Gauss's Law, Duality, and the Hamiltonian Framework of U(1) Lattice Gauge Theory

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36th Annual International Symposium on Lattice Field Theory

Work based on arXiv:1806.08797
(submitted to PRL)

Outline

1. Context of project
2. Recap: Conventional Hamiltonian LGT
3. The emergence of duality
   - Original theory set-up
   - Reconstruction begets duality

Roadmap

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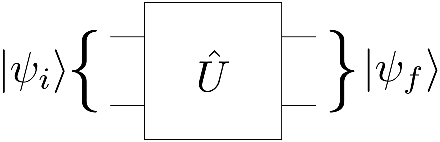
Unitary evolution on a quantum computer

Digital quantum computers (QC):



- Unitary gates $\sim e^{-it\hat{H}}$ of some $\hat{H}$.
- Want to simulate a lattice gauge theory (LGT)
- **How to map its $\hat{H}$ and its Hilbert space $\mathcal{H}$ on to QC?**

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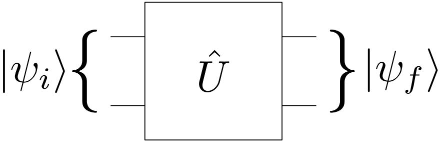

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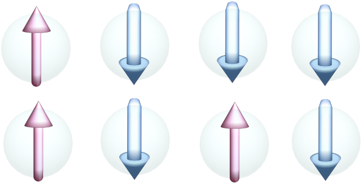
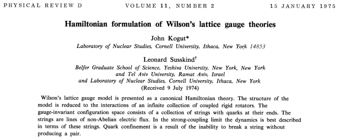
Near-term QC architectures will have very limited capabilities

- **How to most wisely spend those qubits?**



Previous work

- Arena for these questions is the Hamiltonian formalism of LGT.


PHYSICAL REVIEW D VOLUME 11, NUMBER 2 15 JANUARY 1975

Hamiltonian formulation of Wilson's lattice gauge theories

John Kogut*
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853

Leonard Susskind†
Belfer Graduate School of Science, Yeshiva University, New York, New York
and Tel Aviv University, Ramat Aviv, Israel
and Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
(Received 9 July 1974)

Wilson's lattice gauge model is presented as a canonical Hamiltonian theory. The structure of the model is reduced to the interactions of an infinite collection of coupled rigid rotators. The gauge-invariant configuration space consists of a collection of strings with quarks at their ends. The strings are lines of non-Abelian electric flux. In the strong-coupling limit the dynamics is best described in terms of these strings. Quark confinement is a result of the inability to break a string without producing a pair.

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Confinement of quarks*

Kenneth G. Wilson
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(Received 12 June 1974)

A mechanism for total confinement of quarks, similar to that of Schwinger, is defined which requires the existence of Abelian or non-Abelian gauge fields. It is shown how to quantize a gauge field theory on a discrete lattice in Euclidean space-time, preserving exact gauge invariance and treating the gauge fields as angular variables (which makes a gauge-fixing term unnecessary). The lattice gauge theory has a computable strong-coupling limit; in this limit the binding mechanism applies and there are no free quarks. There is unfortunately no Lorentz (or Euclidean) invariance in the strong-coupling limit. The strong-coupling expansion involves sums over all quark paths and sums over all surfaces (on the lattice) joining quark paths. This structure is reminiscent of relativistic string models of hadrons.

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- Taking pure U(1) LGT, we seek **most economical construction**
  - Leads directly to duality transformation

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- Taking pure U(1) LGT, we seek **most economical construction**
  - Leads directly to duality transformation
- Dualities also extensively studied in LGTs and many other areas
  - See, e.g., Anishetty and Sharatchandra 1990; Mathur 2006; Anishetty and Sreeraj 2018

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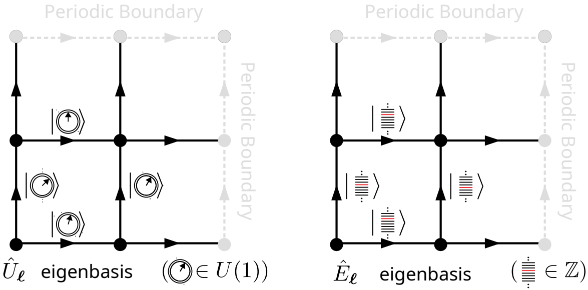
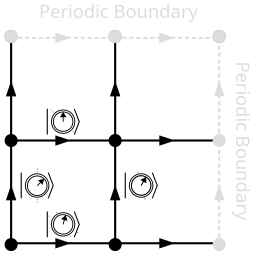
Conventional construction



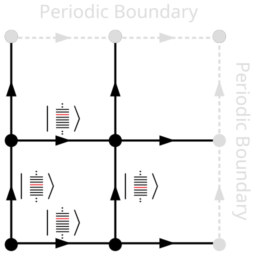
Link operators raise or lower electric field:

$$\hat{U}\,|\equiv\rangle = |\equiv\rangle$$

Conventional construction



$\hat{U}_\ell$ eigenbasis ($\in U(1)$)



$\hat{E}_\ell$ eigenbasis ($\in \mathbb{Z}$)

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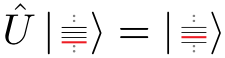


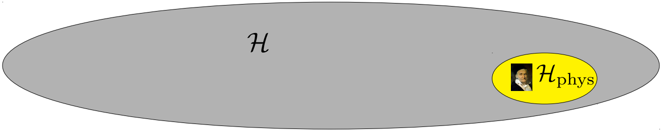
Kogut-Susskind Hamiltonian:

$$H_E = \frac{1}{2a_s} \sum_{\boldsymbol{\ell}} \tilde{g}_t^2 \hat{\mathcal{E}}_{\boldsymbol{\ell}}^2, \qquad H_B = \frac{1}{2a_s} \left[\frac{1}{\tilde{g}_s^2} \sum_{\boldsymbol{p}} \left(2 - \hat{P}_{\boldsymbol{p}} - \hat{P}_{\boldsymbol{p}}^\dagger \right) \right]$$

$$H_E + H_B \xrightarrow{a_s \to 0} H = \frac{1}{2} \int d^D x \, (\mathbf{E}^2 + \mathbf{B}^2)$$

Issues with standard formulation

1. Must impose **Gauss's law** on kets [Kogut and Susskind 1975; Zohar et al. 2017]
   - Most directions in $\mathcal{H}$ unphysical.



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   - If truncating states (by e.g. $|\mathcal{E}_{\boldsymbol{\ell}}| \leqslant \Lambda$ in $U(1)$), makes awkward constraints around cutoff.

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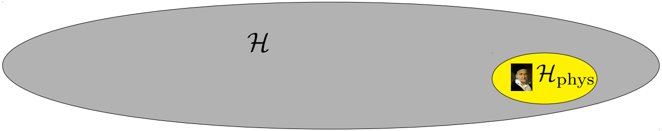

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2. Electric fluctuations large at weak coupling
   - Expect large $\mathbf{E}$ fluctuations as $a_s \to 0$ in $D = 2$ gauge theories and in asymptotically-free theories in $D = 3$
   - Rate of convergence as $a_s \to 0$ unclear when truncating on $\mathbf{E}$

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Starting point for original theory

We start with a symmetric Hamiltonian,[1]

$$\begin{aligned}
\hat{H} &= \hat{H}_E + \hat{H}_B\ , \\
\hat{H}_B &= \frac{1}{2a_s}\left[\frac{1}{\tilde{g}_s^2}\sum_{\boldsymbol{p}}\left(2-\hat{P}_{\boldsymbol{p}}-\hat{P}_{\boldsymbol{p}}^\dagger\right)\right], \\
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- Hilbert space $\mathcal{H}$ and $\hat{H}_B$ *are conventional*
- We exponentiated $\mathbf{E}$:

$$\hat{Q}_{\boldsymbol{\ell}} \equiv e^{i\xi\hat{\mathcal{E}}_{\boldsymbol{\ell}}}\ .$$

- Think of $\xi \ll 1$ as a_t/a_s.

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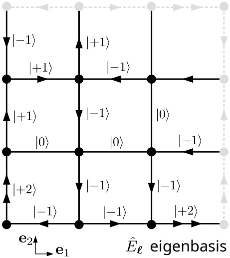
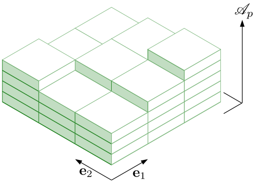
Hilbert space generation



$\hat{E}_\ell$ eigenbasis

$=$



Powers of plaquettes

A basis for $\mathcal{H}_{\text{phys}}$ is generated by acting with plaquettes on trivial state.

$$
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|\Omega\rangle &\equiv \otimes_{\boldsymbol{\ell}} |0\rangle_{\boldsymbol{\ell}} \ , \\
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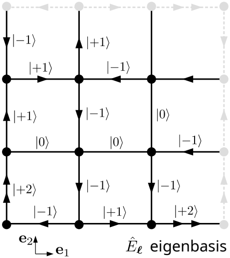

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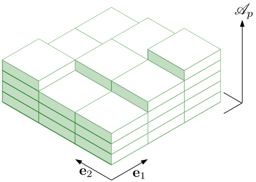

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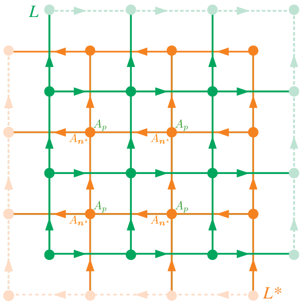
$D = 2$ **for this talk.**

Hilbert space transcription

Take $\mathscr{A}$'s further: Use as *quantum numbers*



Notice:

- Plaquettes $\boldsymbol{p} \sim$ dual sites $\mathbf{n}^\star$.
  $\Rightarrow \mathscr{A}_{\boldsymbol{p}}$ is scalar field $\mathscr{A}_{\mathbf{n}^\star}$ on $L^\star$.
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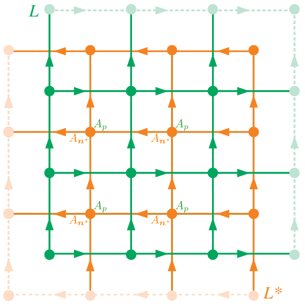

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1.) Identify

$$\prod_{\boldsymbol{p}} \left(\hat{P}_{\boldsymbol{p}}\right)^{\mathscr{A}_{\boldsymbol{p}}} \Bigg|_{\mathscr{A}_{\boldsymbol{p}}=\mathscr{A}_{\mathbf{n}^\star(\boldsymbol{p})}} |\Omega\rangle \quad \longleftrightarrow \quad \otimes_{\mathbf{n}^\star} |\mathscr{A}_{\mathbf{n}^\star}\rangle$$

Hilbert space transcription

2. Define identical local orthonormal bases, $\{|\mathscr{A}_{\mathbf{n}^\star}\rangle\}$, which diagonalize

$$\hat{\mathscr{U}}_{\mathbf{n}^\star} \equiv \sum_{\mathscr{A}_{\mathbf{n}^\star}=-\infty}^{\infty} |\mathscr{A}_{\mathbf{n}^\star}\rangle\, e^{i\xi\mathscr{A}_{\mathbf{n}^\star}}\, \langle\mathscr{A}_{\mathbf{n}^\star}| \ .$$

3. Global basis states:

$$|\mathscr{A}_{L^\star}\rangle \equiv \otimes_{\mathbf{n}^\star} |\mathscr{A}_{\mathbf{n}^\star}\rangle$$

4. (Local) raising operators:

$$\hat{\mathscr{Q}}_{\mathbf{n}^\star} \equiv \sum_{\mathscr{A}_{\mathbf{n}^\star}=-\infty}^{\infty} |\mathscr{A}_{\mathbf{n}^\star}+1\rangle\langle\mathscr{A}_{\mathbf{n}^\star}| \ .$$

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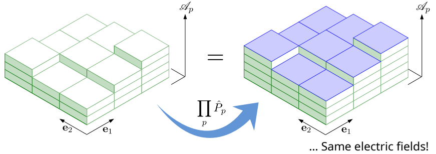
Redundancy:



Since $\prod_{\boldsymbol{p}} \left(\hat{P}_{\boldsymbol{p}}\right) = \hat{1}$, must impose

$$\prod_{\mathbf{n}^\star} \hat{\mathscr{Q}}_{\mathbf{n}^\star} |\mathscr{A}_{L^\star}\rangle = |\mathscr{A}_{L^\star}\rangle$$

on $\mathcal{H}^\star$. This is *magnetic Gauss law*.

The dual formulation

Original		Dual
plaquette, $\boldsymbol{p}$	$\leftrightarrow$	site, $\mathbf{n}^\star$
plaquette operator, $\hat{P}_{\boldsymbol{p}}$	$\leftrightarrow$	site raising operator, $\hat{\mathscr{Q}}_{\mathbf{n}^\star}$
link $\boldsymbol{\ell}$	$\leftrightarrow$	(perpendicular) link, $\boldsymbol{\ell}^\star$
field square, $\mathcal{E}_{\boldsymbol{\ell}}^2$	$\leftrightarrow$	field laplacian, $\hat{\mathscr{U}}_{\mathbf{n}^\star}^\dagger \partial_i^+ \partial_i^- \hat{\mathscr{U}}_{\mathbf{n}^\star}$

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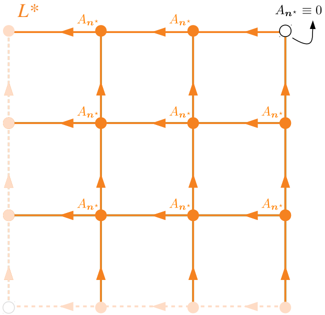
We have $\langle\mathscr{A}'_L|\,\hat{H}\,|\mathscr{A}_L\rangle = \langle\mathscr{A}'_{L^\star}|\,\mathscr{H}\,|\mathscr{A}_{L^\star}\rangle$ for the dual Hamiltonian

$$\boxed{\hat{\mathscr{H}} = \frac{1}{2a_s}\sum_{\mathbf{n}^\star}\left[\frac{1}{\tilde{g}_s^2}\left(2 - \hat{\mathscr{Q}}_{\mathbf{n}^\star} - \hat{\mathscr{Q}}^\dagger_{\mathbf{n}^\star}\right) - \frac{\tilde{g}_t^2}{\xi^2}a_s^2\hat{\mathscr{U}}^\dagger_{\mathbf{n}^\star}\partial_i^+\partial_i^-\hat{\mathscr{U}}_{\mathbf{n}^\star}\right], \qquad (D=2)}$$

(subject to magnetic Gauss).

Solving the dual Gauss law:

1. Fix one $\mathscr{A}_{\mathbf{n}^\star} = 0$.
   ↳ Break translational symmetries
   ↳ $\hat{\mathscr{H}}$ becomes nonlocal
   - Truncation can be done as $|\mathscr{A}_{\mathbf{n}^\star}| \leqslant \Lambda$



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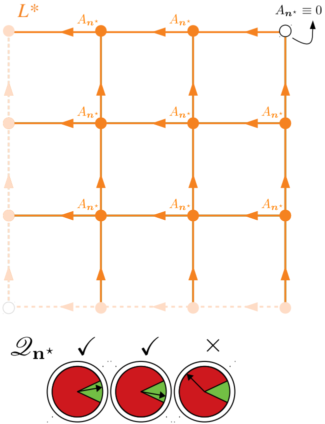
2. Restrict states to subspace on which $\prod_{\mathbf{n}^\star} \hat{\mathscr{Q}}_{\mathbf{n}^\star} = 1$
   - Truncation can be done on argument of $\mathscr{Q}_{\mathbf{n}^\star}$ phases (equivalent to regulating $\mathbf{B}$ in original theory)



Summary

1. Duality transformation **naturally emerges** from building $\boldsymbol{\nabla} \cdot \mathbf{E} = 0$ into $\mathcal{H}$
2. Formulating and truncating dual theory **preferable for weak coupling**

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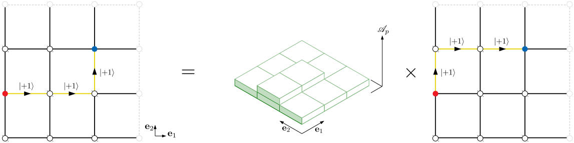

Current/future work

- Putting in matter
  - Want: Local Hilbert spaces, $\hat{H}$ built from local operators
  - How much redundancy?
- Extend to non-Abelian
  - Local field description possible with non-Abelian lattice duality? (prepotential formalism)

Acknowledgments

I thank Natalie Klco and Martin Savage at the Institute for Nuclear Theory for helpful conversations.

This material is based upon work supported by the National Science Foundation Graduate Research Fellowship under Grant No. DGE-1256082.

$\mathbb{E}$ fluctuations at weak coupling

Analogy to SHO: (electric field is momentum, gauge field is coordinate)

$$\begin{aligned} H_E &= \frac{1}{2a_s}\sum_{\boldsymbol{\ell}} \tilde{g}_t^2 \hat{\mathcal{E}}_{\boldsymbol{\ell}}^2 &\sim& \quad \frac{1}{2m}\hat{p}^2 \\ H_B &= \frac{1}{2a_s}\left[\frac{1}{\tilde{g}_s^2}\sum_{\boldsymbol{p}}\left(2-\hat{P}_{\boldsymbol{p}}-\hat{P}_{\boldsymbol{p}}^\dagger\right)\right] &\sim& \quad \frac{k}{2}\hat{x}^2 \end{aligned}$$

Read off

$$m \sim 1/\tilde{g}_t^2, \qquad k \sim 1/\tilde{g}_s^2$$

By dimensional analysis,

$$\langle \hat{p}^2\rangle \propto \sqrt{mk} \sim \frac{1}{\tilde{g}_t\tilde{g}_s}, \qquad \langle \hat{x}^2\rangle \propto \frac{1}{\sqrt{mk}} \sim \tilde{g}_t\tilde{g}_s$$

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Topological sectors

Original formulation (on periodic lattice) has many gauge-invariant states decoupled from $|\Omega\rangle$

- Topological Polyakov loops are gauge-invariant
- Define class representatives,

$$|\boldsymbol{\nu}\rangle \equiv \prod_{i=1}^{d} \left(\hat{W}(C_i)\right)^{\nu_i} |0\rangle \ , \quad \nu_i \in \mathbb{Z} \ .$$

  with $\hat{W}(C_i)$ the product of oriented $\hat{U}_{\boldsymbol{\ell}}$'s along a closed loop C_i wrapping direction i.
- An $\hat{H}$ containing only elementary Wilson loops cannot cause transitions

Fully general state:

$$|\mathscr{A}\rangle_{\boldsymbol{\nu}} = \prod_{\boldsymbol{p}} \left(\hat{P}_{\boldsymbol{p}}\right)^{\mathscr{A}_{\boldsymbol{p}}} |\boldsymbol{\nu}\rangle \ , \ \mathscr{A}_{\boldsymbol{p}} \in \mathbb{Z}$$

Dual Hamiltonian with topology

Since $\boldsymbol{\nu}$'s don't talk to each other, we fix $\boldsymbol{\nu}$. We must adapt $\mathscr{H}$ to get the right matrix elements:

$$\mathscr{H} \quad \rightarrow \quad \mathscr{H}^{\boldsymbol{\nu}} = \mathscr{H}_B + \mathscr{H}_E^{\boldsymbol{\nu}}, \qquad (\mathscr{H}_B \text{ unchanged})$$

$$\mathscr{H}_E^{\boldsymbol{\nu}} = \frac{1}{2a_s} \sum_{\mathbf{n}^\star} \left[-\frac{\tilde{g}_t^2}{\xi^2} a_s^2 \hat{\mathscr{U}}_{\mathbf{n}^\star}^\dagger \Delta \hat{\mathscr{U}}_{\mathbf{n}^\star} \right], \qquad (\mathrm{D} = 2)$$

Here we have generalized to a **covariant Laplacian** $\Delta = \Sigma_{i=1}^2 D_i^+ D_i^-$,

$$D_1^+ F_{\mathbf{n}^\star} = (\mathscr{W}_{\{\mathbf{n}^\star, \mathbf{n}^\star - \mathbf{e}_1\}} F_{\mathbf{n}^\star - \mathbf{e}_1} - F_{\mathbf{n}^\star})/a_s ,$$
$$D_2^+ F_{\mathbf{n}^\star} = (\mathscr{W}_{\{\mathbf{n}^\star, \mathbf{n}^\star + \mathbf{e}_2\}} F_{\mathbf{n}^\star + \mathbf{e}_2} - F_{\mathbf{n}^\star})/a_s ,$$

involving the (dual lattice) **connection**

$$\mathscr{W}_{\boldsymbol{\ell}^\star} = \begin{cases} e^{i\xi\nu_i}, & \text{if } \boldsymbol{\ell} \in C_i\,; \\ 1, & \text{otherwise} \end{cases}$$

Dual Hamiltonian in $d = 3 + 1$

For $D = 3$ spatial dimensions, $\boldsymbol{p} \leftrightarrow \boldsymbol{\ell}^\star$ (rather than $\boldsymbol{p} \leftrightarrow \mathbf{n}^\star$).

- We define $\hat{\mathscr{Q}}_{\boldsymbol{\ell}^\star}$'s and $\hat{\mathscr{U}}_{\boldsymbol{\ell}^\star}$'s on local **dual link Hilbert spaces** by direct analogy.
- Then

$$\hat{\mathscr{H}}_{\boldsymbol{\nu}} = \frac{1}{2a_s}\left[\sum_{\boldsymbol{\ell}^\star} \frac{1}{\tilde{g}_s^2}\left(2 - \hat{\mathscr{Q}}_{\boldsymbol{\ell}^\star} - \hat{\mathscr{Q}}_{\boldsymbol{\ell}^\star}^\dagger\right) + \frac{\tilde{g}_t^2}{\xi^2}\sum_{\boldsymbol{p}^\star}\left(2 - \left(\mathscr{W}_{\boldsymbol{p}^\star}\hat{\mathscr{P}}_{\boldsymbol{p}^\star} + \mathrm{h.c.}\right)\right)\right] \quad (D = 3).$$

  - Dual plaquettes $\hat{\mathscr{P}}_{\boldsymbol{p}^\star}$ are usual products of $\hat{\mathscr{U}}_{\boldsymbol{\ell}^\star}$'s, and

$$\mathscr{W}_{\boldsymbol{p}^\star} = \begin{cases} e^{i\xi\nu_i}, & \text{if } \boldsymbol{\ell} \in C_i\,; \\ 1, & \text{otherwise}\,. \end{cases}$$

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