properties

$$G_{E,n+m} \to G_{E,n} G_{E,m}$$

$$\Downarrow$$

Generators become additive in asymptotically separated subsystems

Used to formulate scattering asymptotic conditions.

Multichannel scattering theory

$$\textbf{Scattering probability} = |S_{fi}|^2 = |\langle \psi_+ | \psi_- \rangle|^2$$

$$|\psi_\pm\rangle = \Omega_\pm |\psi_{0\pm}\rangle$$

$$\Omega_\pm |\psi_{0\pm}\rangle = \lim_{t \to \pm\infty} \sum e^{iHt} \underbrace{\prod_n |\phi_n, \mathbf{p}_n, \mu_n\rangle}_{J} \underbrace{e^{-ie_n t}}_{e^{-iH_0 t}} \underbrace{f_n(\mathbf{p}_n, \mu_n)}_{|\psi_0 \pm\rangle} d\mathbf{p}_n$$

$$= \lim_{t \to \pm\infty} e^{iHt} J e^{-iH_0 t} |\psi_0 \pm\rangle$$

Elements: Cluster properties, subsystem bound states: $|\phi_n\rangle$, wave packets: f_n, dynamics: H, strong limits.

Field theoretic implementation: Haag-Ruelle scattering

(Minkowski case)

$$\Phi(x) = \textbf{interpolating field}$$

$$\tilde{\Phi}(p) = \frac{1}{(2\pi)^2} \int e^{-ip\cdot x} \Phi(x) d^4 x$$

$$\tilde{\Phi}_m(p) = h(p^2)\tilde{\Phi}(p),\ h(-m^2) = 1,\ h(p^2) = 0,\ -p^2 \not\in (m^2-\epsilon, m^2+\epsilon)$$

$$\Phi_m(x) = \frac{1}{(2\pi)^2} \int e^{ip\cdot x} \tilde{\Phi}_m(p) d^4 p$$

$$f_m(x) = \frac{i}{(2\pi)^{3/2}} \int e^{-i\sqrt{\mathbf{p}^2+m^2}t + i\mathbf{p}\cdot\mathbf{x}} \tilde{f}(\mathbf{p}) d\mathbf{p}$$

$$a^\dagger_m(f_m, t) = -i \int d\mathbf{x} \left(\frac{\partial \Phi_m(t,\mathbf{x})}{\partial t} f_m(t,\mathbf{x}) - \Phi_m(t,\mathbf{x}) \frac{\partial f_m(t,\mathbf{x})}{\partial t} - \right)$$

$$\Omega_\pm |\psi_{0\pm}\rangle = s - \lim_{t\to\pm\infty} \Pi_i a^\dagger_{m_i}(f_{m_i}, t)|0\rangle$$

Euclidean formulation of HR scattering - technical issues

- $(M^2 = \nabla^2)$ **One-body solutions must satisfy the time support condition:**

$$\textbf{support}(h(\nabla^2)\langle x|\psi\rangle) = \textbf{support}(\langle x|\psi\rangle)$$

- **Products of one-body solutions must satisfy the relative time support condition ($n = 2$, no spin) .**

$$J : \langle \mathsf{x}_1|\phi_1, \mathbf{p}_1\rangle\langle \mathsf{x}_2|\phi_2, \mathbf{p}_2\rangle =$$

$$h_1(\nabla_1^2)\delta(\mathsf{x}_1^0 - \tau_1)h_2(\nabla_2^2)\delta(\mathsf{x}_2^0 - \tau_2)\frac{1}{(2\pi)^3}e^{i\mathbf{p}_1\cdot\mathbf{x}_1 + i\mathbf{p}_2\cdot\mathbf{x}_2}$$

$$\tau_2 > \tau_1$$

- **Delta functions in Euclidean time $\times f(\mathbf{x})$ are square integrable in $\mathcal{H}_M$!**

- **A sufficient condition for $h_i(\nabla^2)$ to preserve the support condition is for polynomials in ∇^2 to be complete with respect to the inner product on $\mathcal{H}_M$**

$$h_i(\nabla^2) \approx P(\nabla^2)$$

- **The J defined on the previous slide can be used to satisfy the time-support conditions.**

Completeness of $P_n(\nabla^2)$ sufficient to construct $h(m^2)$ without violating positive Euclidean time-support condition.

Proving completeness - Stieltjes moment problem
G_{E2} moments

$$\gamma_n := \int_0^\infty \frac{e^{-\sqrt{m^2+\mathbf{p}^2}\tau}}{2\sqrt{m^2+\mathbf{p}^2}}\rho(m)m^{2n}dm$$

where $\tau = \tau_1 + \tau_2 > 0$.

Carleman's condition

$$\sum_{n=0}^{\infty} |\gamma_n|^{-\frac{1}{2n}} > \infty$$

Satisfied for $\rho(m^2)$ **a tempered distribution** $\Rightarrow$ $P(\nabla^2)$ **complete.**

$$|\gamma_n|^{-\frac{1}{2n}} \sim \frac{1}{n+c}$$

Existence - sufficient condition (Cook)

$$\int_a^\infty \|(HJ - JH_0)U_0(\pm t)|\psi_0\rangle\|_M dt < \infty$$

$$\|(HJ - JH_0)\Phi U_0(\pm t)|\psi_0\rangle\|_M^2 =$$

$$(\psi_0 U_0(\mp t)(J^\dagger H - H_0 J^\dagger)\theta G_E(HJ - JH_0)U_0(\pm t)|\psi_0)_E$$

The effect of using one-body solutions for 2-2 scattering is that the contribution from the disconnected part of G_E to the above is zero. This fails for LSZ scattering.

The connected part is expected to behave like ct^{-3} for large t, satisfying the Cook condition.

Computational tricks for scattering
Invariance principle:

$$\lim_{t\to\pm\infty} e^{iHt} J e^{-iH_0 t}|\psi\rangle = \lim_{t\to\pm\infty} e^{if(H)t} J e^{-if(H_0)t}|\psi\rangle$$

$$f(x) = -e^{-\beta x}$$

$$\lim_{t\to\pm\infty} e^{iHt} J e^{-iH_0 t}|\psi\rangle = \lim_{n\to\infty} e^{\mp ine^{-\beta H}} J e^{i\pm ne^{-\beta H_0}}|\psi\rangle$$

$$\sigma(e^{-\beta H}) \in [0,1] \to$$

$$|e^{inx} - P(x)| < \epsilon \qquad x \in [0,1]$$

$$|||e^{ine^{-\beta H}} - P(e^{-\beta H})||| < \epsilon$$

Matrix elements of $e^{-n\beta H}$ are easy to calculate:

$$\langle \tau, \mathbf{x}|e^{-n\beta H}|\psi\rangle = \langle \tau - n\beta, \mathbf{x}|\psi\rangle$$

Model tests (of computational methods)

$$H = k^2/m - \lambda|g\rangle\langle g|$$

$$(M^2 = 4k^2 + 4m^2 - 4m\lambda|g\rangle\langle g|)$$

$$\langle k|g\rangle = \frac{1}{k^2 + m_\pi^2}$$

Attractive - one pion exchange range, bound state with deuteron mass.

$$e^{-2ine^{-\beta H}} \approx P(e^{-\beta H})$$

$$\langle \mathbf{k}_f|T(E+i0)|\mathbf{k}_i\rangle \approx$$

$$\frac{\langle\psi_f|(I - e^{-ine^{-\beta M_0}}P(e^{-\beta H})e^{-ine^{-\beta H_0}})|\psi_i\rangle}{2\pi i\langle\psi_f|\delta(E-H_0)|\psi_i\rangle}$$

- **Choose sufficiently narrow initial and final wave packets.**
- **Choose sufficiently large n.**
- **Replace $e^{2ine^{-\beta H}}$ by a polynomial approximation.**
- **Calculations formally independent of β, adjust β for faster convergence.**
- **Model allows independent tests of each approximation.**
- **Approximations must be done in the proper order.**

Results

- **Converges to exact sharp momentum transition matrix elements.**

- **Tests converge for** $.050 - 2$ **GeV.**

- **Biggest source of error is the wave packet width.**

Convergence with respect to wave packet width

Table 1

k_0 [GeV]	k_w [GeV]	% error	k_w/k_0
0.1	0.00308607	0.1	0.030
0.3	0.009759	0.1	0.032
0.5	0.0182574	0.1	0.036
0.7	0.0272166	0.1	0.038
0.9	0.0365148	0.1	0.040
1.1	0.0458831	0.1	0.041
1.3	0.0550482	0.1	0.042
1.5	0.0632456	0.1	0.042
1.7	0.0725476	0.1	0.042
1.9	0.0816497	0.1	0.042

Convergence with respect time "n"

Table 2: $k_0 = 2.0$**[GeV]**, $k_w = .09$**[GeV]**

n	Re $\langle\phi\|(S_n - I)\|\phi\rangle$	Im $\langle\phi\|(S_n - I)\|\phi\rangle$
50	-2.60094316473225e-6	1.94120750171791e-3
100	-2.82916859895010e-6	2.35553585404449e-3
150	-2.83171624670953e-6	2.37471383801820e-3
200	-2.83165946257657e-6	2.37492460997990e-3
250	-2.83165905312632e-6	2.37492527186858e-3
300	-2.83165905257121e-6	2.37492527262432e-3
350	-2.83165905190508e-6	2.37492527262493e-3
400	-2.83165905234917e-6	2.37492527262540e-3
ex	-2.83165905227843e-6	2.37492527259701e-3

Table 3: Parameter choices

k_0 [GeV]	$\beta[\mathrm{GeV}^{-1}]$	$k_0 \times \beta$	n
0.1	40.0	4.0	450
0.3	5.0	1.5	330
0.5	3.0	1.5	205
0.7	1.6	1.2	200
0.9	1.05	.945	190
1.1	0.95	1.045	200
1.3	0.85	1.105	200
1.5	0.63	0.945	200
1.7	0.5	0.85	200
1.9	0.42	0.798	200

Table 4: Convergence with respect to Polynomial degree e^{inx}

x	n	deg	poly error %
0.1	200	200	3.276e+00
0.1	200	250	1.925e-11
0.1	200	300	4.903e-13
0.1	630	630	2.069e+00
0.1	630	680	5.015e-08
0.1	630	700	7.456e-11
0.5	200	200	1.627e-13
0.5	200	250	3.266e-13
0.5	630	580	1.430e-14
0.5	630	680	9.330e-13
0.9	200	200	3.276e+00
0.9	200	250	1.950e-11
0.9	200	300	9.828e-13
0.9	630	630	2.069e+00
0.9	630	680	5.015e-08
0.9	630	700	7.230e-11

Table 5: Final calculation

k_0	Real T	Im T	% error
0.1	-2.30337e-1	-4.09325e-1	0.0956
0.3	-3.46973e-2	-6.97209e-3	0.0966
0.5	-6.44255e-3	-3.86459e-4	0.0986
0.7	-1.88847e-3	-4.63489e-5	0.0977
0.9	-7.28609e-4	-8.86653e-6	0.0982
1.1	-3.35731e-4	-2.30067e-6	0.0987
1.3	-1.74947e-4	-7.38285e-7	0.0985
1.5	-9.97346e-5	-2.76849e-7	0.0956
1.7	-6.08794e-5	-1.16909e-7	0.0964
1.9	-3.92110e-5	-5.42037e-8	0.0967

Unfinished business/things to consider

Structure theorem for reflection-positive Euclidean-invariant n>2 point functions.

General form of Bethe-Salpeter kernels that lead to reflection positive four-point functions?

Formulate N-body scattering based on two-body Bethe-Salpeter kernels?

Numerical test of the polynomial approximation to the Haag-Ruelle function $h(m^2)$ for a two-point function with a non-trivial Lehmann weight.

Scattering calculation based on a realistic four-point function.