Multilevel integration for meson propagators

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Signal-to-noise ratio problem

Signal of a two-point function (e.g. for the *rho* meson) **[Parisi; 1983],[Lepage; 1989]**

$$C_\rho(y_0 - x_0) \propto e^{-m_\rho |y_0 - x_0|}$$

while its variance gets contribution from

$$\sigma_\rho^2(C_\rho(y_0 - x_0)) \propto e^{-2m_\pi |y_0 - x_0|}$$

and so the ratio is **exponentially** suppressed at large source-sink distances

$$\frac{C_\rho(y_0 - x_0)}{\sigma_\rho(y_0 - x_0)/\sqrt{N}} \propto \sqrt{N} e^{-(m_\rho - m_\pi)|y_0 - x_0|}$$

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Same problem for

nucleon correlators

$$\frac{C_N(y_0 - x_0)}{\sigma_N(y_0 - x_0)/\sqrt{N}} \propto \sqrt{N} e^{-(m_N - \frac{3}{2} m_\pi)|y_0 - x_0|}$$

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Same problem for

nonzero momentum correlators

$$\frac{C_{\pi,\mathbf{p}}(y_0 - x_0)}{\sigma_{\pi,\mathbf{p}}(y_0 - x_0)/\sqrt{N}} \propto \sqrt{N} e^{-(E_\pi(\mathbf{p}) - m_\pi)|y_0 - x_0|}$$

Multilevel integration

- A way of dealing with an exponentially vanishing signal-to-noise ratio is through a **multilevel integration scheme**
- Exploiting the locality of the theory, regions that are far away can be updated independently

Simplest case: two-level algorithm with two regions

- **Level-0**: n_0 realizations of boundary between two regions (later: "frozen region")
- **Level-1**: the two regions are independently updated n_1 for each of the n_0 boundaries
- $n_0 \times n_1^2$ configurations for the cost of $n_0 \times n_1$
- $n_{reg} > 2 \to n_1^{n_{reg}}$ samples - the error goes like

$$\delta(C) \sim n_1^{-\frac{n_{reg}}{2}} e^{-m_\pi |y_0 - x_0|}$$

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This is feasible if **both the action and the observable can be factorized**.

Solutions in pure-gauge theory:

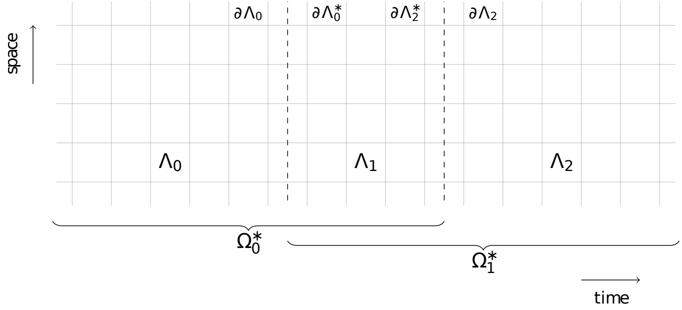
- multihit **[Parisi, Petronzio & Rapuano; 1983]**
- multilevel for Wilson loops and Polyakov loop correlators **[Lüscher & Weisz; 2001]**
- also **[Meyer; 2002]** and **[Giusti & Della Morte; 2008, 2010]**

After Wick's theorem it's not obvious how the QCD action and the observable (propagator) can be factorized.

$\rightarrow$ breakthrough in the development of a multilevel integration scheme for fermions **[Cè,Giusti & Schaefer; 2016,2017]**.

$\rightarrow$ both the determinant and the propagator can be completely factorized

Setup: overlapping domains Ω_0^* and Ω_1^*



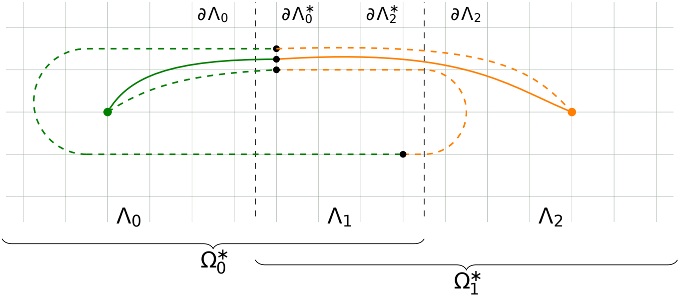
Factorization of the connected propagator

The **exact** propagator can be written as

$$Q^{-1} = -Q^{-1}_{\Omega_1^*} Q_{\Lambda_{1,0}} \frac{1}{1-\omega} Q^{-1}_{\Omega_0^*}$$

with $Q = \gamma_5 D$ and where

$$\omega = Q^{-1}_{\Omega_0^*} Q_{\Lambda_{1,2}} Q^{-1}_{\Omega_1^*} Q_{\Lambda_{1,0}}$$



Fully factorized approximation:

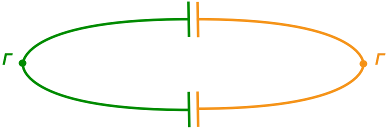
$$Q^{-1} \simeq -Q^{-1}_{\Omega_1^*} Q_{\Lambda_{1,0}} Q^{-1}_{\Omega_0^*}$$

A new method with noise sources

A viable method is needed to compute C^{fact} efficiently in a multilevel integration scheme

$$C_\Gamma^{\text{fact}}(y_0,x_0) = \text{Tr}\left\{ Q^{-1}_{\Omega_0^*}(\cdot,x_0)\Gamma Q^{-1}_{\Omega_0^*}(x_0,\cdot) Q_{\Lambda_{1,0}} Q^{-1}_{\Omega_1^*}(\cdot,y_0)\Gamma Q^{-1}_{\Omega_1^*}(y_0,\cdot) Q_{\Lambda_{1,0}} \right\}$$

Naively one would store $\sim (V_3)^2$ (or $(V_3)^3$ for baryons) complex numbers... $\rightarrow$ not feasible



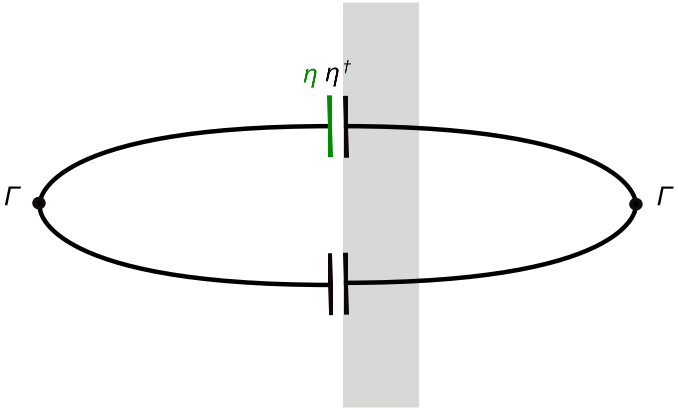
Our new proposal exploits the newly factorized observable by putting noise sources in the **middle** of the lattice

$\rightarrow$ right next to the boundary of the frozen region, i.e.

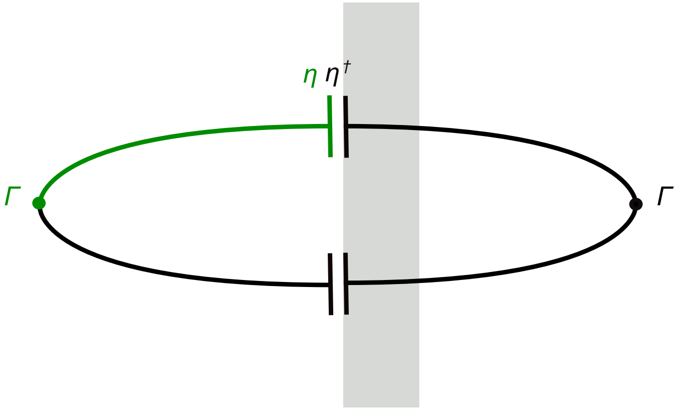
$$C_\Gamma^{\text{fact}}(y_0,x_0) = \frac{1}{N_\eta}\sum_i^{N_s} \text{Tr}\left\{ Q^{-1}_{\Omega_0^*}(\cdot,x_0)\Gamma Q^{-1}_{\Omega_0^*}(x_0,\cdot)\eta_i\eta_i^\dagger Q_{\Lambda_{1,0}} Q^{-1}_{\Omega_1^*}(\cdot,y_0)\Gamma Q^{-1}_{\Omega_1^*}(y_0,\cdot) Q_{\Lambda_{1,0}} \right\}$$

$\rightarrow$ the meson propagator can be computed **sequentially** $\rightarrow$ only $\sim V_3$ complex numbers to be saved

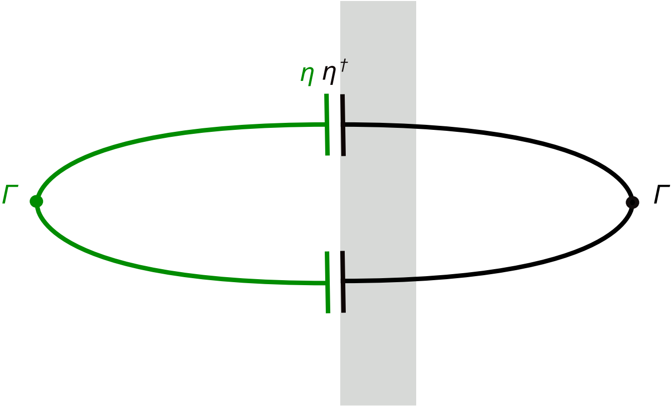
$$\mathrm{Tr}\left\{ Q^{-1}_{\Omega^*_1} Q_{\Lambda_{1,0}} Q^{-1}_{\Omega^*_0} \Gamma Q^{-1}_{\Omega^*_0} \, \eta \, \eta^\dagger \, Q_{\Lambda_{1,0}} Q^{-1}_{\Omega^*_1} \Gamma \right\}$$



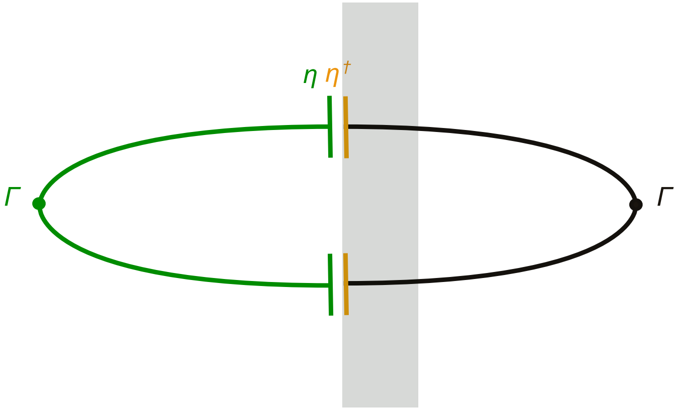
$$\mathrm{Tr}\left\{ Q^{-1}_{\Omega^*_1}\, Q_{\Lambda_{1,0}}\, Q^{-1}_{\Omega^*_0}\, \Gamma\, Q^{-1}_{\Omega^*_0}\, \eta\; \eta^\dagger\, Q_{\Lambda_{1,0}}\, Q^{-1}_{\Omega^*_1}\, \Gamma \right\}$$



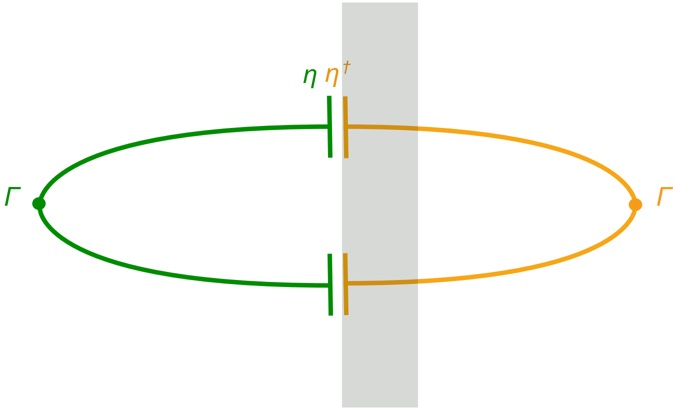
$$\mathrm{Tr}\left\{ Q^{-1}_{\Omega^*_1} Q_{\Lambda_{1,0}} Q^{-1}_{\Omega^*_0} \Gamma Q^{-1}_{\Omega^*_0} \eta \; \eta^\dagger Q_{\Lambda_{1,0}} Q^{-1}_{\Omega^*_1} \Gamma \right\}$$



$$\mathrm{Tr}\left\{ Q^{-1}_{\Omega^*_1}\, Q_{\Lambda_{1,0}}\, Q^{-1}_{\Omega^*_0} \Gamma\, Q^{-1}_{\Omega^*_0}\, \eta\, \eta^{\dagger}\, Q_{\Lambda_{1,0}}\, Q^{-1}_{\Omega^*_1} \Gamma \right\}$$



$$\mathrm{Tr}\left\{ Q^{-1}_{\Omega^*_1}\, Q_{\Lambda_{1,0}}\, Q^{-1}_{\Omega^*_0}\, \Gamma\, Q^{-1}_{\Omega^*_0}\, \eta\, \eta^\dagger\, Q_{\Lambda_{1,0}}\, Q^{-1}_{\Omega^*_1}\, \Gamma \right\}$$



A test in quenched QCD

The quenched theory suffers from the S-N ratio problem as well $\rightarrow$ perfect playground for a multilevel test

β	L/a	T/a	κ	c_{SW}	m_{PS} (MeV)	a (fm)
6.2	32	96	0.1352	1.61375	580	0.068

$\rightarrow$ periodic boundary conditions

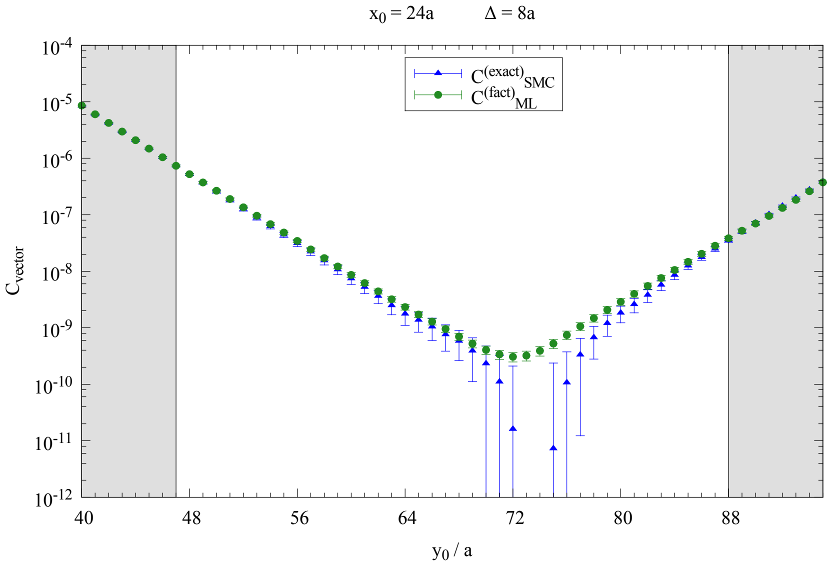
Multilevel simulation with

- two active regions connected by two frozen regions of width $\Delta = 8a$
- $n_0 = 50$ level-0 updates $\times$ $n_1 = 16$ level-1 updates
- $N_\eta = 40$ noise sources
- $\Gamma = \gamma_i$ with $i = 1, 2, 3$

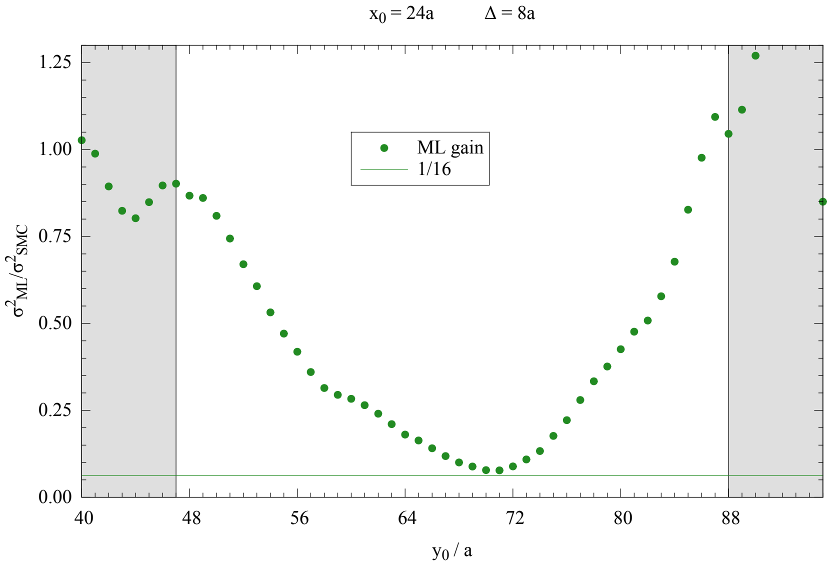
The factorized meson propagator is an approximation: also the rest (whose size depends on Δ) has to be computed. Strategy of the calculation:

$$\langle C_\rho \rangle = \langle C_\rho^{\text{fact}} \rangle_{\text{ML}} + \langle C_\rho^{\text{ex}} - C_\rho^{\text{fact}} \rangle_{\text{SMC}}$$

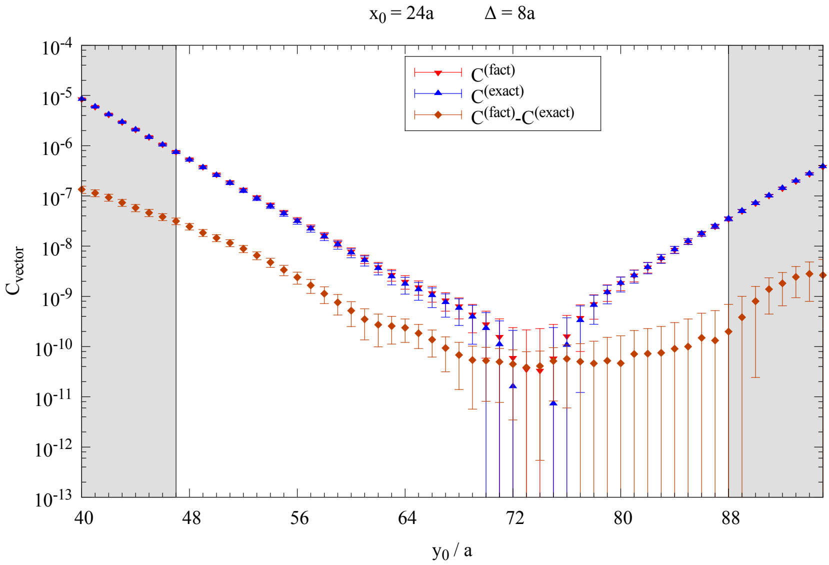
Vector-vector correlator



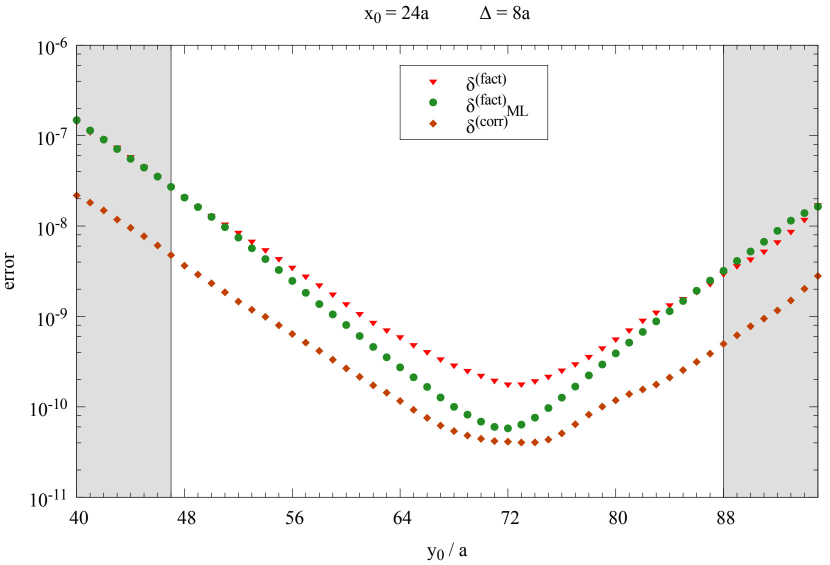
Vector-vector correlator: ML gain



The correction term



The correction term: error



Conclusions

- new method to directly compute the factorized form of the connected propagator **sequentially** (with 2 active regions)
- clear improvement with respect to previous method (projection to deflated modes)
- theoretical gain of multilevel (~ 16 in the cost) reached far away enough from the frozen region
- the rest of the propagator is computed with standard MC $\rightarrow$ in this case it is quite small $\sim 3-5\%$ for $\Delta = 8$
- if it is large and/or with large error $\rightarrow$ multilevel also on the rest
- next: advancements for disconnected propagator by Tim

Thanks for your attention!

Factorization of the quark propagator

- Decomposition of Hermitian Wilson-Dirac operator $Q = \gamma_5 D$ **[Lüscher; 2003]**

$$Q = \begin{pmatrix} Q_{\Gamma} & Q_{\partial\Gamma} \\ Q_{\partial\Gamma^*} & Q_{\Gamma^*} \end{pmatrix}$$

- Schur complement

$$S_{\Gamma} = Q_{\Gamma} - Q_{\partial\Gamma} Q_{\Gamma^*}^{-1} Q_{\partial\Gamma^*}$$

If $\Gamma = \Lambda_0$ and $\Gamma^* = \Omega_1^*$ we can write

$$Q^{-1} = \begin{pmatrix} S_{\Lambda_0}^{-1} & -S_{\Lambda_0}^{-1} Q_{\Lambda_{1,0}} Q_{\Omega_1^*}^{-1} \\ -Q_{\Omega_1^*}^{-1} Q_{\Lambda_{1,0}} S_{\Lambda_0}^{-1} & Q_{\Omega_1^*}^{-1} + Q_{\Omega_1^*}^{-1} Q_{\Lambda_{1,0}} S_{\Lambda_0}^{-1} Q_{\Lambda_{1,0}} Q_{\Omega_1^*}^{-1} \end{pmatrix}$$

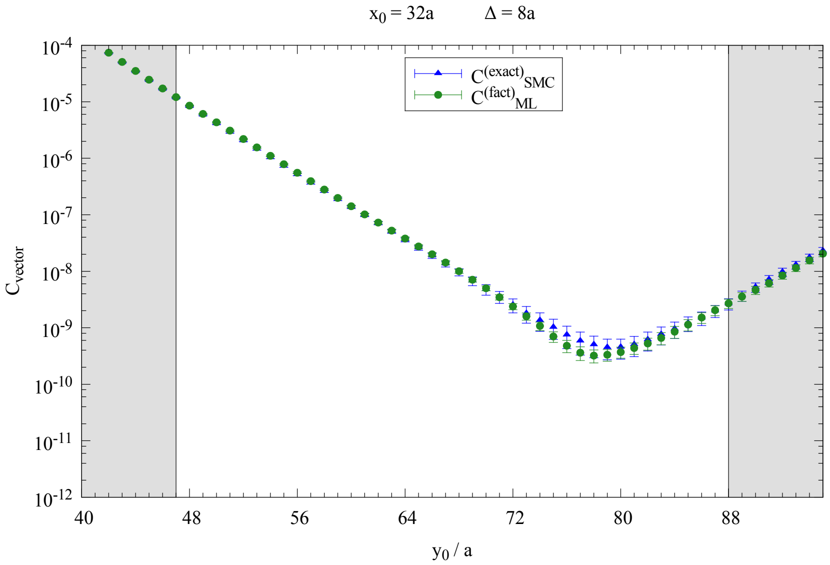
and noting that

$$S_{\Lambda_0}^{-1} = P_{\Lambda_0} Q^{-1} P_{\Lambda_0} = P_{\Lambda_0} Q_{\Omega_0^*}^{-1} P_{\Lambda_0} + \ldots$$

we get (bottom left element)

$$Q^{-1} \simeq -Q_{\Omega_1^*}^{-1} Q_{\Lambda_{1,0}} Q_{\Omega_0^*}^{-1}$$

Vector-vector correlator



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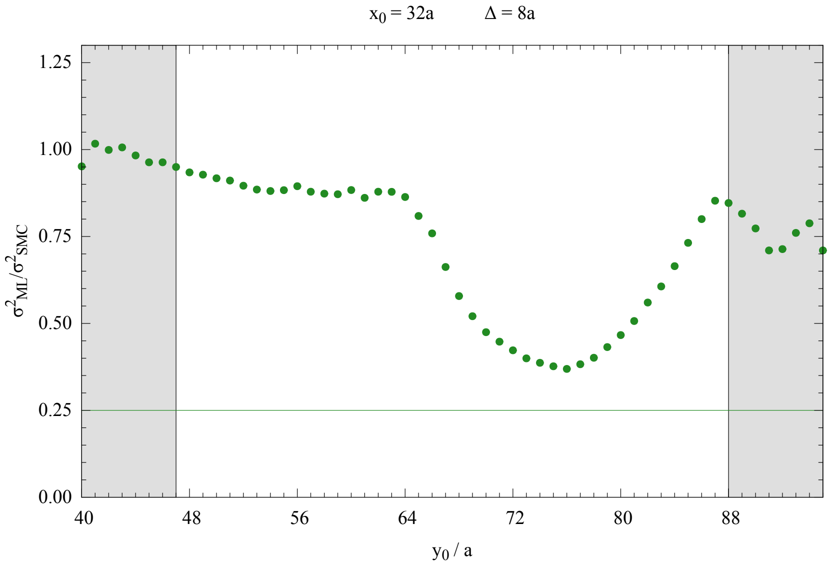

Factorizing the meson correlator

Factorization of the two–point function was previously attempted in different ways:

- placing two different noise sources to cut the two quark lines $\rightarrow$ in practice the variance is extremely large, due to the presence of disconnected diagrams which do not benefit from volume averaging

- cut the two lines with N orthonormal vectors which can be used to project the propagator $\rightarrow$ memory problem under control $(V_3)^2 \rightarrow N^2$
- it is a viable (but expensive) possibility. successfully implemented using
  - local deflation subspace
  - generating eigenvectors of the Dirac operator restricted to a block of width 2Δ