

The leading hadronic contribution to $\sin^2 \theta_W$ using covariant coordinate-space methods

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motivation – the running of the electroweak mixing angle

the leading hadronic contribution to the running of $\sin^2 \theta_W$ is

[Jegerlehner 1986; 2011]

$$\Delta_{\text{had}} \sin^2 \theta_W(Q^2) = -\frac{e^2}{\sin^2 \theta_W} [\Pi^{\gamma Z}(Q^2) - \Pi^{\gamma Z}(0)],$$

proportional to the subtracted vacuum polarization

$$(Q_\mu Q_\nu - \delta_{\mu\nu} Q^2) \Pi^{\gamma Z}(Q^2) = \Pi_{\mu\nu}^{\gamma Z}(Q^2) = \int d^4x e^{iQx} \langle j_\mu^Z(x) j_\nu^\gamma(0) \rangle$$

of the γ current and the (vector part of the) Z current

$$\begin{aligned} j_\mu^\gamma &= \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d - \frac{1}{3} \bar{s} \gamma_\mu s + \frac{2}{3} \bar{c} \gamma_\mu c, \\ j_\mu^3 &= \frac{1}{4} \bar{u} \gamma_\mu u - \frac{1}{4} \bar{d} \gamma_\mu d - \frac{1}{4} \bar{s} \gamma_\mu s + \frac{1}{4} \bar{c} \gamma_\mu c, \\ j_\mu^Z &= j_\mu^3 - \sin^2 \theta_W j_\mu^\gamma, \end{aligned}$$

- can be extracted from phenomenology using dispersion relations
- or can be computed *ab initio* on the lattice

[Burger *et al.* 2015; Gülpers *et al.* 2015]

the γZ correlator on the lattice

on the lattice, the flavour component are isolated naturally
⇒ advantage over phenomenological analysis

$$\begin{aligned}\langle j_\mu^Z(x) j_\nu^Z(0) \rangle &= \left(\frac{1}{4} - \frac{5}{9} \sin^2 \theta_W \right) \langle j_\mu^\ell(x) j_\nu^\ell(0) \rangle_{\text{con}} \\ &\quad + \left(\frac{1}{12} - \frac{1}{9} \sin^2 \theta_W \right) \langle j_\mu^s(x) j_\nu^s(0) \rangle_{\text{con}} + \left(\frac{1}{6} - \frac{4}{9} \sin^2 \theta_W \right) \langle j_\mu^c(x) j_\nu^c(0) \rangle_{\text{con}} \\ &\quad + \frac{1}{9} \sin^2 \theta_W \langle j_\mu^{\ell+As}(x) j_\nu^{\ell-s}(0) \rangle_{\text{dis}},\end{aligned}$$

- with $A = \frac{3}{4 \sin^2 \theta_W} - 1 \approx 2.14$
- neglecting disconnected charm contribution

$$\begin{aligned}\langle j_\mu^f(x) j_\nu^f(y) \rangle_{\text{con}} &= - \left\langle \text{tr} \left\{ D_f^{-1}(x, y) \gamma_\mu D_f^{-1}(y, x) \gamma_\nu \right\} \right\rangle, \\ \langle j_\mu^f(x) j_\nu^f(y) \rangle_{\text{dis}} &= \left\langle \text{tr} \left\{ D_f^{-1}(x, x) \gamma_\mu \right\} \text{tr} \left\{ D_f^{-1}(y, y) \gamma_\nu \right\} \right\rangle.\end{aligned}$$

the time-momentum representation (TMR) method

introduced for the HVP contribution to a_μ

[Bernecker, Meyer 2011; Francis et al. 2013]

$$\Pi^{\gamma Z}(Q^2) - \Pi^{\gamma Z}(0) = \int_0^\infty dx_0 G^{\gamma Z}(x_0) \left[x_0^2 - \frac{4}{Q^2} \sin^2\left(\frac{Qx_0}{2}\right) \right],$$
$$G^{\gamma Z}(x_0) = -\frac{1}{3} \int d^3x \sum_{k=1}^3 \langle j_k^Z(x) j_k^\gamma(0) \rangle,$$

⇒ using correlators from $N_f = 2 + 1$ Mainz effort in computing the a_μ^{HVP} [talk by Wittig, Fri 16:50]

- w.r.t. the a_μ^{HVP} case, the kernel has a shorter range
- expect $\Delta_{\text{had}} \sin^2 \theta_W$ to be more sensitive at cut-off effects, especially at high Q^2
- but much simpler large-distance systematic
⇒ no loss of signal in the tail of the connected correlator

⇒ we chose $\Delta_{\text{had}} \sin^2 \theta_W$ to test a new method

Lorentz-covariant coordinate-space (CCS) method

rewrite the subtracted γ -Z vacuum polarization as

[Meyer 2017]

$$\Pi^{\gamma Z}(Q^2) - \Pi^{\gamma Z}(0) = \int d^4x G_{\mu\nu}^{\gamma Z}(x) H_{\mu\nu}(x) = \int d^4x g_i(x) \mathcal{H}_i(|x|),$$
$$G_{\mu\nu}^{\gamma Z}(x) = \langle j_\mu^Z(x) j_\nu^Z(0) \rangle,$$
$$g_1(x) = -\delta_{\mu\nu} G_{\mu\nu}^{\gamma Z}(x), \quad g_2(x) = -\frac{x_\mu x_\nu}{x^2} G_{\mu\nu}^{\gamma Z}(x),$$

where the CCS kernel has a simple Lorentz structure

$$H_{\mu\nu}(x) = -\delta_{\mu\nu} \mathcal{H}_1(|x|) + \frac{x_\mu x_\nu}{x^2} \mathcal{H}_2(|x|), \quad \mathcal{H}_i(|x|) = x^2 \bar{\mathcal{H}}_i(|Q||x|),$$

$$\bar{\mathcal{H}}_1(z) = \frac{z^2}{4608} [24 {}_2F_3(1, 1; 2, 3, 3; -z^2/4) - 20 {}_2F_3(1, 1; 2, 3, 4; -z^2/4) \\ + 3 {}_2F_3(1, 1; 2, 3, 5; -z^2/4)],$$

$$\bar{\mathcal{H}}_2(z) = \frac{z^2}{1152} [6 {}_2F_3(1, 1; 2, 3, 3; -z^2/4) - 8 {}_2F_3(1, 1; 2, 3, 4; -z^2/4) \\ + 4 {}_2F_3(1, 1; 2, 4, 4; -z^2/4) - {}_2F_3(1, 1; 2, 4, 5; -z^2/4)].$$

- kernel known analytically, also for $Q^2 \Pi'(Q^2)$, a_μ^{HVP}

numerical tests

we tested the methods on three CLS ensembles

[Bruno *et al.* 2015]

(tree-level Lüscher-Weisz gauge action, non-perturbatively $\mathcal{O}(a)$ -improved Wilson fermions, open BCs in time)

- on all three ensembles, $\beta = 3.55$ and $a \simeq 0.065$ fm

[Bruno, Korzec, Schaefer 2017]

	L/a	L [fm]	m_π [MeV]	m_K [MeV]	$m_\pi L$
N203	48	3.1	340	440	5.4
N200	48	3.1	280	460	4.4
D200	64	4.1	200	480	4.2

same correlators used with the TMR and the CCS methods

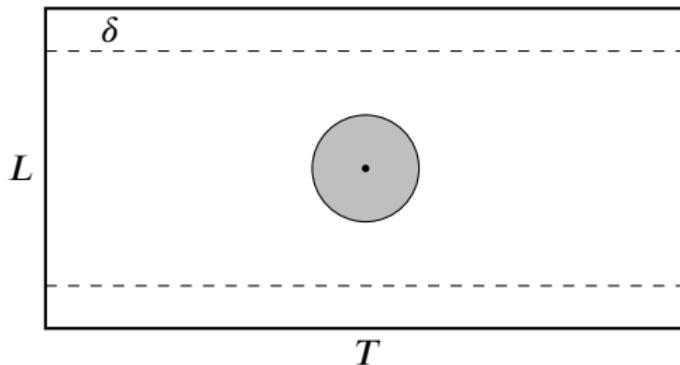
- connected contribution computed with inversion on local sources \Rightarrow point-to-all
(only partial statistics shown here)
- disconnected contribution estimated using hierarchical probing
512 vectors \times 2 random sources
 \Rightarrow use the fast Fourier transform to build the **all-to-all** two-point function

[Stathopoulos *et al.* 2013]

integration strategy

sum $g_i(x)\mathcal{H}_i(|x|)$ incrementally over S^3 shells, up to r

$$\Delta_{\text{had}} \sin^2 \theta_W(Q^2, r) = -\frac{e^2}{\sin^2 \theta_W} \int_{|x| < r} d^4x g_i(x) \mathcal{H}_i(|x|)$$

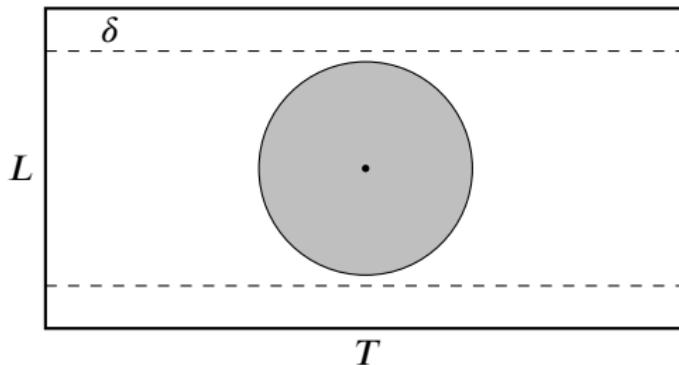


- keep distance δ from space boundaries to control finite- L effects
- correct for missing lattice points for $|x| > L/2 - \delta$
⇒ just a geometric factor, assuming $g_i(x)$ constant on S^3
- working on estimating finite-volume effects with NLO χ PT

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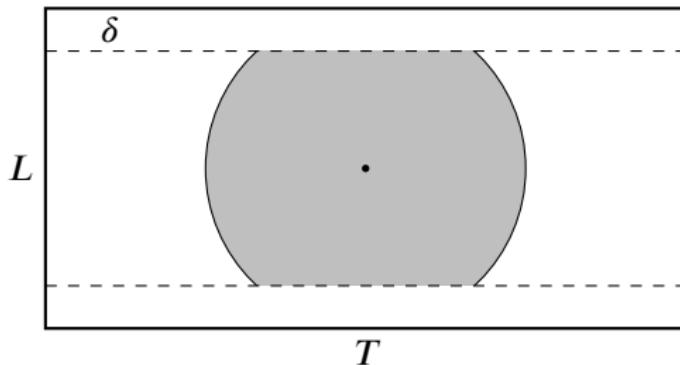


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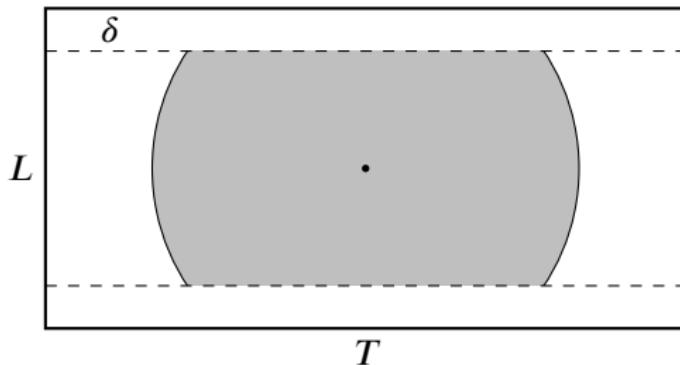


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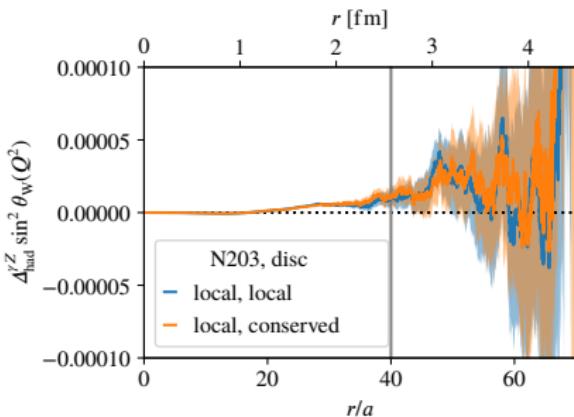
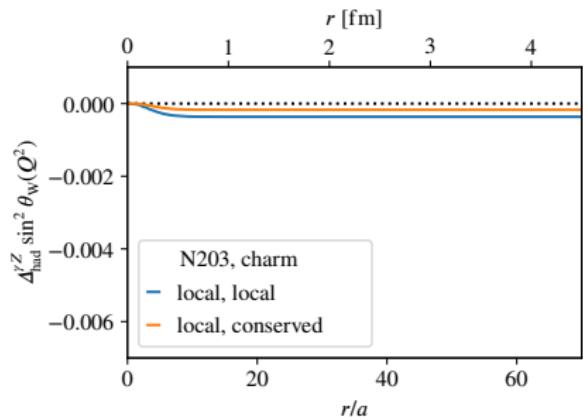
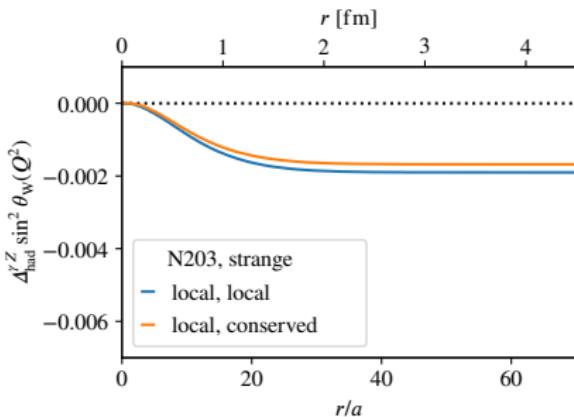
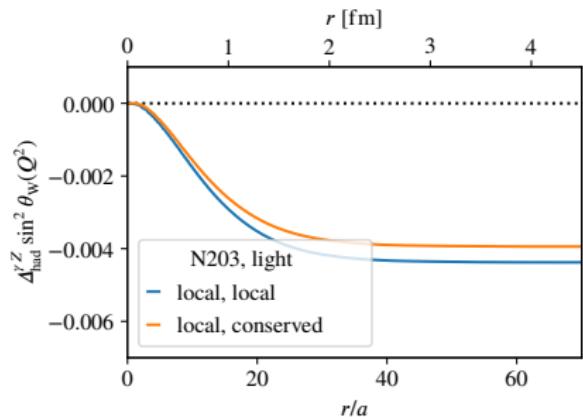
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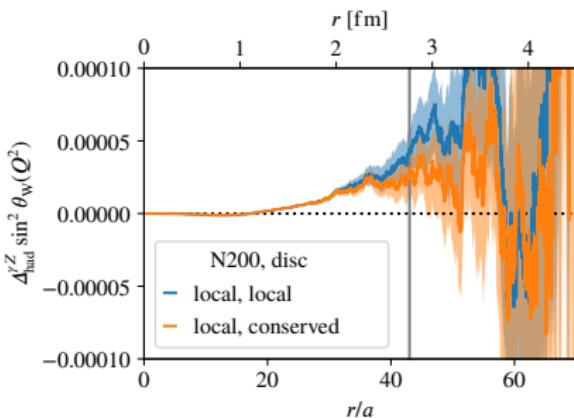
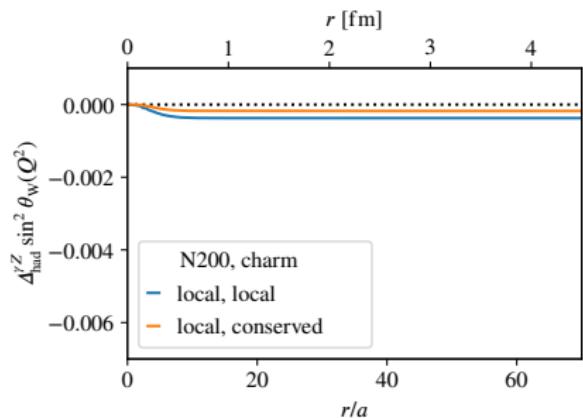
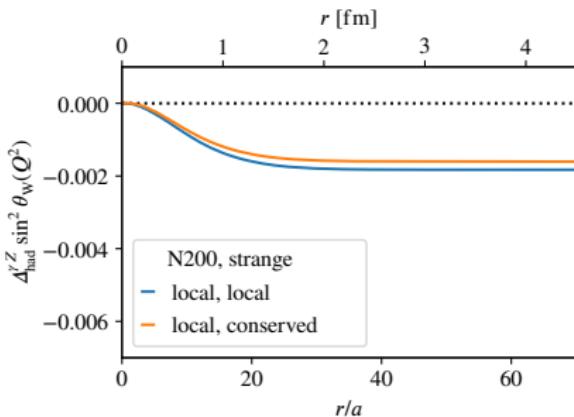
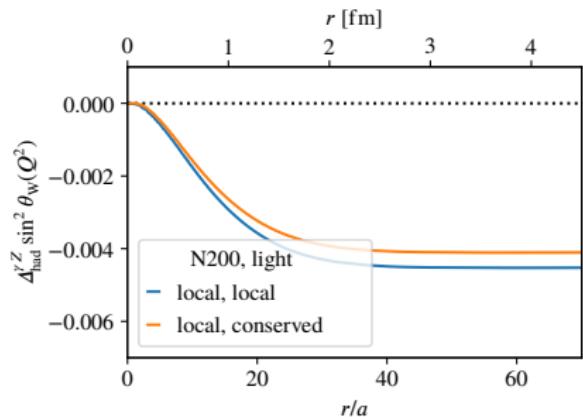
N203 ($m_\pi = 340$ MeV, $L = 3.1$ fm) at $Q^2 = 4$ GeV 2

[preliminary]



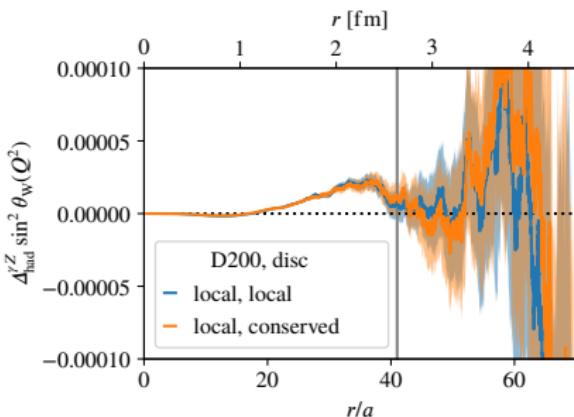
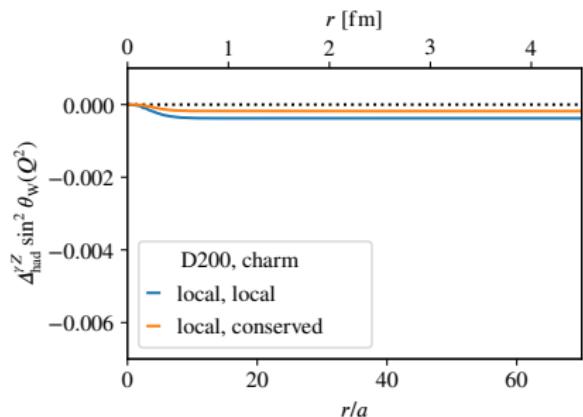
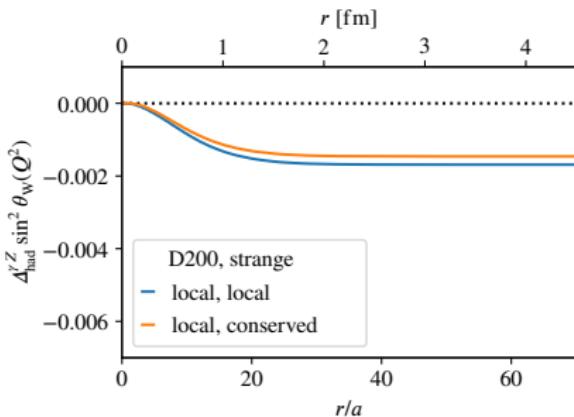
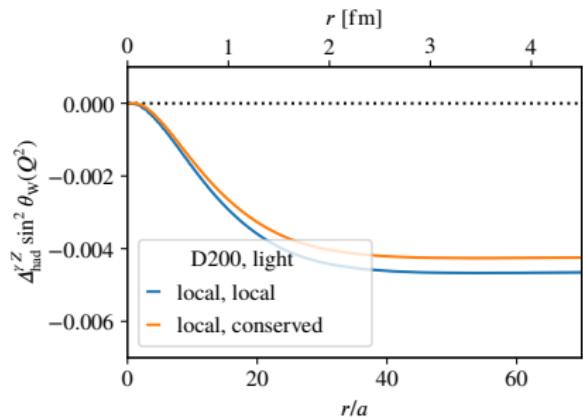
N200 ($m_\pi = 280$ MeV, $L = 3.1$ fm) at $Q^2 = 4$ GeV 2

[preliminary]

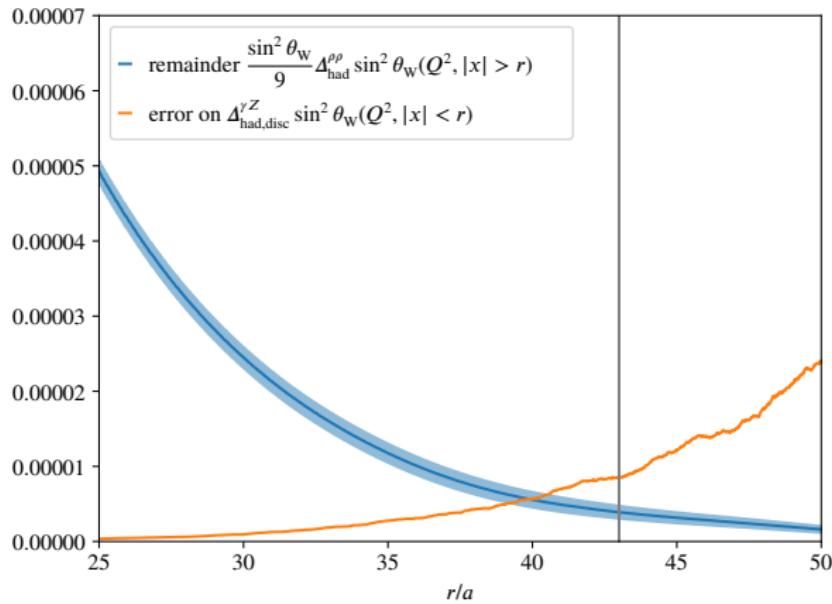


D200 ($m_\pi = 200$ MeV, $L = 4.1$ fm) at $Q^2 = 4$ GeV 2

[preliminary]



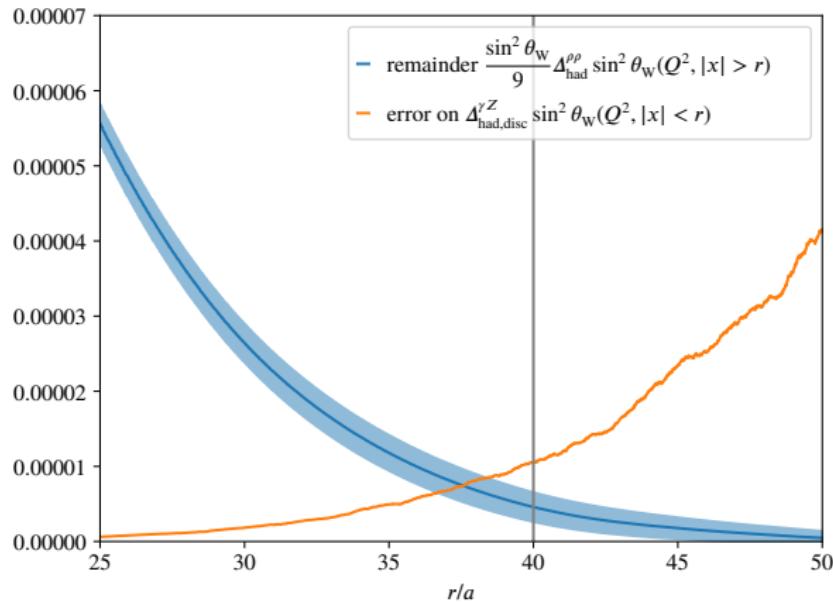
r -cut on the disconnected contribution on N203 ($m_\pi = 340$ MeV, $L = 3.1$ fm) at $Q^2 = 4$ GeV 2



- $G_{\text{disc}}^{\gamma Z}(x) \xrightarrow{x \rightarrow \infty} \frac{\sin^2 \theta_W}{9} G^{\rho\rho}(x)$, that is purely connected
- 0.000 01 is 0.1-0.2 % of the total $\Delta_{\text{had}} \sin^2 \theta_W(Q^2)$

[Gülpers *et al.* 2015]

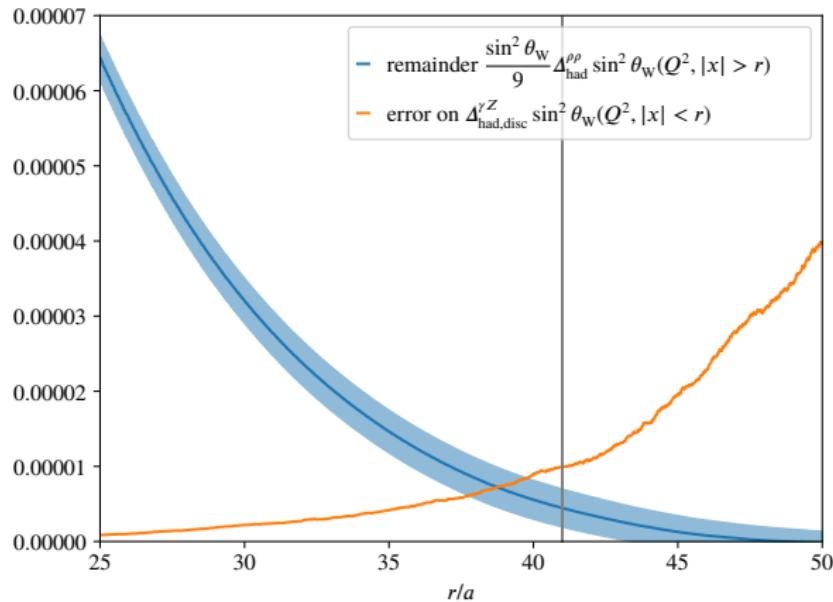
r -cut on the disconnected contribution on N200 ($m_\pi = 280$ MeV, $L = 3.1$ fm) at $Q^2 = 4$ GeV 2



- $G_{\text{disc}}^{\gamma Z}(x) \xrightarrow{x \rightarrow \infty} \frac{\sin^2 \theta_W}{9} G^{\rho\rho}(x)$, that is purely connected
- 0.000 01 is 0.1-0.2 % of the total $\Delta_{\text{had}} \sin^2 \theta_W(Q^2)$

[Gülpers et al. 2015]

r -cut on the disconnected contribution on D200 ($m_\pi = 200$ MeV, $L = 4.1$ fm) at $Q^2 = 4$ GeV 2



- $G_{\text{disc}}^{\gamma Z}(x) \xrightarrow{x \rightarrow \infty} \frac{\sin^2 \theta_W}{9} G^{\rho\rho}(x)$, that is purely connected
- 0.000 01 is 0.1-0.2 % of the total $\Delta_{\text{had}} \sin^2 \theta_W(Q^2)$

[Gülpers et al. 2015]

$\mathcal{O}(a)$ improvement

$$V_\mu(x) \Rightarrow V_\mu(x) + ac_V \partial_\alpha T_{\mu\alpha}(x), \quad T_{\mu\alpha}(x) = \bar{\psi}(x) [\gamma_\mu, \gamma_\alpha] \psi(x).$$

integrating by part and using translation invariance, we take the derivative of the kernel.

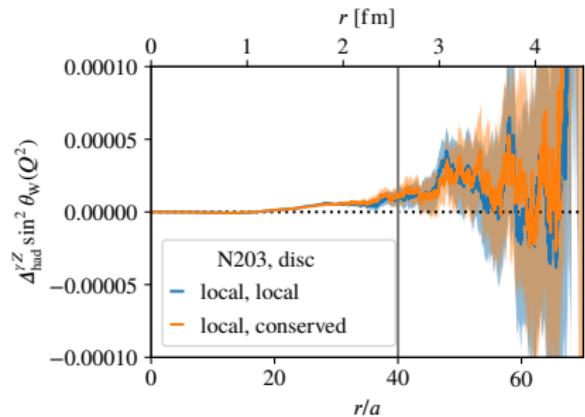
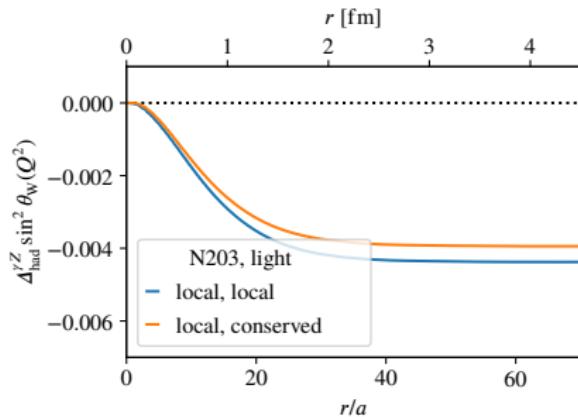
$$\begin{aligned} \int d^4x \left\{ \langle V_\mu(x) V_\nu(0) \rangle H_{\mu\nu}(x) - a [c_V \langle T_{\mu\alpha}(x) V_\mu(0) \rangle - c_V \langle V_\mu(x) T_{\mu\alpha}(0) \rangle] \partial_\mu H_{\mu\nu}(x) \right\} \\ = \int d^4x \{ g_1(x) \mathcal{H}_1(|x|) + g_2(x) \mathcal{H}_2(|x|) + g_3(x) \mathcal{H}_3(|x|) \} \end{aligned}$$

where

$$\begin{aligned} g_3(x) &= \frac{ax_\alpha}{x^2} [c_V \langle T_{\mu\alpha}(x) V_\mu(0) \rangle - c_V \langle V_\mu(x) T_{\mu\alpha}(0) \rangle], \\ \mathcal{H}_3(|x|) &= |x| \mathcal{H}'_1(|x|) + \mathcal{H}_2(|x|). \end{aligned}$$

$\mathcal{O}(a)$ improvement

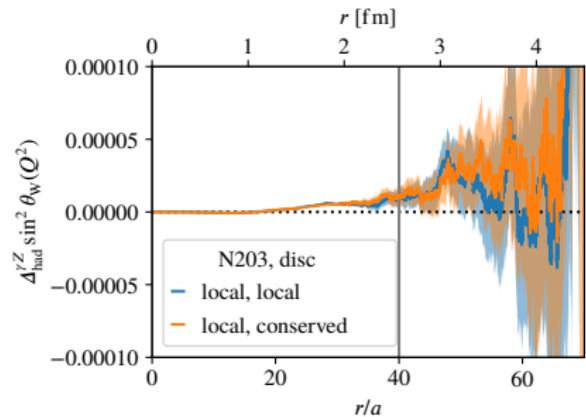
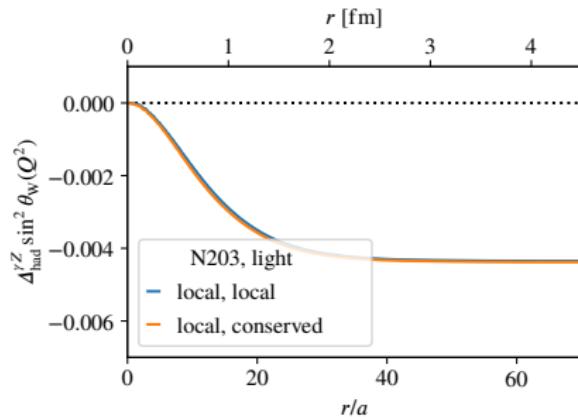
on N203 at $Q^2 = 4 \text{ GeV}^2$ without $\mathcal{O}(a)$ improvement



- the local, conserved currents combination is mostly affected

$\mathcal{O}(a)$ improvement

on N203 at $Q^2 = 4 \text{ GeV}^2$ with $\mathcal{O}(a)$ improvement



- the local, conserved currents combination is mostly affected

non-transverse kernel

$H_{\mu\nu}(x)$ is a transverse tensor

$$\partial_\mu H_{\mu\nu}(x) = 0 \quad \Rightarrow \quad |x|\mathcal{H}'_1(|x|) = |x|\mathcal{H}'_2(|x|) + 3\mathcal{H}_2(|x|).$$

Since $\partial_\mu G_{\mu\nu}(x) = 0$ we have the freedom to add a non-transverse part to the kernel

$$\partial_\mu [x_\nu \mathcal{F}(|x|)] = \delta_{\mu\nu} \mathcal{F}(|x|) + \frac{x_\mu x_\nu}{x^2} |x| \mathcal{F}'(|x|),$$

- the surface term $\int d^4x \partial_\mu [G_{\mu\nu} x_\nu \mathcal{F}]$ vanishes in infinite volume.

With the choice $\mathcal{F}(|x|) = (\alpha + \gamma)\mathcal{H}_1(|x|) - \gamma\mathcal{H}_2(|x|)$, we have

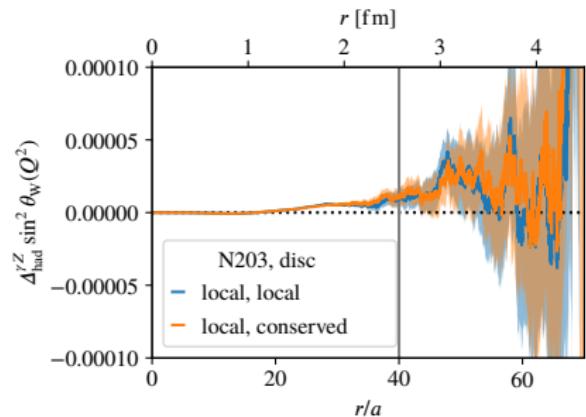
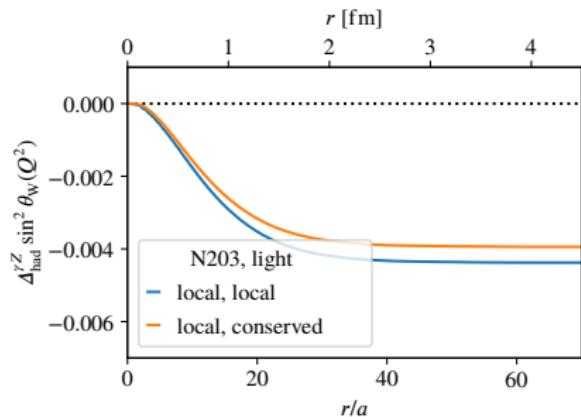
$$\begin{aligned} H_{\mu\nu}^{(\alpha,\gamma)}(x) &= -\delta_{\mu\nu} [(1 - \alpha - \gamma)\mathcal{H}_1(|x|) + \gamma\mathcal{H}_2(|x|)] \\ &\quad + \frac{x_\mu x_\nu}{x^2} [(1 + 3\gamma)\mathcal{H}_2(|x|) + \alpha|x|\mathcal{H}'_1(|x|)], \end{aligned}$$

so that

$$\begin{aligned} \gamma = -1/3 \quad H_{\mu\nu}^{(0,-1/3)}(x) &= -\delta_{\mu\nu} [4\mathcal{H}_1(|x|) - \mathcal{H}_2(|x|)]/3, \\ \gamma = 1 \quad H_{\mu\nu}^{(0,1)}(x) &= -\delta_{\mu\nu} \mathcal{H}_2(|x|) + 4 \frac{x_\mu x_\nu}{x^2} \mathcal{H}_2(|x|), \\ \alpha = 1 \quad H_{\mu\nu}^{(1,0)}(x) &= \frac{x_\mu x_\nu}{x^2} [\mathcal{H}_2(|x|) + |x|\mathcal{H}'_1(|x|)]. \end{aligned}$$

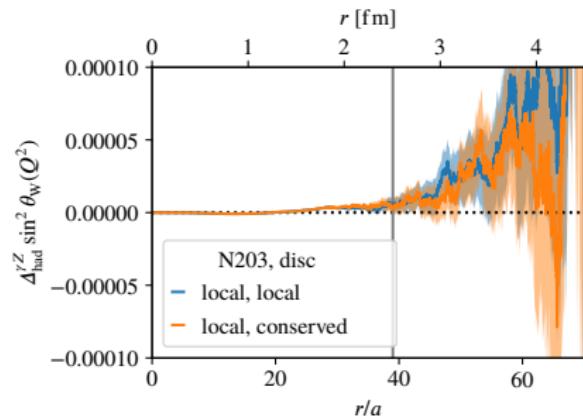
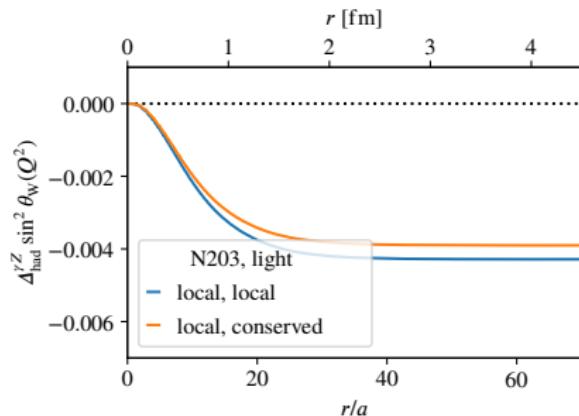
non-transverse kernel

on N203 with the standard transverse kernel



non-transverse kernel

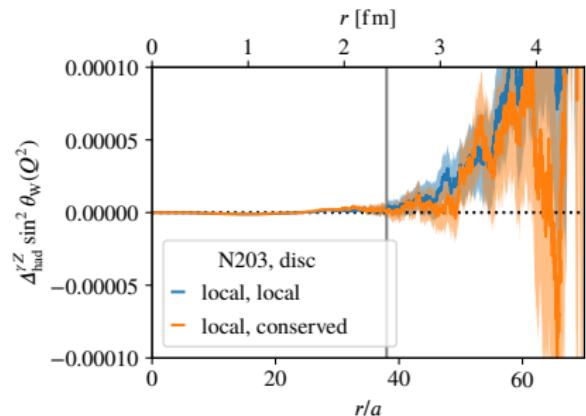
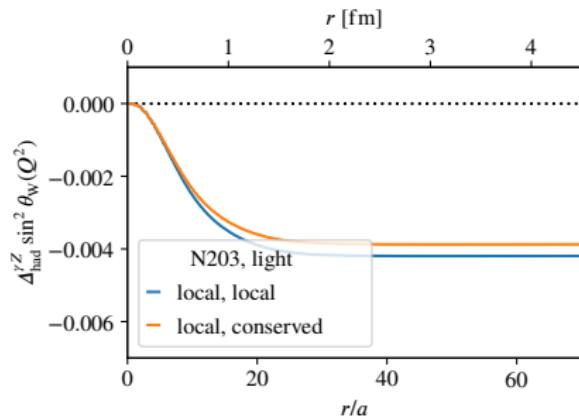
on N203 with $\gamma = 1.0$



- the kernel is shorter range
- the disconnected contribution integration can be cut at a smaller r
- finite-volume effects?

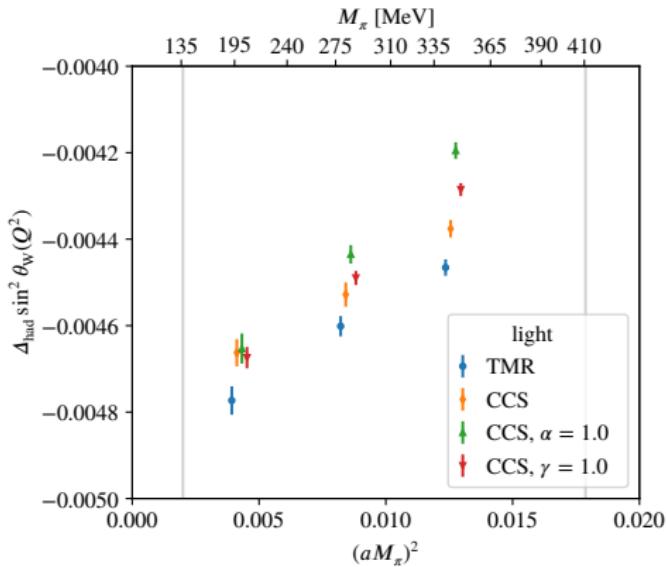
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preliminary results



with the TMR method

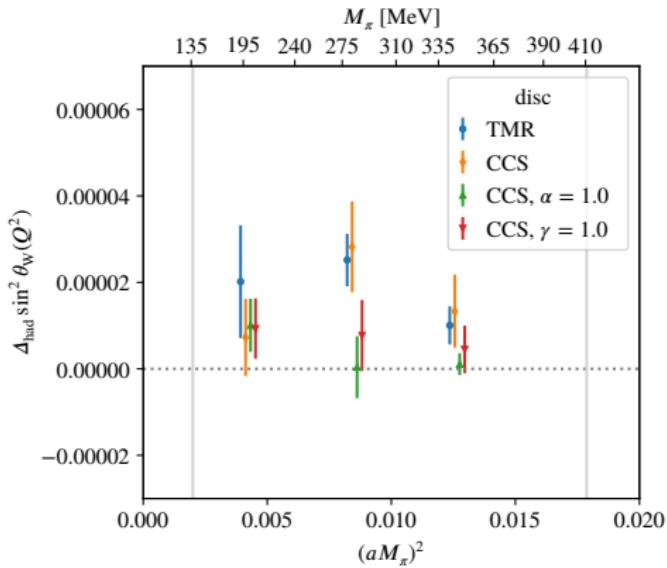
	$\times 10^{-6}$	N203	N200	D200
light	-4466(19)	-4601(23)	-4773(33)	
strange	-1961(7)	-1888(6)	-1757(4)	
charm	-443(2)	-451(2)	-455(1)	
disc	10(4)	25(6)	20(13)	
total	-6859(22)	-6916(26)	-6965(35)	

with the CCS method

	$\times 10^{-6}$	N203	N200	D200
light	-4376(20)	-4528(28)	-4663(32)	
strange	-1901(6)	-1828(5)	-1689(3)	
charm	-364(1)	-373(1)	-377(1)	
disc	13(8)	28(10)	8(10)	
total	-6628(24)	-6700(30)	-6721(34)	

- unimproved local, local discretization
- at $Q^2 = 4 \text{ GeV}^2$
- statistical errors only

preliminary results



with the TMR method

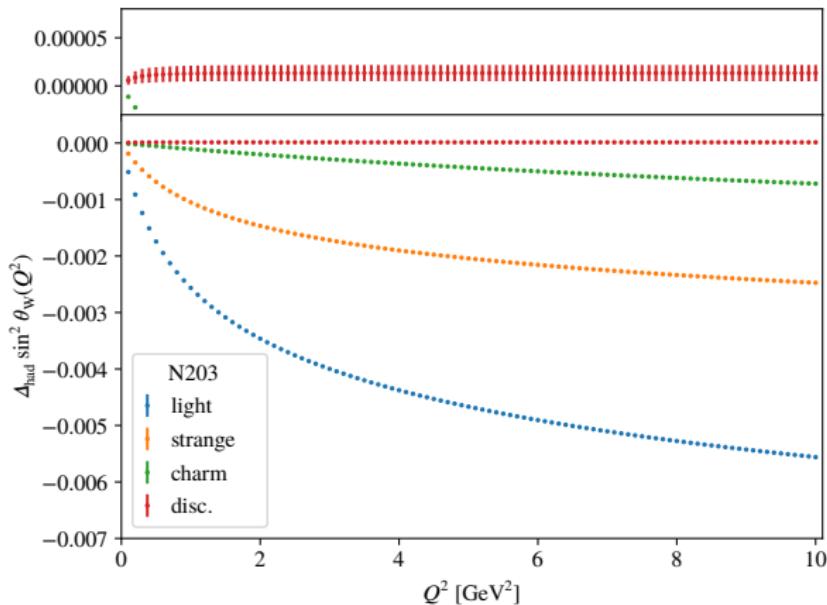
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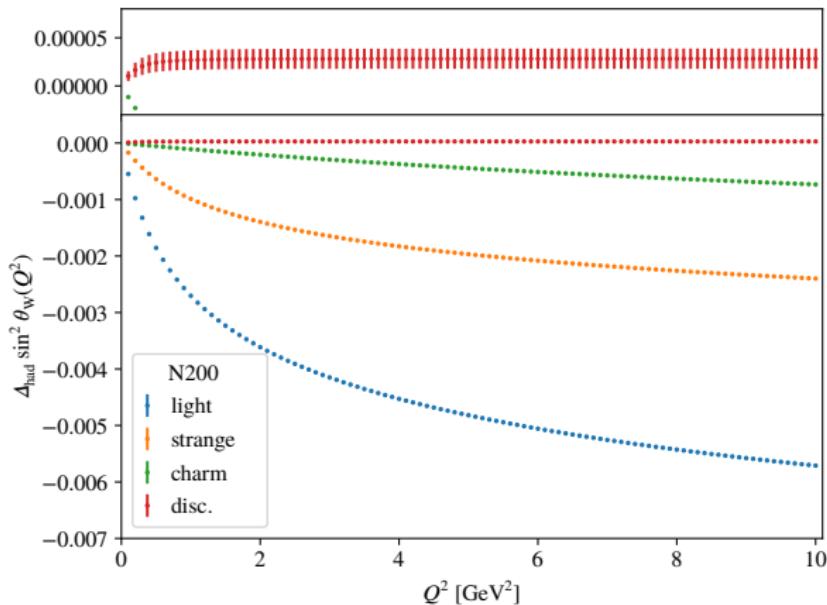
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preliminary results – running with Q^2



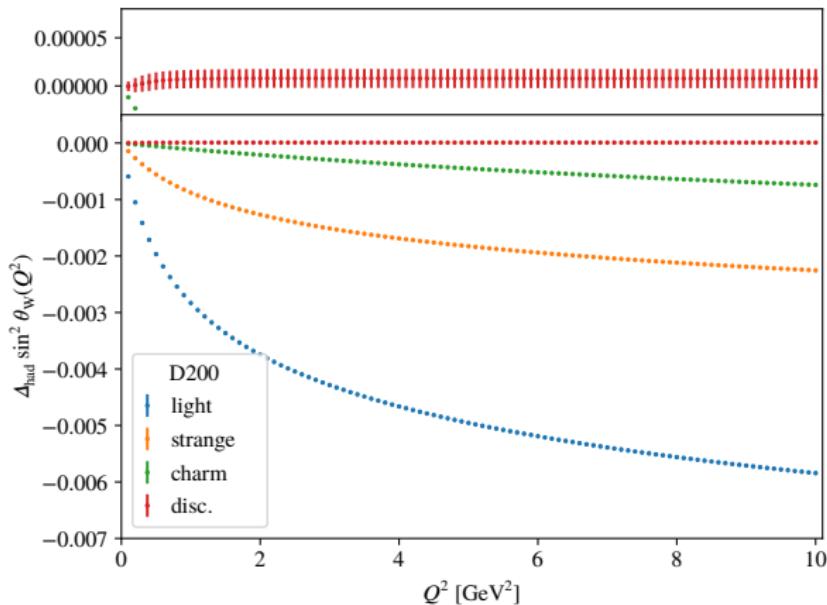
- the disconnected contribution is constant with Q^2

preliminary results – running with Q^2



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conclusions & outlook

the leading hadronic contribution to the running $\sin^2 \theta_W$ can be computed on the lattice

- with sub-percent errors, competitive with phenomenology
- including the disconnected contribution, with $\approx 0.2\%$ determination
- lattice provides flavour separation
- increase the statistic on the connected contributions
- more ensembles \Rightarrow chiral and continuum limit
- include systematics (e.g. scale setting, finite volume)

the Lorentz-covariant coordinate space (CCS) method works

- for connected contributions from local sources \Rightarrow error compatible with TMR
- for the disconnected contribution, the error is reduced on the bigger volume using non-transverse kernel, up to a factor of 2 compared to TMR
- significant finite-volume effects may be present
 \Rightarrow model and correct for finite-volume effects
- apply to a_μ^{HVP} , $\Delta_{\text{had}}\alpha_{\text{em}}$, $\Delta_{\text{had}}\alpha_2$, ...
- ensemble with bigger volume? spacetime factorization?

thanks
for your attention!



questions?

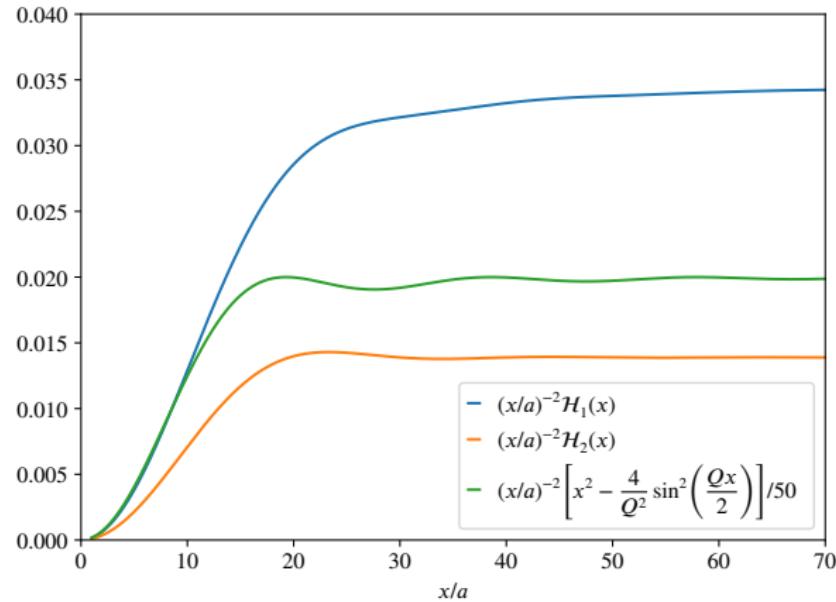
backup

statistics

	N203		N203		N203	
	#cfg	#src	#cfg	#src	#cfg	#src
light	1504	15	1712	10	1080	15
strange	752	5	856	5	1080	5
charm	94	5	107	5	135	5
disc	752	2×512	856	2×512	270	2×512

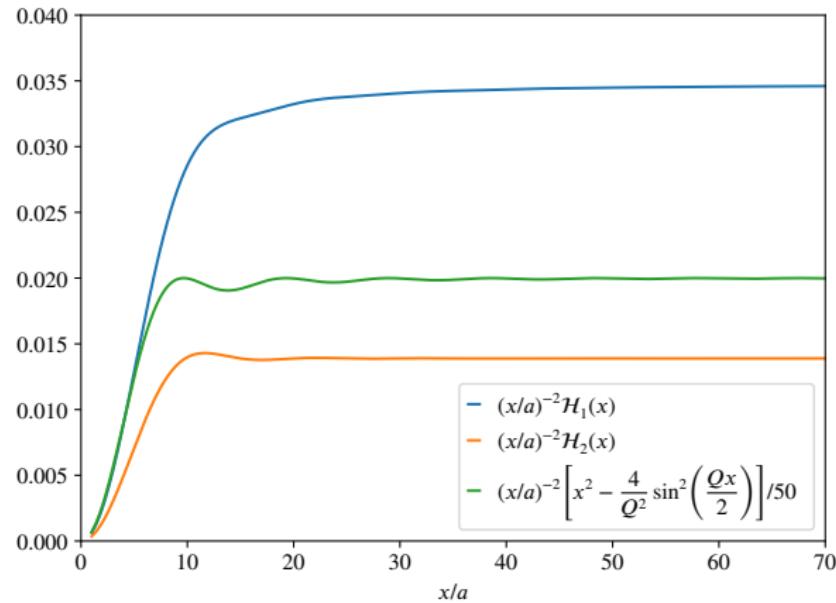
CCS kernel

at $Q^2 = 1 \text{ GeV}^2$



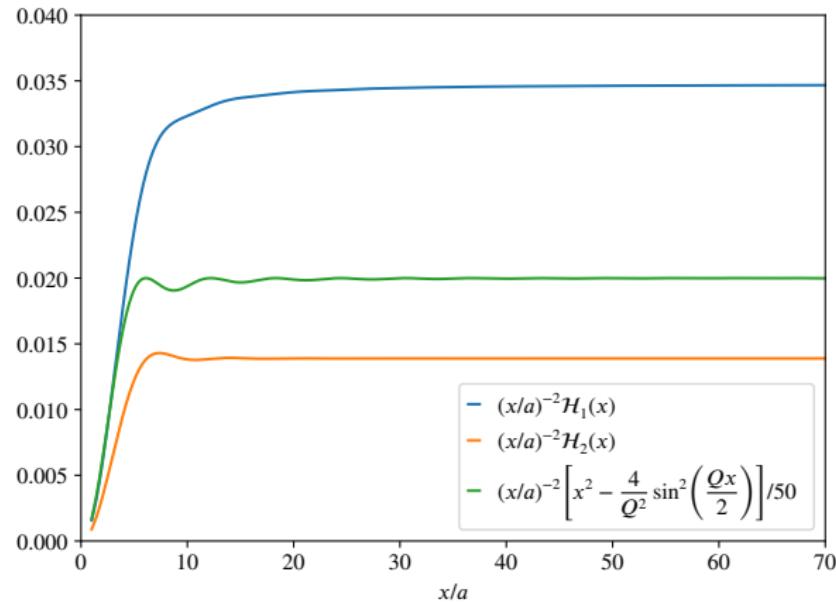
CCS kernel

at $Q^2 = 4 \text{ GeV}^2$



CCS kernel

at $Q^2 = 10 \text{ GeV}^2$



asymptotic of the disconnected contribution

$$G^{\gamma Z}(x) \xrightarrow{x \rightarrow \infty} \left(\frac{1}{2} - \sin^2 \theta_W \right) G^{\rho\rho}(x)$$

$$\begin{aligned} \frac{G_{\text{disc}}^{\gamma Z}(x)}{G^{\rho\rho}(x)} &\underset{x \rightarrow \infty}{\sim} \frac{G^{\gamma Z}(x) - \left(\frac{1}{2} - \sin^2 \theta_W \right) G^{\rho\rho}(x)}{G^{\rho\rho}(x)} \\ &+ \frac{1}{9} \sin^2 \theta_W - \left(\frac{1}{6} - \frac{2}{9} \sin^2 \theta_W \right) \frac{G_{\text{con}}^s(x)}{G_{\text{con}}^\ell(x)} \end{aligned}$$