Progress on relativistic three-particle quantization condition

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Based on arXiv:1803.04169 (published in PRD), arXiv:1808:XXXXX, and work in progress

Outline

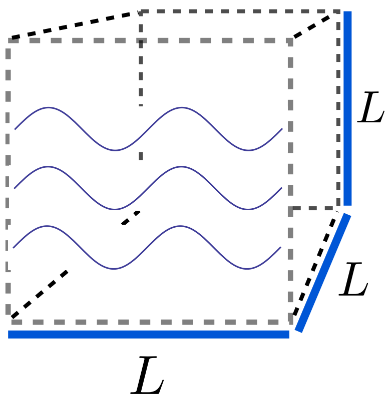
- Motivation
- Status
- Completing the formalism: including resonant subchannels
- Numerical results from the isotropic approximation
- Numerical results including higher partial waves

Motivation

- Studying resonances with three particle decay channels
  - $\omega(782, I^G J^{PC} = 0^- 1^{--}) \to 3\pi$ (no resonant subchannels)
  - $a_2(1320, I^G J^{PC} = 1^- 2^{++}) \to \rho\pi \to 3\pi$
  - $N(1440) \to \Delta\pi \to N\pi\pi$
  - $X(3872) \to J/\Psi\pi\pi$
- Calculating weak decay amplitudes involving 3 or more particles, e.g. $K \to 3\pi$, $D \to 2\pi$, 4π, …
- Determining NNN interactions

Methodology & Status

2 & 3 particle spectrum from LQCD



Quantization conditions

$$\det\left[F_2^{-1} + \mathcal{K}_2\right]$$

$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right]$$

Intermediate scattering quantities

Integral equations in infinite volume

Scattering amplitudes

$$\mathcal{M}_2\,,\quad \mathcal{M}_3\,,\quad \mathcal{M}_{23}\,,\ldots$$

Methodology & Status

Quantization conditions

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Intermediate scattering quantities

Integral equations in infinite volume

- Three approaches
  - Relativistic [Briceño, Hansen, SRS]
  - NREFT [Hammer, Pang, Rusetsky]
  - Finite-volume Khuri-Treiman [Döring, Mai]
- Each have pros and cons
  - Intermediate scattering quantities differ
  - All require partial-wave truncation
  - Similar challenges for numerical implementation

Status of relativistic approach

- Original work applied to scalars with G-parity & no subchannel resonances [Hansen, SRS: 1408.5933 & 1504.04248]

$$\det\left[F_3^{-1} + \mathcal{K}_{\mathrm{df},3}\right]$$

- Second major step: removing G-parity constraint, allowing 2↔3 processes [Briceño, Hansen, SRS: 1701.07465]

$$\det\left[\begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix}^{-1} + \begin{pmatrix} \mathcal{K}_{22} & \mathcal{K}_{23} \\ \mathcal{K}_{32} & \mathcal{K}_{\mathrm{df},33} \end{pmatrix}\right] = 0$$

Completing the formalism

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- Final major step: allowing subchannel resonance (i.e. pole in $\mathcal{K}_2$) [Briceño, Hansen, SRS: 1808.XXXXX]

$$\det\left[\begin{pmatrix} F_{\tilde{2}\tilde{2}} & F_{\tilde{2}3} \\ F_{3\tilde{2}} & F_{33} \end{pmatrix}^{-1} + \begin{pmatrix} \mathcal{K}_{\mathrm{df},\tilde{2}\tilde{2}} & \mathcal{K}_{\mathrm{df},\tilde{2}3} \\ \mathcal{K}_{\mathrm{df},3\tilde{2}} & \mathcal{K}_{\mathrm{df},33} \end{pmatrix}\right] = 0$$

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resonance + particle channel (not physical)

$$\det\left[\begin{pmatrix} F_{\tilde{2}\tilde{2}} & F_{\tilde{2}3} \\ F_{3\tilde{2}} & F_{33} \end{pmatrix}^{-1} + \begin{pmatrix} \mathcal{K}_{\mathrm{df},\tilde{2}\tilde{2}} & \mathcal{K}_{\mathrm{df},\tilde{2}3} \\ \mathcal{K}_{\mathrm{df},3\tilde{2}} & \mathcal{K}_{\mathrm{df},33} \end{pmatrix}\right] = 0$$

Determined by K_2 & Lüscher finite-volume zeta functions

Infinite-volume quantities related to $\mathcal{M}_2$ & $\mathcal{M}_3$ by known integral equations

Formalism to-do list

- Multiple poles in K_2
- Nondegenerate particles with spin
- Connecting formalism for resonances to that for stable particles (e.g. raising m_q stabilizes ρ)

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$$\boxed{\text{All are straightforward!}}$$

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Isotropic low-energy approximation

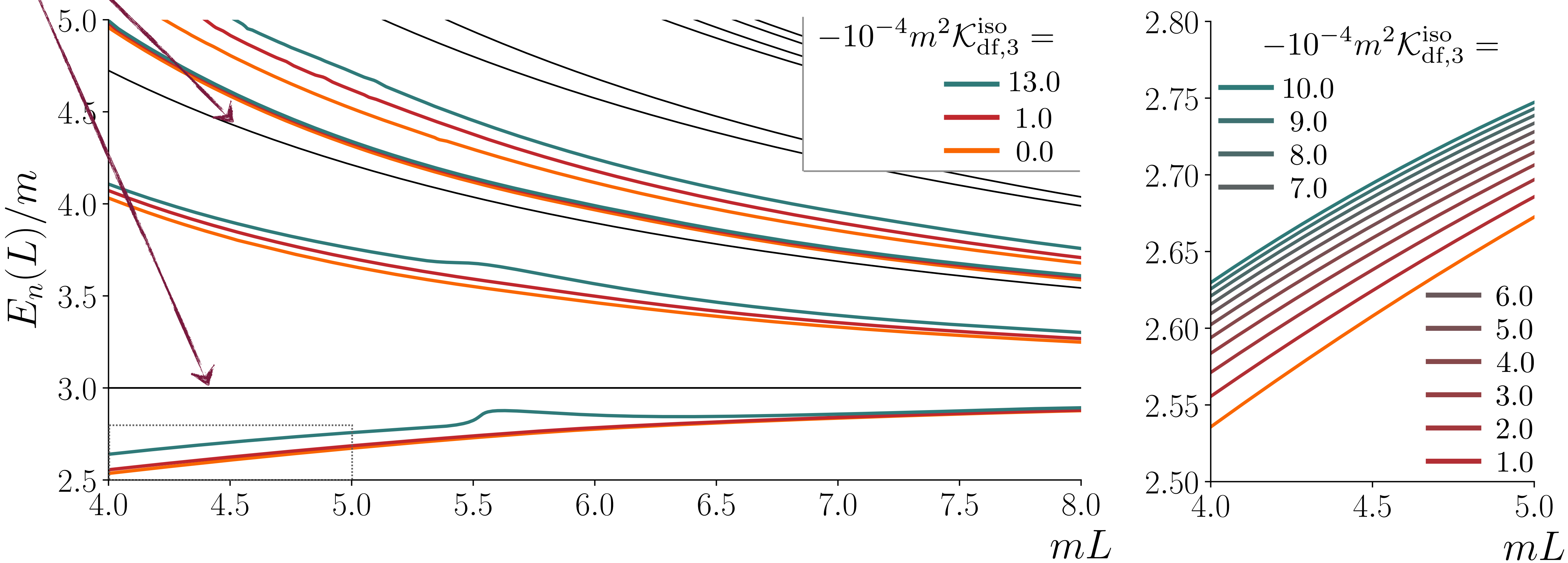
[Briceño, Hansen & SRS, 1803.04169]

- Scalar particles with G parity so no $2\leftrightarrow3$ transitions and no subchannel resonances (e.g. $3\ \pi^+$)
- 2-particle interactions are purely s-wave, and determined by the scattering length alone (which can be arbitrarily negative, $a\to-\infty$)
- Point-like three-particle interaction $\mathcal{K}_{\mathrm{df},3}$, independent of momenta
- Reduces problem to 1-dim. quantization condition, although intermediate matrices involve finite-volume momenta up to cutoff $|k|\sim m$
- Analog in our formalism of the approximations used in other approaches: [Hammer, Pang, Rusetsky, 1706.07700; Mai & Döring, 1709.08222; Döring et al., 1802.03362; Mai & Döring, 1807.04746]

Impact of $K_{df,3}$ on spectrum

noninteracting levels

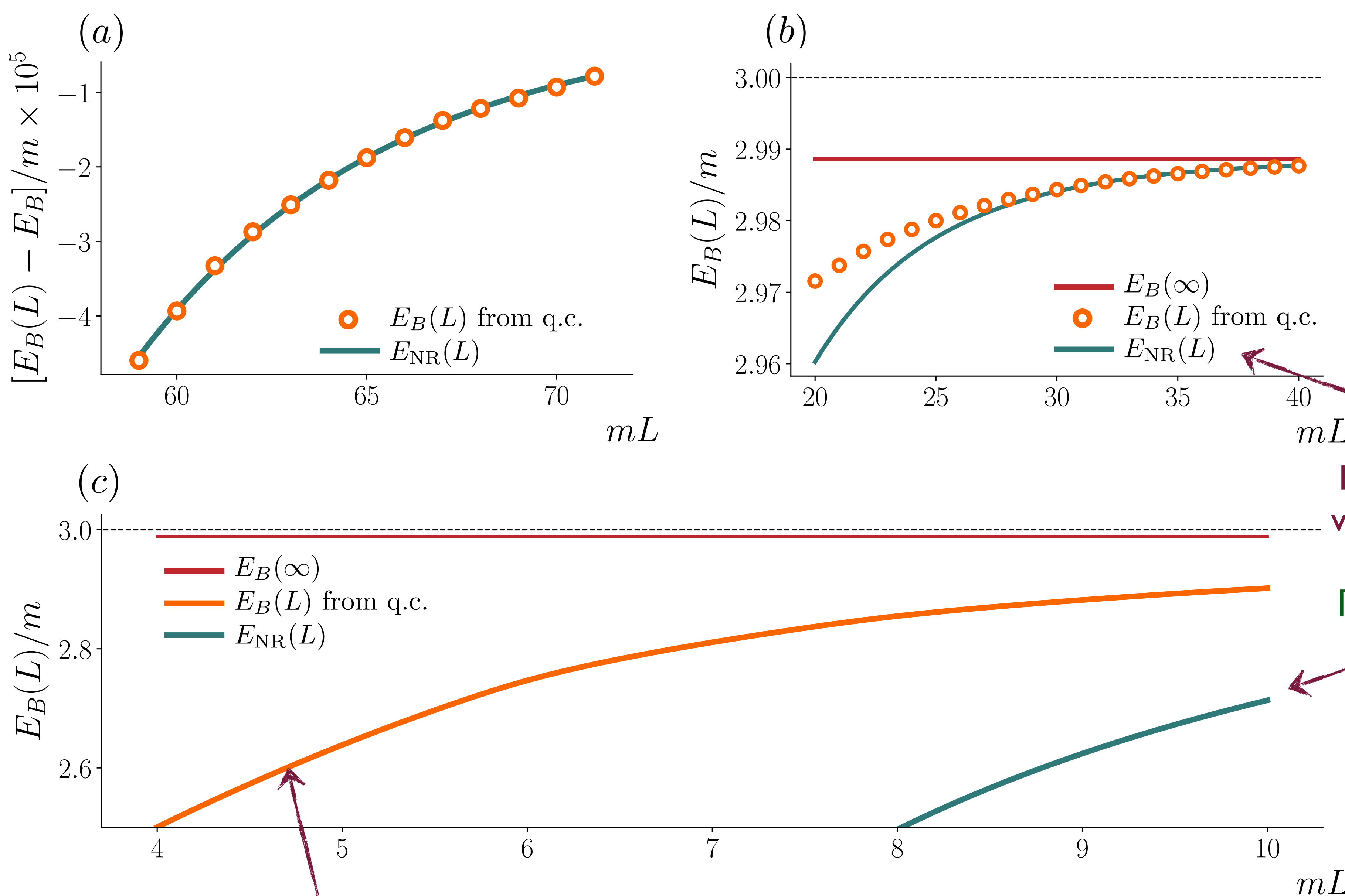
am=−10 (strong attractive interaction)



Local 3-particle interaction has significant effect on energies, especially in region of simulations (mL<5), and thus can be determined

Volume-dependence of 3-body bound state

am=−10⁴ & $m^2 K_{df,3}^{iso}$=2500 (unitary regime)



Prediction of asymptotic volume-dependence from NRQM
[Meißner, Rîos, Rusetsky]

Need quantization condition to determine finite-volume effects for realistic values of mL

Bound state wave-function

- Work in unitary regime (ma=−10⁴) and tune $\mathcal{K}_{\text{df},3}$ so 3-body bound state at E_B=2.98858 m
- Solve integral equations numerically to determine $\mathcal{M}_{\text{df},3}$ from $\mathcal{K}_{\text{df},3}$
- Determine wavefunction from residue at bound-state pole

$$\mathcal{M}_{\text{df},3}^{(u,u)}(k,p) \sim -\frac{\Gamma^{(u)}(k)\Gamma^{(u)}(p)^*}{E^2 - E_B^2}$$

- Compare to analytic prediction from NRQM in unitary limit [Hansen & SRS, 1609.04317]

$$|\Gamma^{(u)}(k)_{\text{NR}}|^2 = |c||A|^2 \frac{256\pi^{5/2}}{3^{1/4}} \frac{m^2\kappa^2}{k^2(\kappa^2 + 3k^2/4)} \frac{\sin^2\left(s_0 \sinh^{-1}\frac{\sqrt{3}k}{2\kappa}\right)}{\sinh^2\frac{\pi s_0}{2}}$$

Known constant

Determined by fit to volume-dependence of bound-state energy

Known constant

Bound state wave-function

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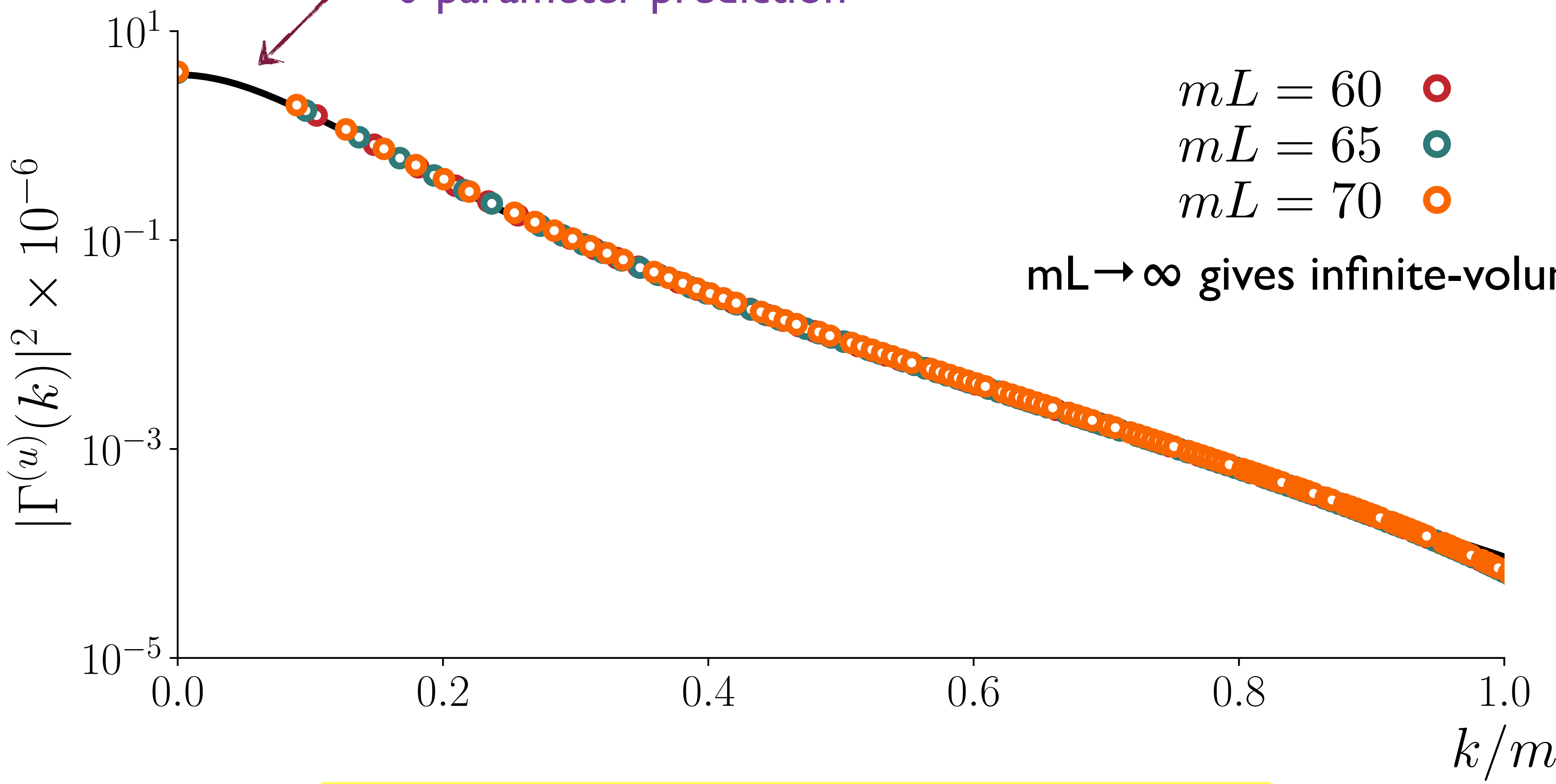
0-parameter prediction



mL→∞ gives infinite-volume result

Works over many orders of magnitude
to expected accuracy

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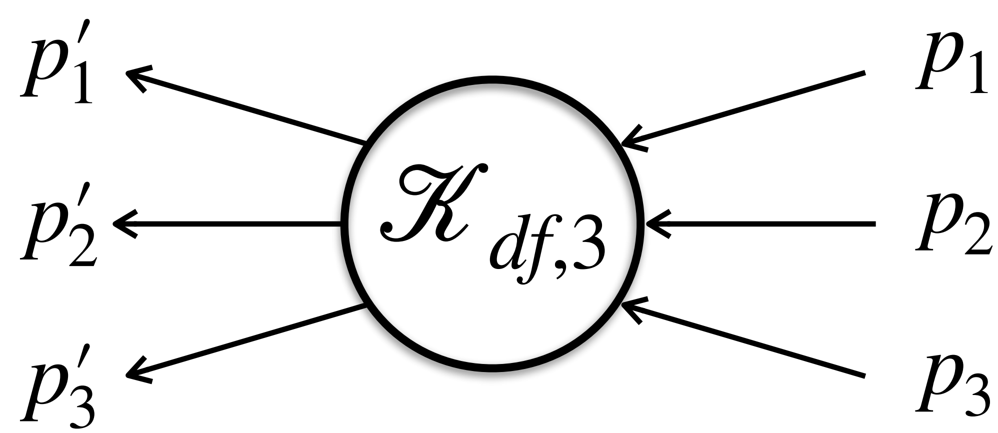
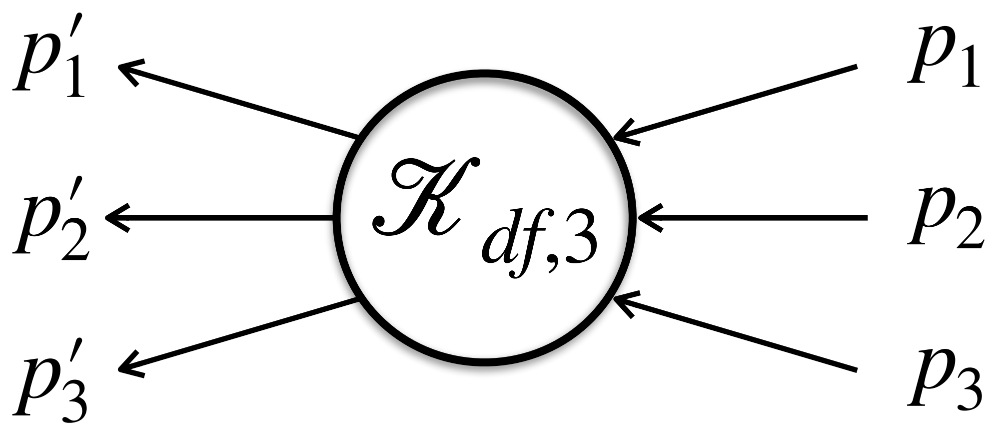
Beyond the isotropic approximation

[Tyler Blanton, Fernando Romero-Lopez & SRS, in progress]

- In 2-particle case, assume s-wave dominance at low energies, then systematically add in higher waves (suppressed by q^{2l})
- We are implementing the same general approach for $\mathcal{K}_{df,3}$, making use of the facts that it is relativistically invariant and completely symmetric under initial- & final-state permutations, and expanding about threshold
- We work in the G-parity invariant theory with 3 identical scalars, so the first channel beyond s-wave has l=2 (d-wave)



Beyond the isotropic approximation



$$\Delta = s - 9m^2$$

$$\Delta_1 = (p_2 + p_3)^2 - 4m^2 \text{ etc .}$$

$$\Delta_1' = (p_2' + p_3')^2 - 4m^2 \text{ etc .}$$

$$t_{ij} = (p_i - p_j')^2$$

$$\mathcal{K}_{\text{df},3} = \mathcal{K}_{\text{df},3}^{\text{iso}}(E) + c_A \mathcal{K}_{3A} + c_B \mathcal{K}_{3B} + \mathcal{O}(\Delta^3)$$

$$\mathcal{K}_{\text{df},3}^{\text{iso}} = c_0 + c_1 \Delta + c_2 \Delta^2$$

c_0 is the leading term—
only term kept in isotropic approx

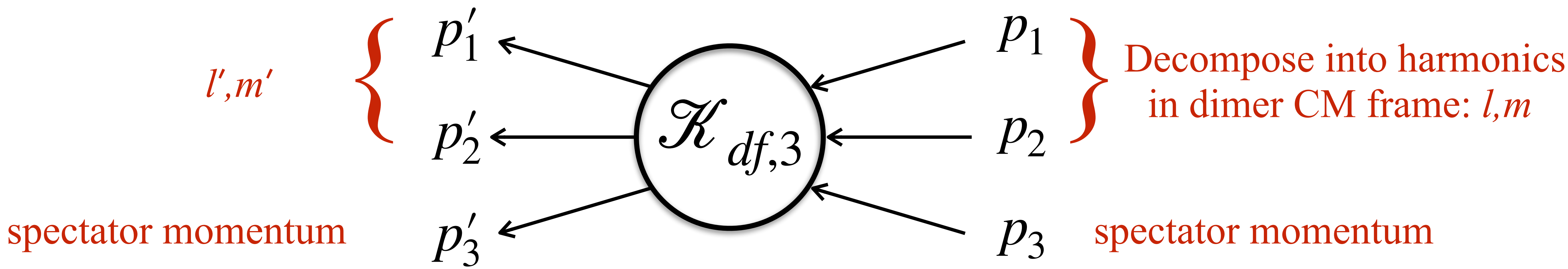
$$\mathcal{K}_{3A} = \sum_{i=1}^{3} \left(\Delta_i^2 + \Delta_i'^2\right)$$

c_1 is coefficient of the <u>only</u> linear term

$$\mathcal{K}_{3B} = \sum_{i,j=1}^{3} t_{ij}^2$$

Only three coefficients needed at quadratic order:
c_2 , c_A & c_B
Many fewer than the 7 angular variables + s dependence present at arbitrary energy!

Decomposing into spectator/dimer basis



$$\mathcal{K}_{3A},\quad \mathcal{K}_{3B} \quad \Rightarrow \quad l'=0,2 \ \&\ l=0,2$$

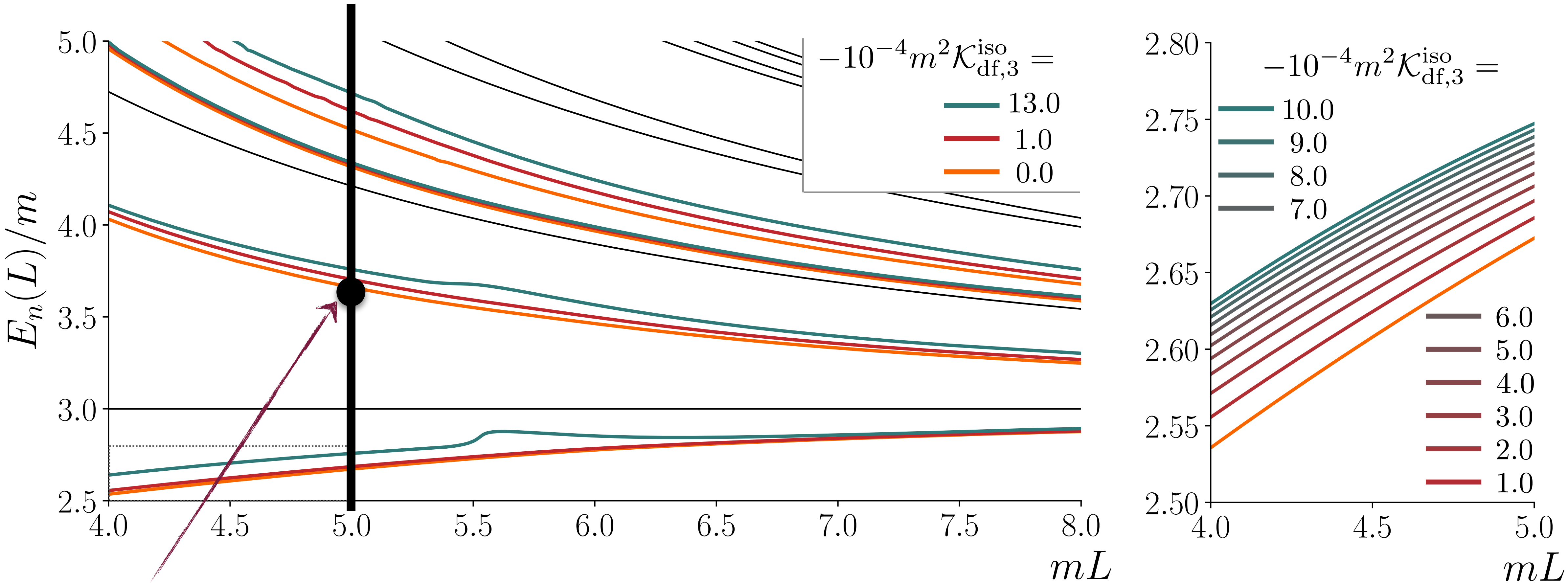
For consistency, need $\mathcal{K}_2^{(0)} \sim 1+q^2+q^4$ & $\mathcal{K}_2^{(2)} \sim q^4$

$$\frac{1}{\mathcal{K}_2^{(0)}} = \frac{1}{16\pi E_2}\left[\frac{1}{a_0} + r_0\frac{q^2}{2} + P_0 r_0^3 q^4\right] \qquad \frac{1}{\mathcal{K}_2^{(2)}} = \frac{1}{16\pi E_2}\frac{1}{q^4}\frac{1}{a_2^5}$$

Implemented quantization condition through quadratic order, for P=0, including projection onto overall cubic group irreps

First results including l=2

$$\mathcal{K}_{\mathrm{df},3} = 0\,,\ a_0 = -10\,,\ r_0 = 0.5\,,\ P_0 = 0.5\,,\ \ -1.5 \le a_2 \le 0.1$$



What happens to
this level as
a_2 is turned on?

More in progress!

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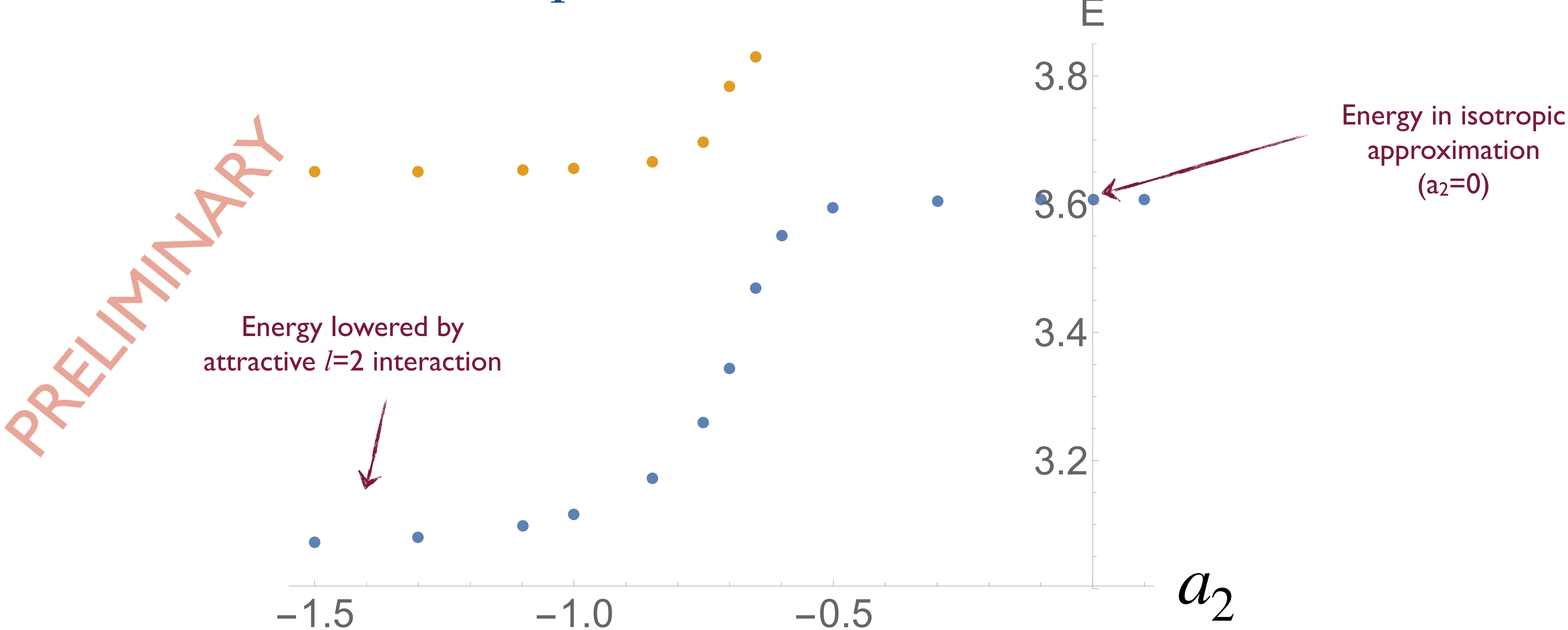
Spectrum for mL=5



More in progress!

Any questions?