The perturbative SU(N) one-loop running coupling in the twisted gradient flow scheme

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   - Expanding the coupling in perturbation theory

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   - Rewriting the observable in integral form
   - Identifying and regulating the divergences
   - Running coupling and lambda parameter

3. Numerical Results

4. Conclusions

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Introduction

- In the last years, the gradient flow has become quite the popular tool to work in Yang-Mills theories.
- Computations in perturbation theory using the gradient flow, however, are comparatively scarce, with results being obtained by Lüscher for the running of the coupling at infinite volume, as well as other relevant results by Harlander *et al.*, by Ishikawa *et al.* or by Dalla Brida *et al.*
- In our case, our goal is to compute the running of the 't Hooft coupling constant in perturbation theory on the twisted torus, using a particular choice of boundary conditions and choice of regularisation that we will explain along this talk.

References

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Twisted boundary conditions

- We considered a $SU(N)$ pure gauge theory defined on an asymmetrical d-dimensional torus with sides of length l_μ **in the continuum**, with twisted boundary conditions in d_t dimensions and periodic ones in the rest.
- We chose to work in $d = 4$ and $d_t = 2$, and used the following twist:

$$A_\mu(x + l_\nu \hat{\nu}) = \Gamma_\nu A_\mu(x) \Gamma_\nu^\dagger, \qquad \Gamma_\mu \in SU(N), \quad k, l_g = N^{2/d_t} \in \mathbb{Z}$$
$$\Gamma_\mu \Gamma_\nu = \exp\{2\pi i \epsilon_{\mu\nu} k / l_g\} \Gamma_\nu \Gamma_\mu, \quad \epsilon_{01} = -\epsilon_{10} = 1, \epsilon_{\mu\nu} = 0 \text{ otherwise}$$

- In the periodic directions, the Γ_μ matrices are simply the identity.

Gonzalez-Arroyo *et al* '83

Choice of basis

- A solution for those boundary conditions can be obtained bulding a momentum-dependent basis $\hat{\Gamma}(q)$ for the fields from the Γ_μ matrices:

$$A_\mu(x) = V^{-\frac{1}{2}} \sum_q{}' \hat{A}_\mu(q) e^{iqx} \,\hat{\Gamma}(q), \quad V = \prod_\mu l_\mu$$

- As we picked k and N coprime, there are N^2 independent $\hat{\Gamma}$ matrices from which to build a basis for the $SU(N)$ fields. Tracelessness forces us to exclude the identity, which eliminates zero modes (modulo N) in the twisted directions. This is indicated by a prime in the sum.
- In twisted directions, the momenta are quantised in terms of $l_\mu l_g$, and in the rest in terms of l_μ only. For maximum symmetry, we chose a torus of length l in the twisted directions and $\tilde{l} = l_g l$ in the rest, so that all momenta are quantised equally: $q_\mu = 2\pi\tilde{l}^{-1} m_\mu, \quad m_\mu \in \mathbb{Z}$:

Gradient flow

- To define the running coupling, we used the gradient flow.
- We introduced a flow time parameter t and a gauge field $B_\mu(x,t)$, along with the field strength and covariant derivative $G_{\nu\mu}$ and D_ν:

$$B_\mu(x, t = 0) = A_\mu(x), \quad \partial_t B_\mu(x,t) = D_\nu G_{\nu\mu}(x,t) + \xi D_\mu \partial_\nu B_\nu(x,t)$$

- For $t > 0$, observables built from the expectation values of B are renormalised quantities, so we defined the 't Hooft coupling as:

$$\lambda_{TGF}(\tilde{l}) = g^2(\tilde{l})N = \left. \frac{< t^2 E(t) >}{N\mathcal{F}(c)} \right|_{t=c^2\tilde{l}^2/8}$$

Where $E(t) = \frac{1}{2}\mathrm{Tr}(G^2_{\mu\nu}(x,t))$ is the action density of the theory, $\mathcal{F}(c)$ was set up so that $\lambda_{TGF} = \lambda_0 + \mathcal{O}(\lambda_0^2)$, and c is a scheme-defining parameter relating the energy scale to the size of the torus: $1/\mu = \sqrt{8t} = c\tilde{l}$ Ramos '14

Perturbative expansion

- The procedure is analogous to the infinite volume one (Lüscher '10), only integrals are replaced by sums and our choice of basis comes with different structure constants $[\hat{\Gamma}(p), \hat{\Gamma}(q)] = iF(p,q)\hat{\Gamma}(p+q)$:

$$F(p,q) = -\sqrt{\frac{2}{N}} \sin(\frac{1}{2}\theta_{\mu\nu} p_\mu q_\nu), \quad \theta_{\mu\nu} = \frac{\bar{k}\tilde{l}^2}{2\pi l_g}\tilde{\epsilon}_{\mu\nu}, \quad \begin{array}{c} k\bar{k} = 1 \bmod l_g \\ \epsilon_{\mu\lambda}\tilde{\epsilon}_{\lambda\nu} = \delta_{\mu\nu} \end{array}$$

- We expand the gauge potential in powers of g_0 in momentum space:

$$B_\mu = \sum_k g_0^k B_\mu^{(k)}, \quad B_\mu^{(k)}(x,t) = V^{-\frac{1}{2}} \sum_q{}' B_\mu^{(k)}(q,t)\, e^{iqx}\, \hat{\Gamma}(q)$$

- We then plug this expansion into the flow equation, set $\xi = 1$, and solve them order by order to get results of the form:

$$B_\mu^{(1)}(p,t) = e^{-p^2 t} A_\mu(p), \quad B_\mu^{(i)}(p,t) = \int_0^t ds e^{-(t-s)p^2} R_\mu^{(i)}(p,s)$$

Outline

Starting point

- We wished to compute the observable up to order $\mathcal{O}(\lambda_0^4)$:

$$\mathcal{E} \equiv N^{-1} < E(t) >= \frac{1}{2N} < \mathrm{Tr}(G_{\mu\nu}^2(x,t)) >$$

- We expressed $G_{\mu\nu}$ in terms of the B_μ fields, expanded the fields in perturbation theory, plugged in the solutions to the flow equations to relate them to the A_μ fields, and used the standard Feynman rules to obtain the corresponding expectation values.
- We obtained seven different terms contributing to $\mathcal{E} = \sum_{i=0}^{6} \mathcal{E}_i$.
- One of terms is of order $\mathcal{O}(\lambda_0)$, and the rest are of order $\mathcal{O}(\lambda_0^2)$. For instance, the term $\mathcal{E}_5$ is:

$$\lambda_0^2 \tilde{l}^{-2d}(1-d) \int_0^t ds \sum_{q,r} NF^2(q,r) e^{-(t+s)(q^2+r^2)-(t-s)p^2} \frac{5r^2+qr}{p^2 q^2}$$

Integral form of the observable

- The perturbative expansion of $\mathcal{E}$ at $\mathcal{O}(\lambda_0^2)$ can be written as:

$$\mathcal{E} \equiv \lambda_0 \mathcal{E}^{(0)}(t) + \lambda_0^2 \mathcal{E}^{(1)}(t) + \mathcal{O}(\lambda_0^3)$$

$$\mathcal{E}^{(0)} = \frac{1}{2}\lambda_0 \tilde{l}^{-d}(d-1)\sum_q{}' e^{-2tq^2}$$

- For the subleading term, we rewrote the denominators using Schwinger's parametrisation, and the numerators as flow time derivatives, and were able to rewrite it after a bit of algebra as the sum of twelve basic integrals:

$$\mathcal{E}^{(1)}(t) = 2(d-2)(I_1 + I_2) - 4(d-1)I_3 + 4(3d-5)I_4$$
$$+ 6(d-1)(I_5 - I_6) - 2(d-2)(d-1)I_7 + \frac{1}{2}(d-2)^2 I_8 +$$
$$(d-2)^2 I_9 - 2(d-1)(I_{10} + I_{11}) - 4(d-1)I_{12}$$

Example

- As an example, one of the simplest integrals is, introducing three auxiliary variables $t' = 8t/(c\tilde{l})^2$, $\hat{c} = \pi c^2/2$, $\hat{\theta} = \bar{k}/l_g$, and a prefactor $\mathcal{N} = \hat{c}^2/32\pi^2\tilde{l}^{(2d-4)}$:

$$I_{10}(t') = \int_0^\infty dz \int_0^{t'} dx x \, \partial_{t'} \Phi \left(2t' + z, 2t', x\right)$$
$$\Phi(s, u, v) = \mathcal{N} \sum_{m,n \in \mathbb{Z}^d} e^{-\pi \hat{c}(sm^2 + un^2 + 2vmn)} (1 - \text{Re } e^{-2\pi i \hat{\theta} n \tilde{\epsilon} m})$$

- These Φ functions can be written in terms of Siegel Theta functions, often implemented in computational software such as Mathematica:

$$\Phi(s, u, v) = \mathcal{N} \text{Re}(\Theta(\hat{c}s, \hat{c}u, \hat{c}v, 0) - \Theta(\hat{c}s, \hat{c}u, \hat{c}v, \hat{\theta}))$$

Structure of the divergences

- Some of the integrals in $\mathcal{E}^{(1)}$ are UV divergent. After a bit of algebra, all integrals can be made such that all divergences occur for $u = 0$, with $v \propto u$ and with $\alpha \equiv s - v^2/u > 0$
- Using Poisson resummation and defining $n' \equiv n_\mu - \hat{\theta}\tilde{\epsilon}_{\mu\nu} m_\nu$ we have:

$$\Theta(s,u,v,\hat{\theta}) = (\hat{c}u)^{-\frac{d}{2}} \sum_{m,n} e^{-\pi\hat{c}(s-\frac{v^2}{u})m^2 - \frac{\pi}{\hat{c}u}n'^2 - i\frac{2\pi v}{u}mn}$$

- Singularities arise when $\hat{\theta}\tilde{\epsilon}_{\mu\nu} m_\nu \in \mathbb{Z}$ in all directions. In our integrals, this happens in two situations, leading to singularities of the form:

$$\Theta \sim (\hat{c}u)^{-\frac{d}{2}} \sum_m \exp(-\pi\hat{c}\alpha m^2)$$

- For $\hat{\theta} = 0$, there are divergences for $n'_\mu = n_\mu = 0$ in all directions.
- For $\hat{\theta} \neq 0$, there are divergences for $n_\mu = 0$ and $m_\nu = 0$ mod l_g in all twisted directions.

Dealing with the divergences I

- To deal with the divergences, we isolated the terms containing them, subtracted them from the quantity to compute numerically, and then obtained them analitically in dimensional regularisation.
- For the $\hat{\theta} \neq 0$ divergences, we defined an auxiliary H function so that the divergences are reabsorbed into a $\hat{\theta} = 0$ term:

$$\Phi(s, u, v, \hat{\theta}) = H(s, u, v, 0) - H(s, u, v, \hat{\theta})$$

$$H(s, u, v, \hat{\theta}) = \mathcal{N} \sum_{n} \sum_{m}{}' \mathrm{Re}\ e^{-\pi \hat{c}(sm^2 + un^2 + 2vmn) - 2\pi i \hat{\theta} m \tilde{\epsilon} n}$$

- For $\hat{\theta} = 0$, recalling that $\alpha = s - \frac{v^2}{u}$, we define two functions:

$$\mathcal{A}(\hat{c}\alpha) \equiv \alpha^{d/2} \sum_{m \in \mathbb{Z}^d}{}' e^{-\pi \alpha m^2}, \quad \phi^{\infty}(s, u, v) \equiv \mathcal{N} \hat{c}^{-d} (u\alpha)^{-d/2}$$

Dealing with the divergences II

- Taking $\hat{\theta} = 0$ and $n = 0$ in H and integrating the result leads to an integral $I^{(0)}$ containing the UV divergences plus a finite part:

$$\phi^{(0)}(s,u,v) = \mathcal{A}(\hat{c}\alpha)\phi^{\infty}(s,u,v), \quad I_i^{(0)} = \int \phi^{(0)}(s,u,v)$$

  We construct a finite quantity subtracting $I_i^{(0)}$ from I_i, and all that is left is to split and compute the finite and divergent parts of $I_i^{(0)}$.

- Expanding $A(\hat{c}\alpha)$ around $u = 0$, asymptotically we have:

$$I_i^{\text{div}} = \mathcal{A}(2\hat{c}t')I_i^{\infty}, \quad I_i^{\infty} = \int \phi^{\infty}(s,u,v)$$

  The divergence is contained in I_i^{∞}, which is proportional to the infinite volume integral, and can be computed in dimensional regularisation.

- To deal with this divergence, we simply computed the finite quantities $I_i^{(0)} - I_i^{\text{div}}$ and $\mathcal{A}(2\hat{c}t')$ numerically, and obtained I_i^{div} analytically.

Dealing with the divergences - Summary

- To summarise the previous two slides, we simply separated the integrals into three parts:

$$I_i = (I_i - I_i^{(0)}) + (I_i^{(0)} - I_i^{\text{div}}) + I_i^{\text{div}}$$

- The first two terms, $I_i - I_i^{(0)}$ and $I_i^{(0)} - I_i^{\text{div}}$, were computed numerically, preparing a C++ integration program expressly for the former, and using Mathematica to compute the latter.
- The last term, I_i^{div}, was obtained through a combination of numerical and analytical approaches: the momentum part, contained in $\mathcal{A}(2\hat{c}t')$, was obtained numerically using Mathematica, while the I_i^{∞} terms were obtained analitically using dimensional regularisation.

Running coupling and lambda parameter

- Combining everything that has been mentioned, we end up with:

$$\mathcal{E} = \frac{\lambda_0 (d-1)\mathcal{A}(2\hat{c}t')}{2(8\pi t)^{d/2}} \left\{ 1 + \lambda_0 \frac{(32\pi t)^{\epsilon}}{16\pi^2} \left(\frac{11}{3\epsilon} + \frac{52}{9} - 3\log 3 + C_1(t') \right) \right\}$$

- Inserting the the bare coupling in terms of the $\overline{\mathrm{MS}}$ one at a scale $\mu = 1/c\tilde{l}$, this yields the following coupling and Λ parameter:

$$\lambda_{TGF}(\tilde{l}) = \lambda_{\overline{\mathrm{MS}}} \left(1 + \frac{\lambda_{\overline{\mathrm{MS}}}}{16\pi^2} \left(\frac{11}{3}\gamma_E + \frac{52}{9} - 3\log 3 + C_1(t'=1) \right) \right)$$

$$\log \left(\frac{\Lambda_{TGF}}{\Lambda_{\overline{\mathrm{MS}}}} \right) = \frac{1}{32\pi^2 b_0} \left(\frac{11}{3}\gamma_E + \frac{52}{9} - 3\log 3 + C_1\left(t'=1\right) \right)$$

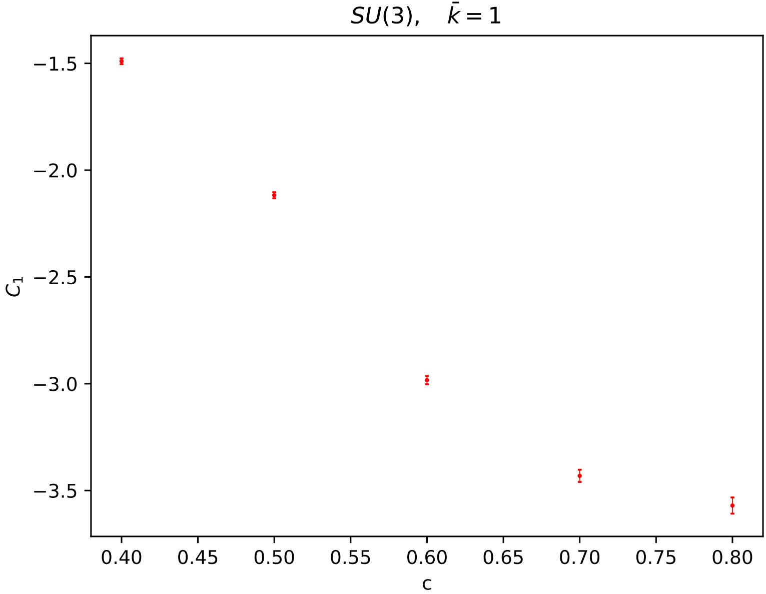
The large volume limit is obtained taking $c \to 0$, for which $C_1 = 0$ and we recover Lüscher's infinite volume results. Lüscher '10

Outline

Implementation

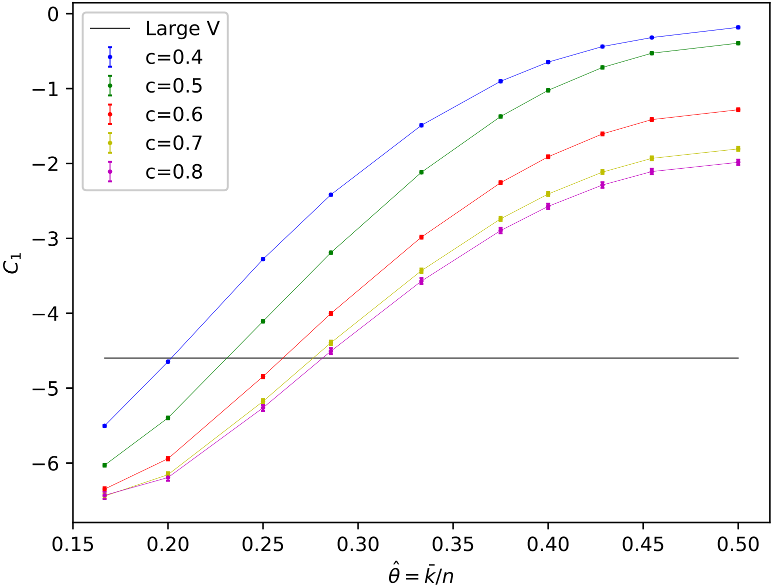
- Our goal was to obtain the finite constant $C_1(t' = 1)$, in order to obtain the running coupling and lambda parameter.
- We ran the computations for the case $d_t = 2$, for a series of values of c ranging from 0.4 to 0.8, and for eleven different combinations of $\bar{k}$ and N, in order to study its dependence on both c and $\hat{\theta} = \bar{k}/N$.
- For the first finite term $I_i - I_i^{(0)}$, we prepared a numerical C++ code to compute the sums over momenta and integrate over them using a trapezoidal rule integration algorithm for all twelve basic integrals.
- A large part of the results of the C++ code were cross-checked using an independent Mathematica code whenever possible.
- The second finite term, $I_i^{(0)} - I_i^{\text{div}}$, was computed using Mathematica. The momentum part $\mathcal{A}(2\hat{c}t')$ was obtained using Mathematica as well, whereas I_i^{∞} was determined analitically.

SU(3) results



$SU(3), \quad \bar{k} = 1$
−1.5
−2.0
−2.5
−3.0
−3.5
C_1
0.40
0.45
0.50
0.55
0.60
0.65
0.70
0.75
0.80
c

General results



Outline

Summary and conclusions

- We considered a $SU(N)$ pure gauge theory in the continuum, defined in an asymmetrical four-dimensional finite torus with twisted boundary conditions in one plane and periodical ones in the rest.
- We expanded the gauge fields in perturbation theory, using the gradient flow to define a renormalised 't Hooft coupling with effective length $\tilde{l} = Nl$ as the energy scale for the running, l being the physical size of the smaller, twisted sides of the torus.
- We rewrote the observable used to define the coupling in terms of twelve integrals. We devised a way to regularise these integrals and computed them for a range of values of $\hat{\theta}$ and c.
- We obtained the running coupling and Λ parameter as a function of $\hat{\theta}$ and c, and recovered the correct limits in large volume.
- In the future, we would like to repeat this computation on the lattice, defining and computing the running coupling in terms of $\tilde{l}$, as well as estimate the extent of finite volume corrections in relation to the matter of volume independence conjectures.

Thank you.