Renormalization on the fuzzy sphere

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“Lattice 2018” at the Kellogg Hotel and Conference Center on July 23rd, 2018

1. Introduction 1

◆Matrix model: a nonperturbative formulation of string theory
Numerical simulation is useful for studying this model.

◆Noncommutative space
appears in various contexts of string theory.
Ex) matrix model, string field theory, ...

➡ It is important to elucidate the difference between field theories on ordinary spaces and noncommutative spaces.

1. Introduction 2

◆Star product (Moyal product)

In field theories on noncommutative spaces,
the product is noncommutative and nonlocal.
For instance, on the noncommutative plane

$$f(x) \star g(x) = e^{\frac{i\theta}{2}\left(\partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1}\right)} f(x) g(y) \Big|_{y=x}$$

➡ This property gives rise to IR divergence originating from UV divergence. = UV/IR mixing [Minwalla-Raamsdonk-Seiberg ('99)]
So, it is non-trivial whether field theories on noncommutative spaces are renormalizable or not due to the UV/IR mixing.

1. Introduction 3

◆Renormalization in a scalar field theory on the fuzzy sphere

typical example of the compact noncommutative space

We calculate 2-point and 4-point correlation functions nonperturbatively by Monte Carlo simulation.

While it is non-trivial whether the theory is renormalizable or not in the perturbation theory,
we find the theory is nonperturbatively renormalizable when we take the commutative limit where the effect of the noncommutativity remains in quantum theory.

Contents

2. Review of a scalar field theory on the fuzzy sphere

Action

Field theory on the fuzzy sphere is realized by the following matrix model.

◆Action of matrix model ($\boldsymbol{\Phi}$: $\boldsymbol{N} \times \boldsymbol{N}$ Hermitian matrix, $\boldsymbol{N = 2j + 1}$)

UV cutoff

$$S_{\mathrm{MM}} = \frac{1}{N} \mathrm{tr} \left(-\frac{1}{2} \left[\hat{L}_i, \Phi \right]^2 + \frac{\mu^2}{2} \Phi^2 + \frac{\lambda}{4} \Phi^4 \right)$$

($\hat{\boldsymbol{L}}_{\boldsymbol{i}}$: generators of the SU(2) algebra with the spin-$\boldsymbol{j}$ rep., $\boldsymbol{N} \times \boldsymbol{N}$ matrices)

Classically agrees ($\boldsymbol{N} \to \infty$).

In quantum theory, does not agree.

= UV/IR anomaly [Chu-Madore-Steinacker ('01)]

⇒ It is non-trivial whether this theory is renormalizable or not.

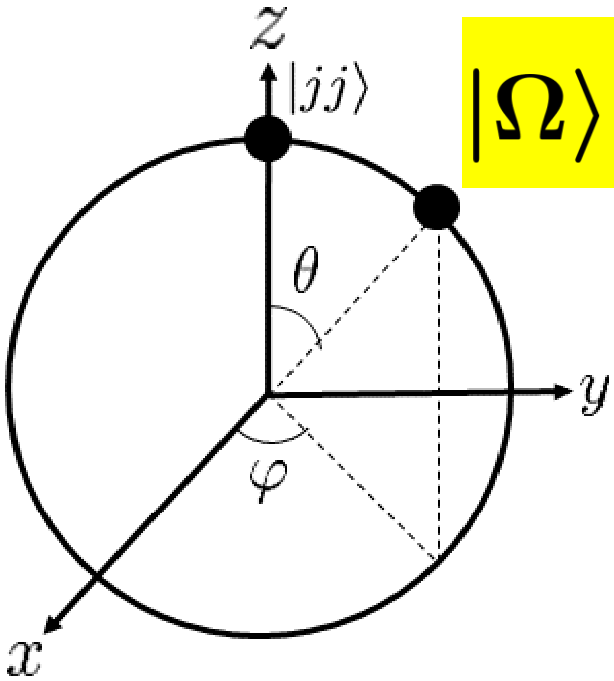
◆Action of the scalar field on the commutative sphere

$$S_{\mathrm{C}} = R^2 \int \frac{d\Omega}{4\pi} \left(-\frac{1}{2R^2} [\mathcal{L}_i \phi(\Omega)]^2 + \frac{\mu^2}{2} \phi(\Omega)^2 + \frac{\lambda}{4} \phi(\Omega)^4 \right)$$

($\boldsymbol{\phi(\Omega)}$: scalar field , $\boldsymbol{\mathcal{L}_i}$: angular momentum operators, $\boldsymbol{i} = 1, 2, 3$)

Bloch coherent state and Berezin symbol

◆Bloch coherent state is localized around the point $\boldsymbol{\Omega} = (\boldsymbol{\theta},\ \boldsymbol{\varphi})$ and its width is $1/\sqrt{\boldsymbol{j}}$



$$|\boldsymbol{\Omega}\rangle = \underbrace{\boldsymbol{e}^{i\boldsymbol{\theta}\left(\hat{\boldsymbol{L}}_1 \sin\varphi - \hat{\boldsymbol{L}}_2 \cos\varphi\right)}}_{\text{rotation matrix}} \underbrace{|\boldsymbol{jj}\rangle}_{\substack{\text{corresponds to} \\ \text{the north pole} \\ \text{Highest-weight state}}}$$

classical $(\boldsymbol{N} \to \infty)$ correspondence

◆Berezin symbol $\langle \boldsymbol{\Omega} | \boldsymbol{\Phi} | \boldsymbol{\Omega} \rangle = \varphi(\boldsymbol{\Omega}) \longleftrightarrow \underbrace{\phi(\boldsymbol{\Omega})}_{\text{field}}$

Define of correlation function

◆Define $\boldsymbol{n}$−point correlation function in the matrix model

Berezin symbol: $\langle \Omega | \Phi | \Omega \rangle = \varphi(\Omega) \longleftrightarrow \underline{\phi(\Omega)}$ field

$$\langle \varphi(\Omega_1)\varphi(\Omega_2)\cdots\varphi(\Omega_n)\rangle = \frac{\int d\Phi \varphi(\Omega_1)\varphi(\Omega_2)\cdots\varphi(\Omega_n)e^{-S_{\mathrm{MM}}}}{\int d\Phi e^{-S_{\mathrm{MM}}}},$$

where

$$S_{\mathrm{MM}} = \frac{1}{N}\mathrm{tr}\left(-\frac{1}{2}\left[\hat{L}_i, \Phi\right]^2 + \frac{\mu^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4\right), \quad d\Phi = \prod_{i=1}^{N} d\Phi_{ii} \prod_{1\le j<k\le N} d\,\mathrm{Re}\,\Phi_{jk} d\,\mathrm{Im}\,\Phi_{jk}$$

$$\langle \varphi(\Omega_1)\varphi(\Omega_2)\cdots\varphi(\Omega_n)\rangle \longleftrightarrow \langle \phi(\Omega_1)\phi(\Omega_2)\cdots\phi(\Omega_n)\rangle$$

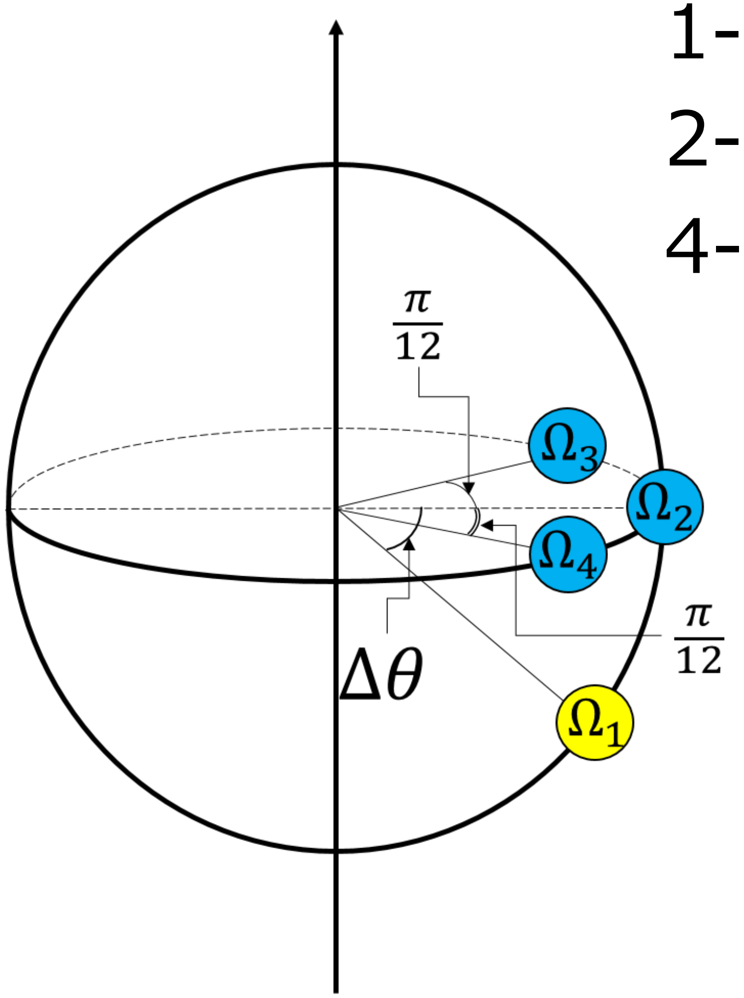
3. Calculation of correlation functions and renormalization

◆Correlation functions calculated by Monte Carlo simulation

1-point function: $\langle \varphi(\Omega_1) \rangle$,

2-point function: $\langle \varphi(\Omega_i)\varphi(\Omega_j) \rangle \ \ (1 \le i < j \le 4)$,

4-point function: $\langle \varphi(\Omega_1)\varphi(\Omega_2)\varphi(\Omega_3)\varphi(\Omega_4) \rangle$



$$\Omega_1 = \left(\frac{\pi}{2} + \Delta\theta, 0\right)$$

$\Delta\boldsymbol{\theta}$ is taken in steps of 0.1 in the range $0 \le \Delta\boldsymbol{\theta} \le 1.5$.

$$\begin{aligned} \Omega_2 &= \left(\frac{\pi}{2}, 0\right) \\ \Omega_3 &= \left(\frac{\pi}{2}, \frac{\pi}{12}\right) \\ \Omega_4 &= \left(\frac{\pi}{2}, -\frac{\pi}{12}\right) \end{aligned}$$

Fixed on the equator

Renormalization

◆Renormalization

$\Phi = \sqrt{Z}\underline{\Phi_r}$ (Z: the factor of the wave function renormalization)

renormalized matrix

⬇

$\varphi_r(\Omega) = \langle \Omega | \Phi_r | \Omega \rangle$: the renormalized Berezin symbol

$$\langle \varphi(\Omega_1) \rangle = \sqrt{Z} \langle \varphi_r(\Omega_1) \rangle ,$$

$$\langle \varphi(\Omega_i)\varphi(\Omega_j) \rangle = Z \langle \varphi_r(\Omega_i)\varphi_r(\Omega_j) \rangle \ \ (1 \leq i < j \leq 4),$$

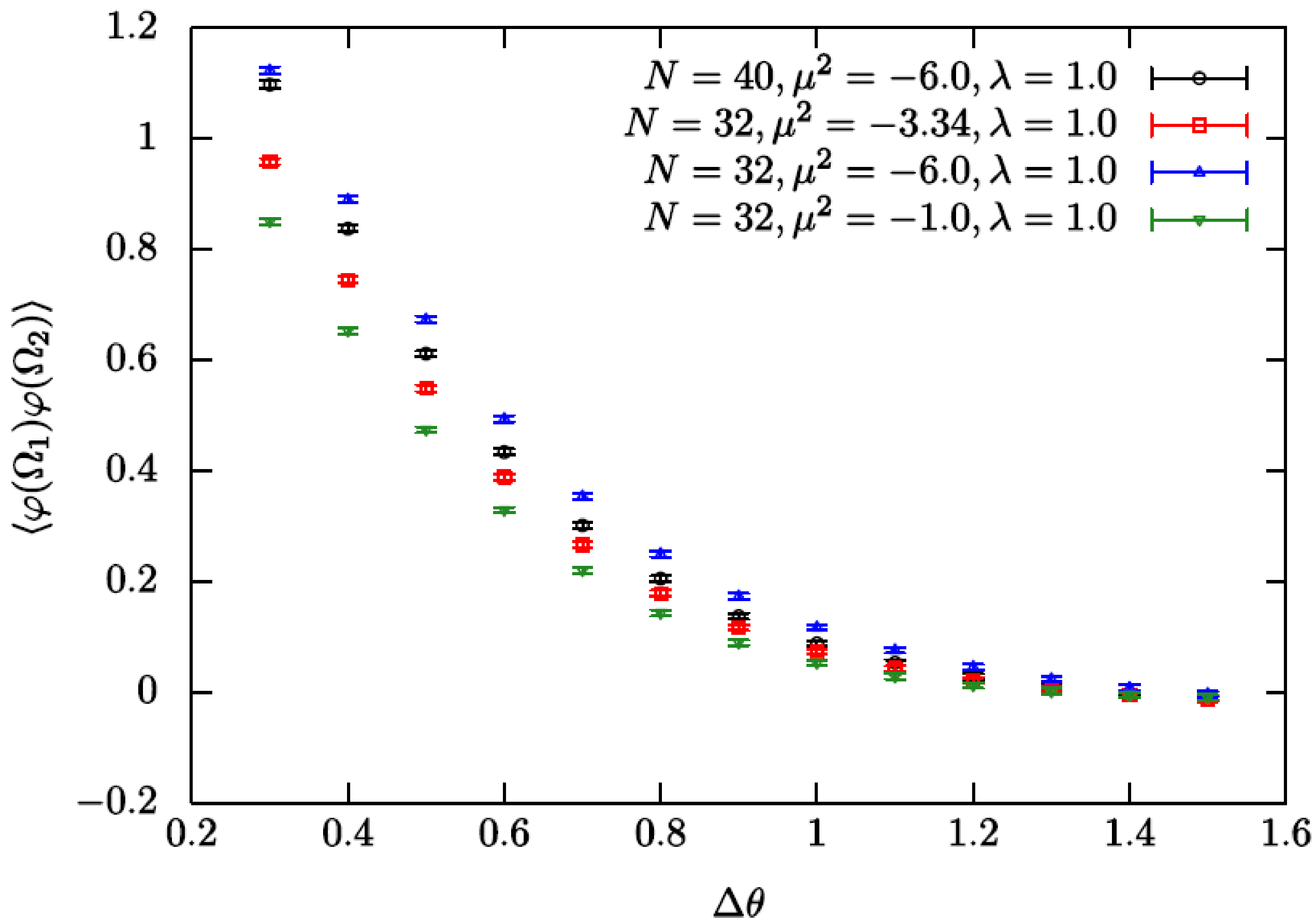
$$\langle \varphi(\Omega_1)\varphi(\Omega_2)\varphi(\Omega_3)\varphi(\Omega_4) \rangle_c = Z^2 \langle \varphi_r(\Omega_1)\varphi_r(\Omega_2)\varphi_r(\Omega_3)\varphi_r(\Omega_4) \rangle_c$$

In the following,
we show that renormalized correlation functions are independent of the matrix size N, which is the UV cutoff, by tuning 1-parameter.

Renormalization with λ fixed (λ=1.0)

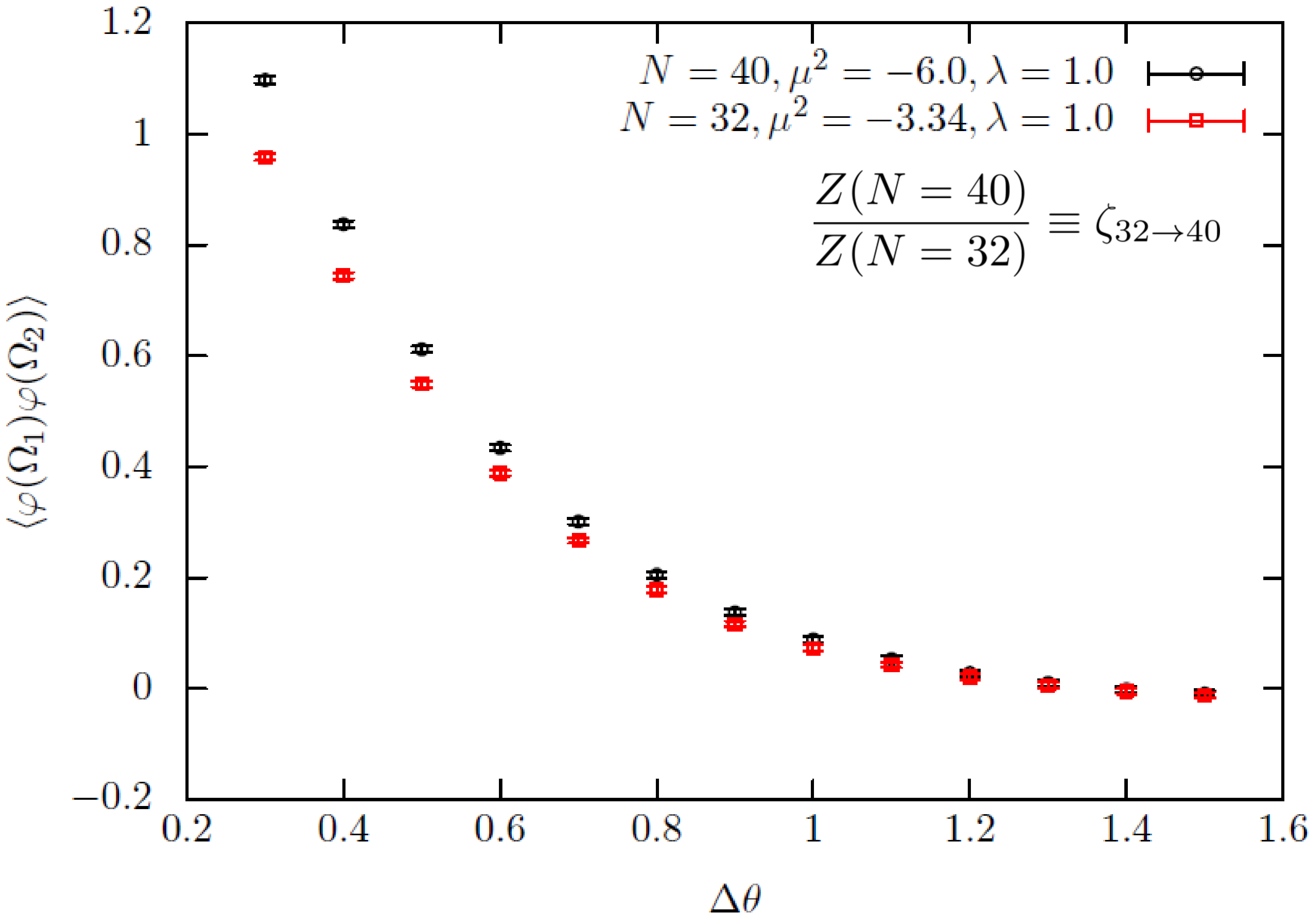
2-point function (N=40 and 32, λ=1.0)

$$\langle \varphi(\Omega_1)\varphi(\Omega_2)\rangle = Z\,\langle \varphi_r(\Omega_1)\varphi_r(\Omega_2)\rangle$$



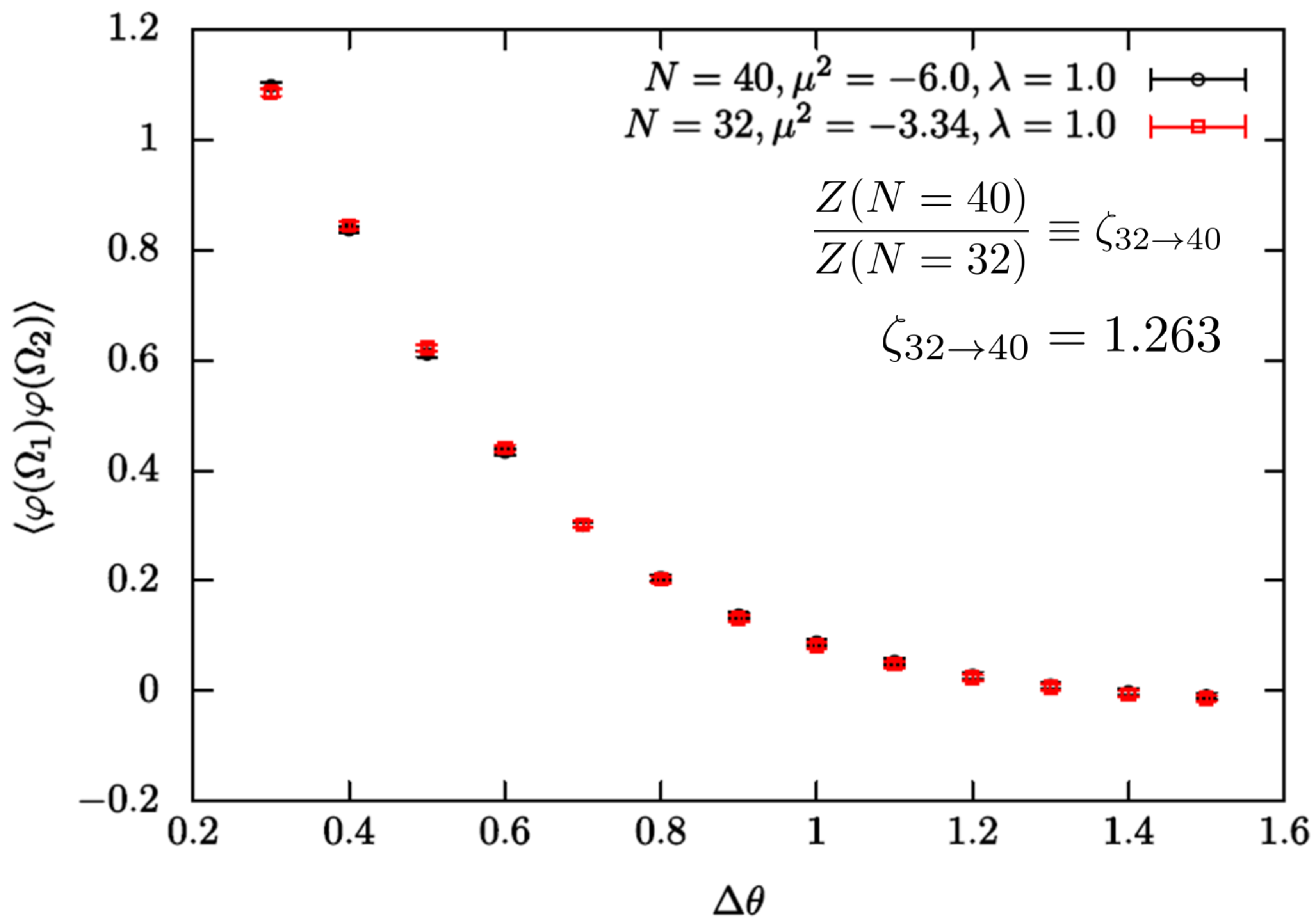
2-point function（N=40 and 32, λ=1.0）

$$\langle \varphi(\Omega_1)\varphi(\Omega_2)\rangle = Z\,\langle \varphi_r(\Omega_1)\varphi_r(\Omega_2)\rangle$$



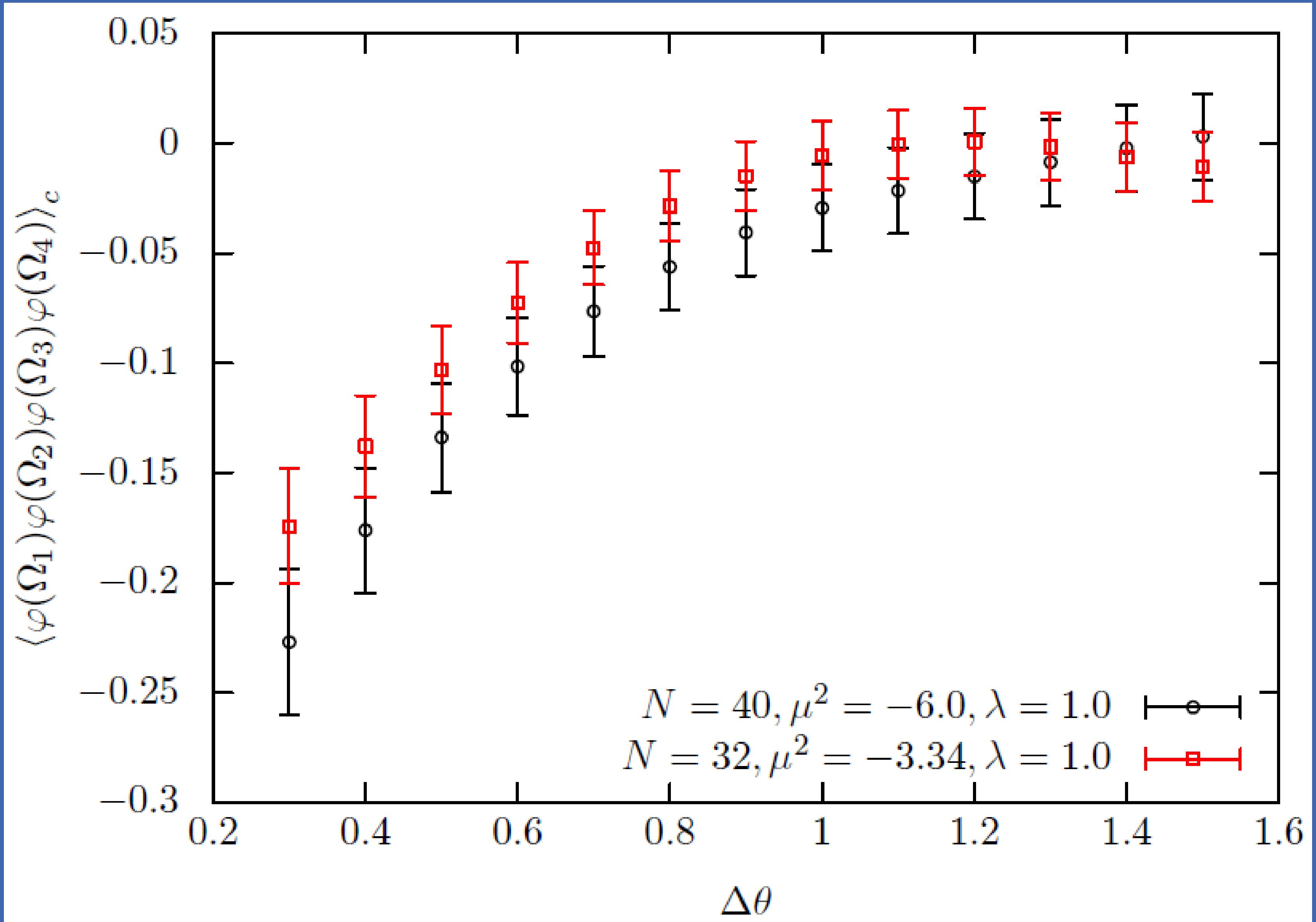
2-point function（N=40 and 32, λ=1.0）

$$\langle\varphi(\Omega_1)\varphi(\Omega_2)\rangle = Z\,\langle\varphi_r(\Omega_1)\varphi_r(\Omega_2)\rangle$$



Connected 4-point function（N=40 and 32, λ=1.0)

$$\langle\varphi(\Omega_1)\varphi(\Omega_2)\varphi(\Omega_3)\varphi(\Omega_4)\rangle_c = Z^2\,\langle\varphi_r(\Omega_1)\varphi_r(\Omega_2)\varphi_r(\Omega_3)\varphi_r(\Omega_4)\rangle_c$$



Connected 4-point function（N=40 and 32, λ=1.0）

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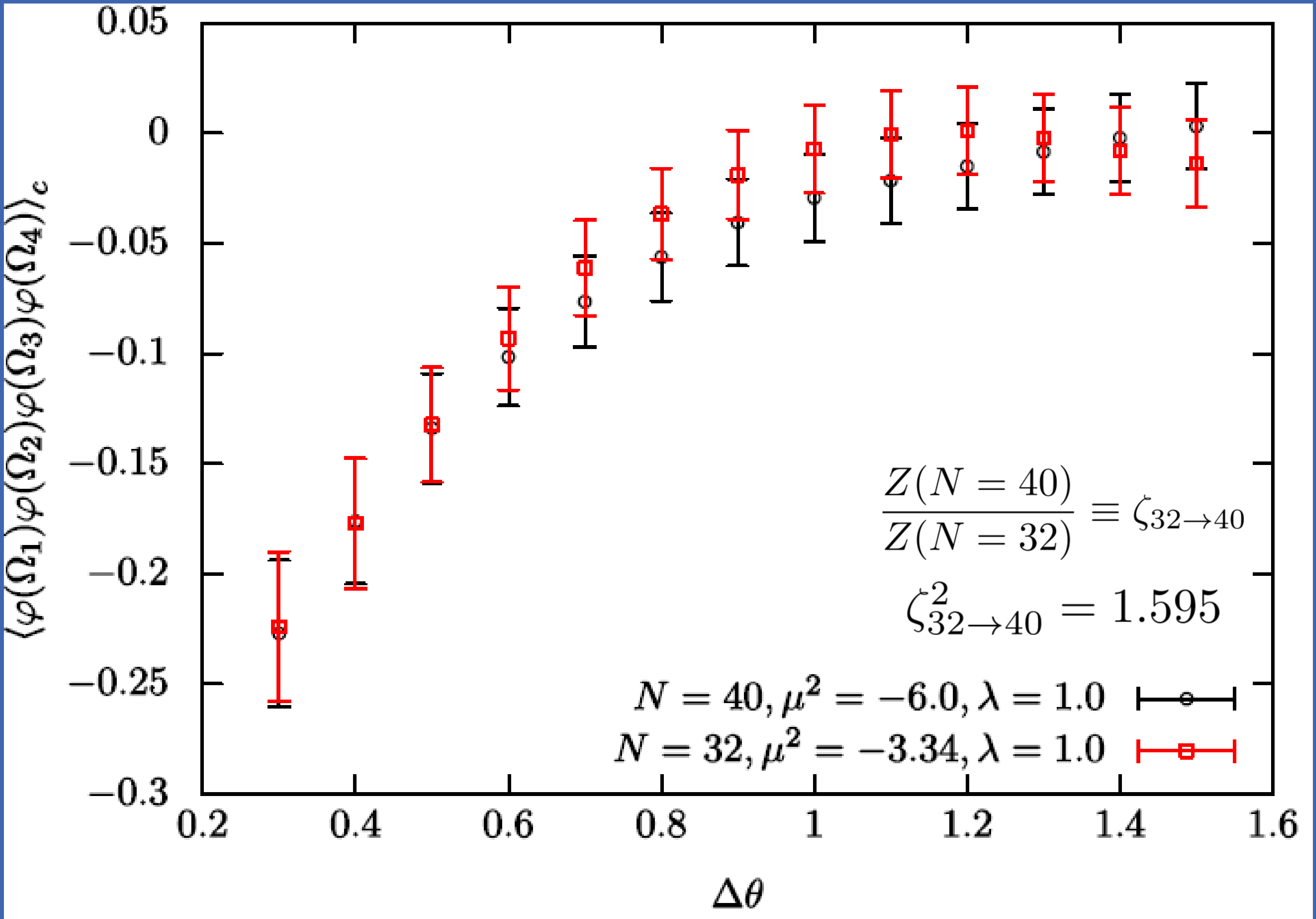
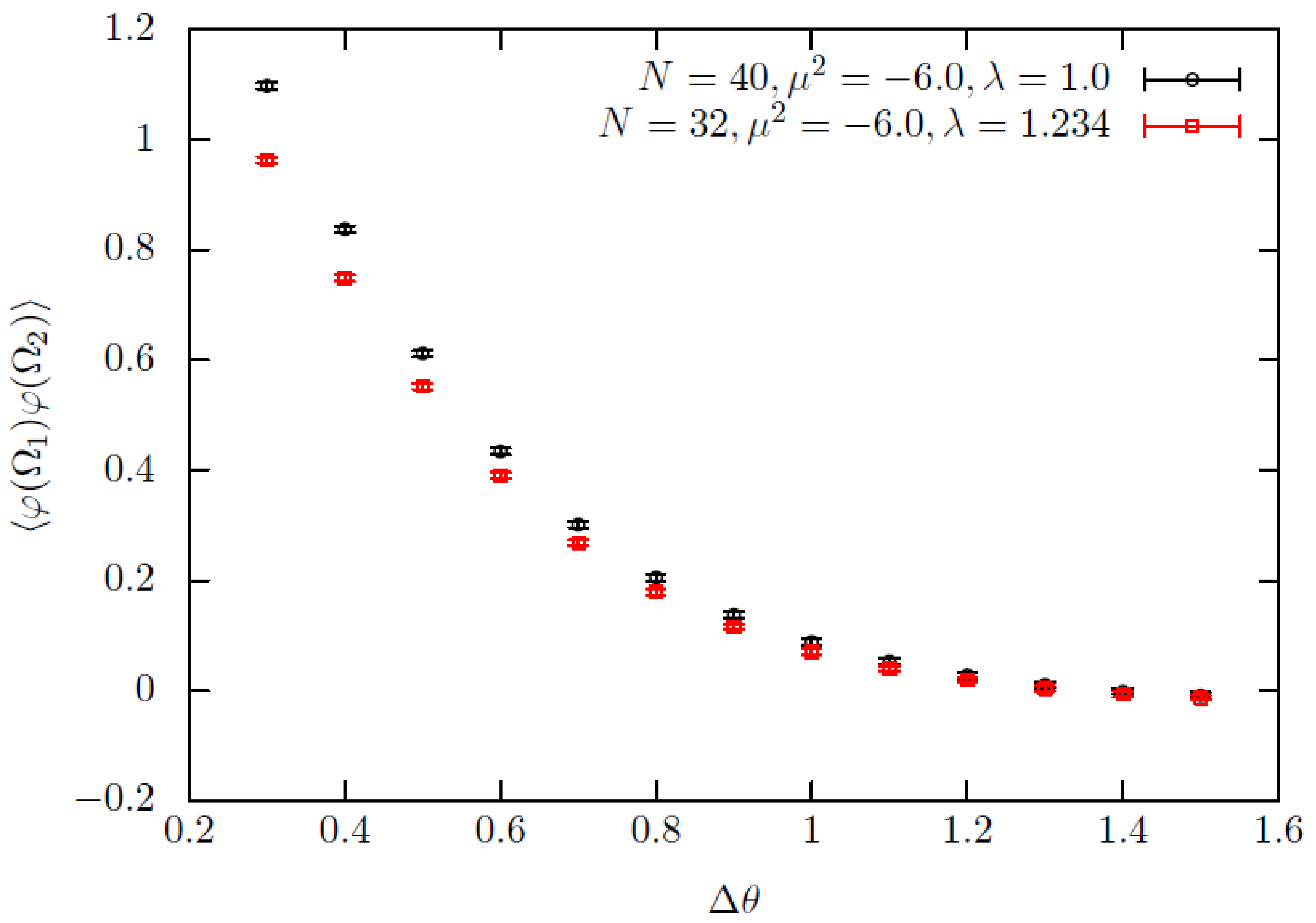


Renormalization with μ^2 fixed (μ^2=-6.0)

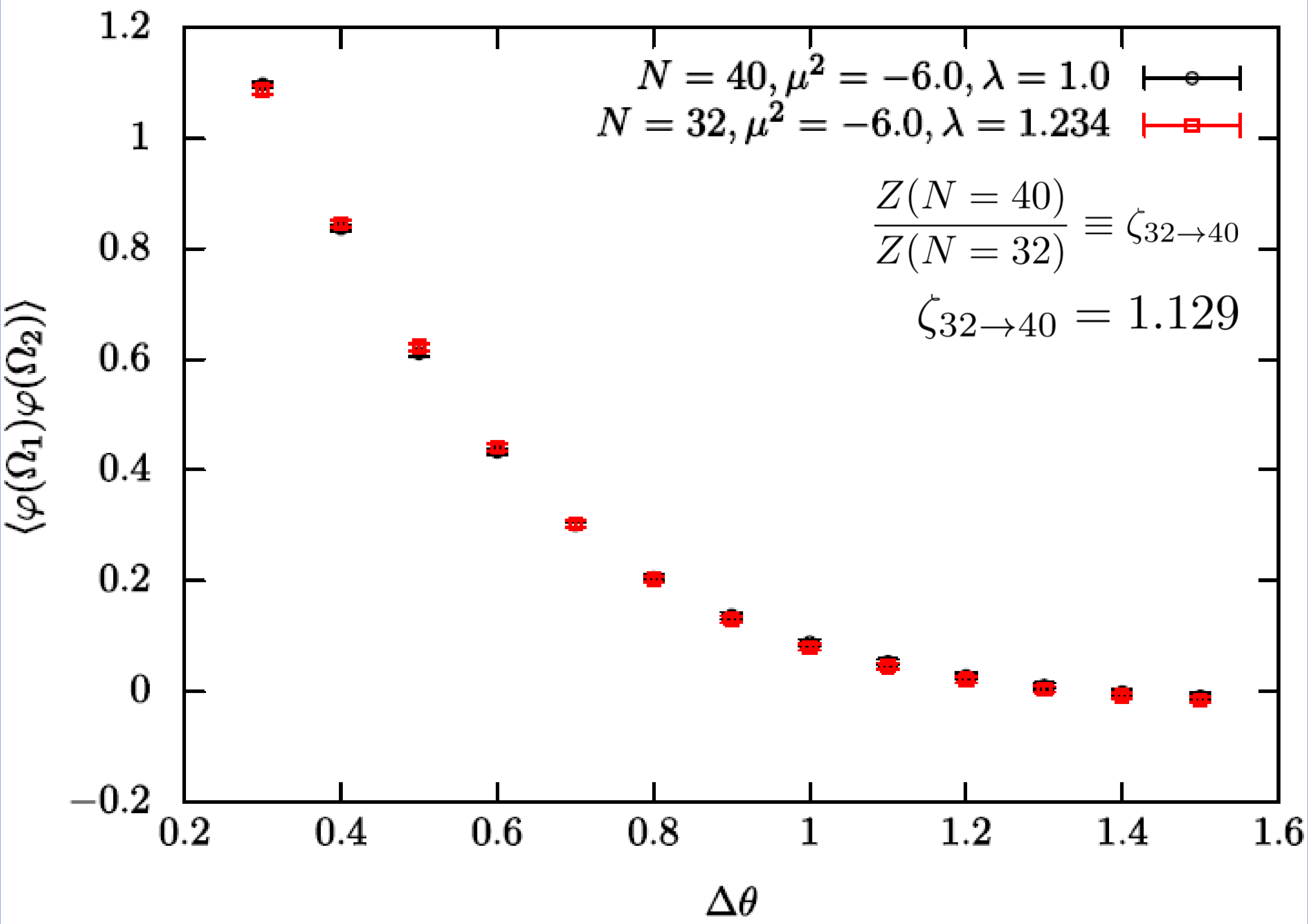
2-point function（N=40 and 32, μ^2=-6.0）

$$\langle \varphi(\Omega_1)\varphi(\Omega_2)\rangle = Z\,\langle \varphi_r(\Omega_1)\varphi_r(\Omega_2)\rangle$$



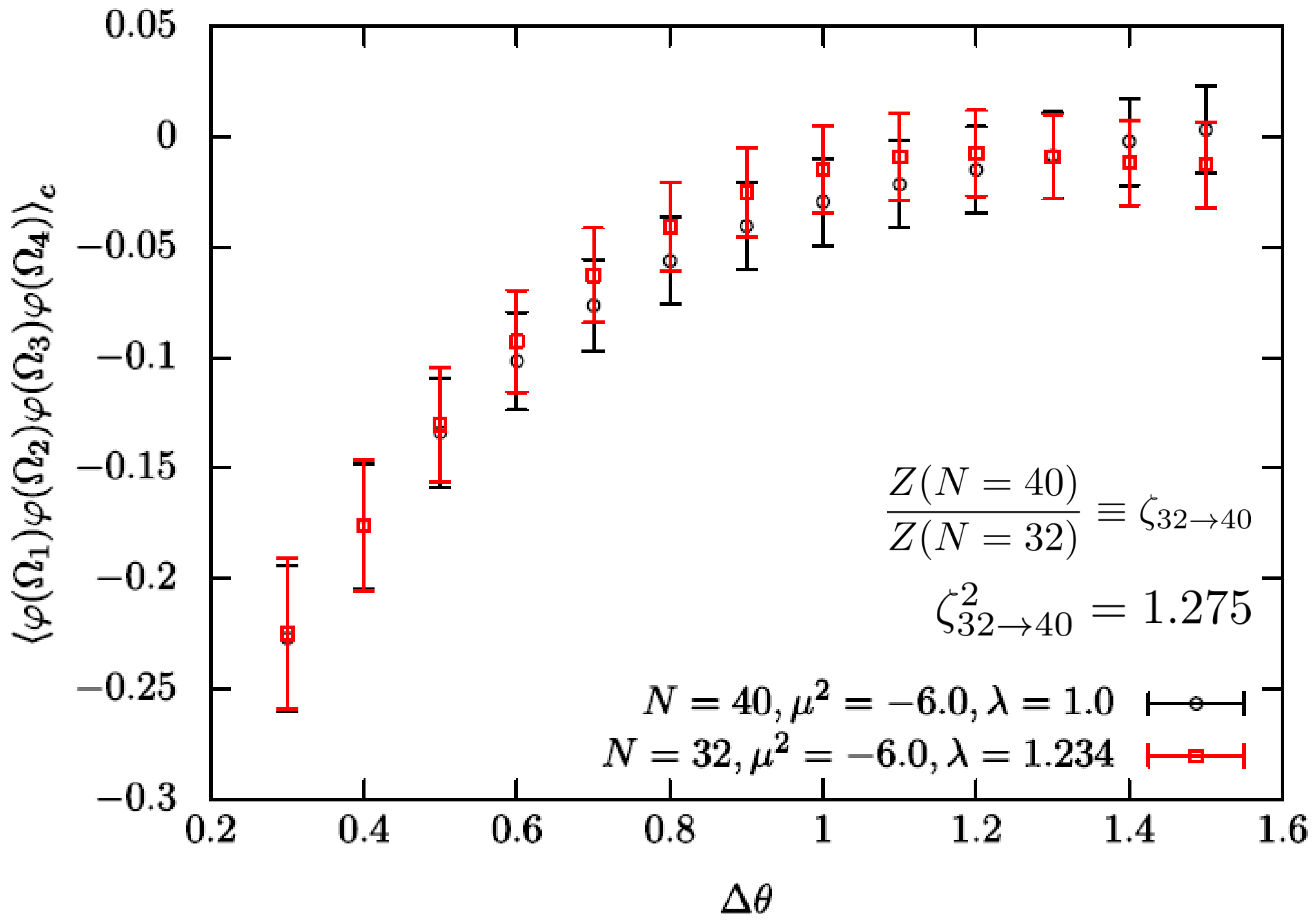
2-point function（N=40 and 32, μ^2=-6.0）

$$\langle \varphi(\Omega_1)\varphi(\Omega_2)\rangle = Z\,\langle \varphi_r(\Omega_1)\varphi_r(\Omega_2)\rangle$$



Connected 4-point function（N=40 and 32, μ^2=−6.0）

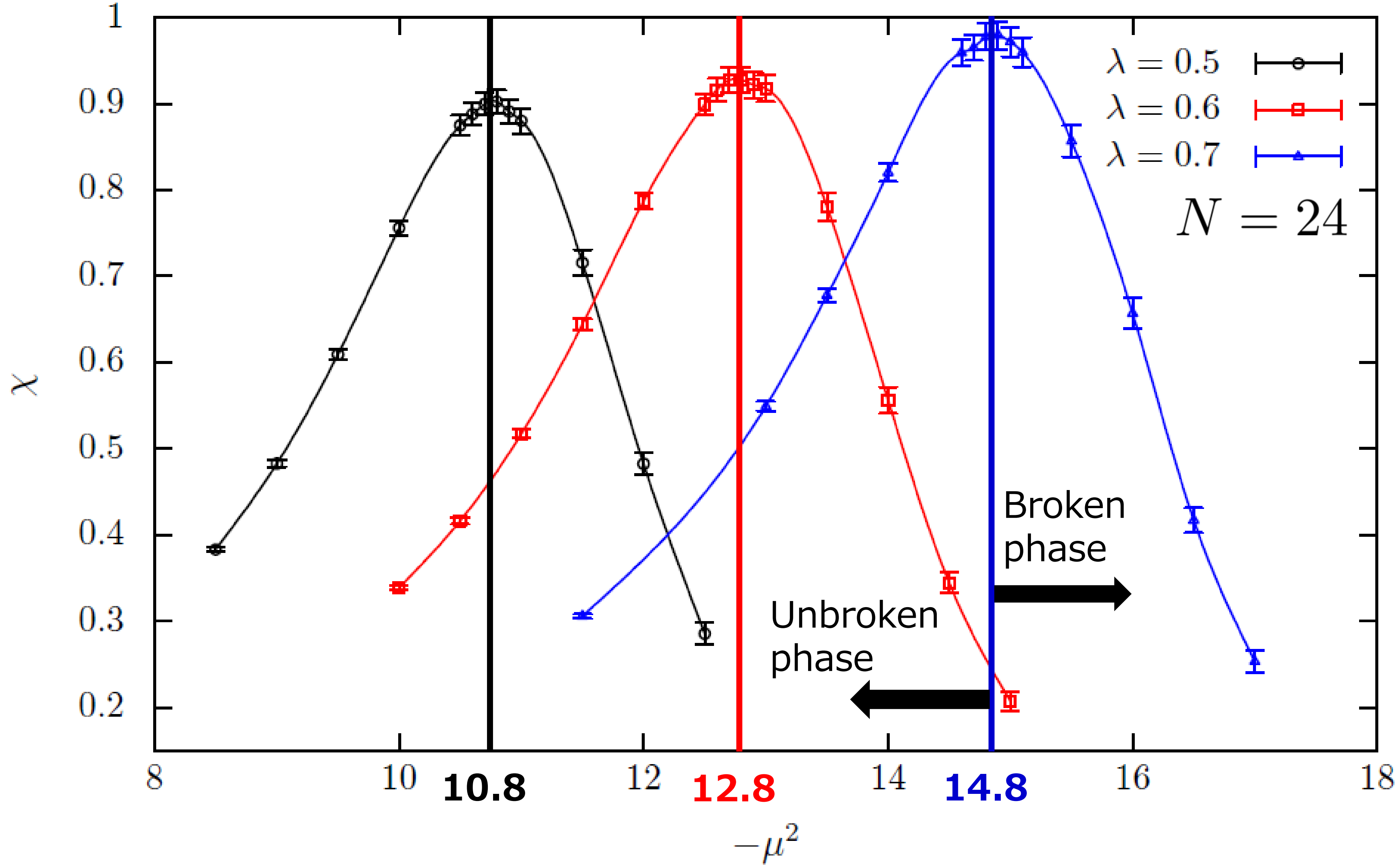
$$\langle\varphi(\Omega_1)\varphi(\Omega_2)\varphi(\Omega_3)\varphi(\Omega_4)\rangle_c = Z^2\langle\varphi_r(\Omega_1)\varphi_r(\Omega_2)\varphi_r(\Omega_3)\varphi_r(\Omega_4)\rangle_c$$



4. Critical behavior of correlation functions $(\boldsymbol{N = 24})$

Susceptibility χ: order parameter of Z_2 symmetry ($\Phi \to -\Phi$)

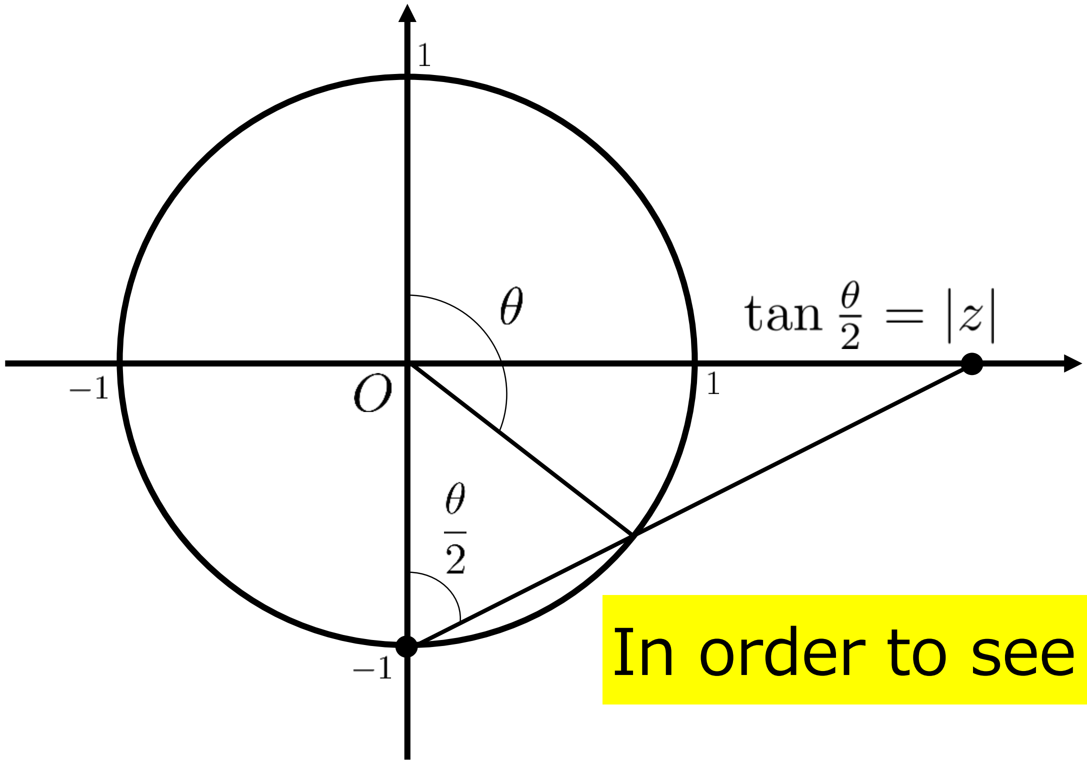
$$\chi = \left\langle \left(\frac{1}{N} \mathrm{Tr}\boldsymbol{\Phi} \right)^2 \right\rangle - \left\langle \frac{1}{N} \left| \mathrm{Tr}\boldsymbol{\Phi} \right| \right\rangle^2$$



Stereographic projection

In order to see the behavior of correlation functions on the phase boundary, we introduce a stereographic projection,

$\boldsymbol{z = \tan\frac{\theta}{2} e^{i\varphi}}$, which maps a sphere to the complex plane.



$$\Omega_1 = \left(\frac{\pi}{2} + 0.1m, 0\right) \to \boxed{z_m = \tan\left[\frac{1}{2}\left(\frac{\pi}{2} + 0.1m\right)\right]}$$

with $1 \leq m \leq 15$

$$\Omega_2 = \left(\frac{\pi}{2}, 0\right) \to z = 1$$

$$\Omega_3 = \left(\frac{\pi}{2}, \frac{\pi}{3}\right) \to z = e^{i\frac{\pi}{3}}$$

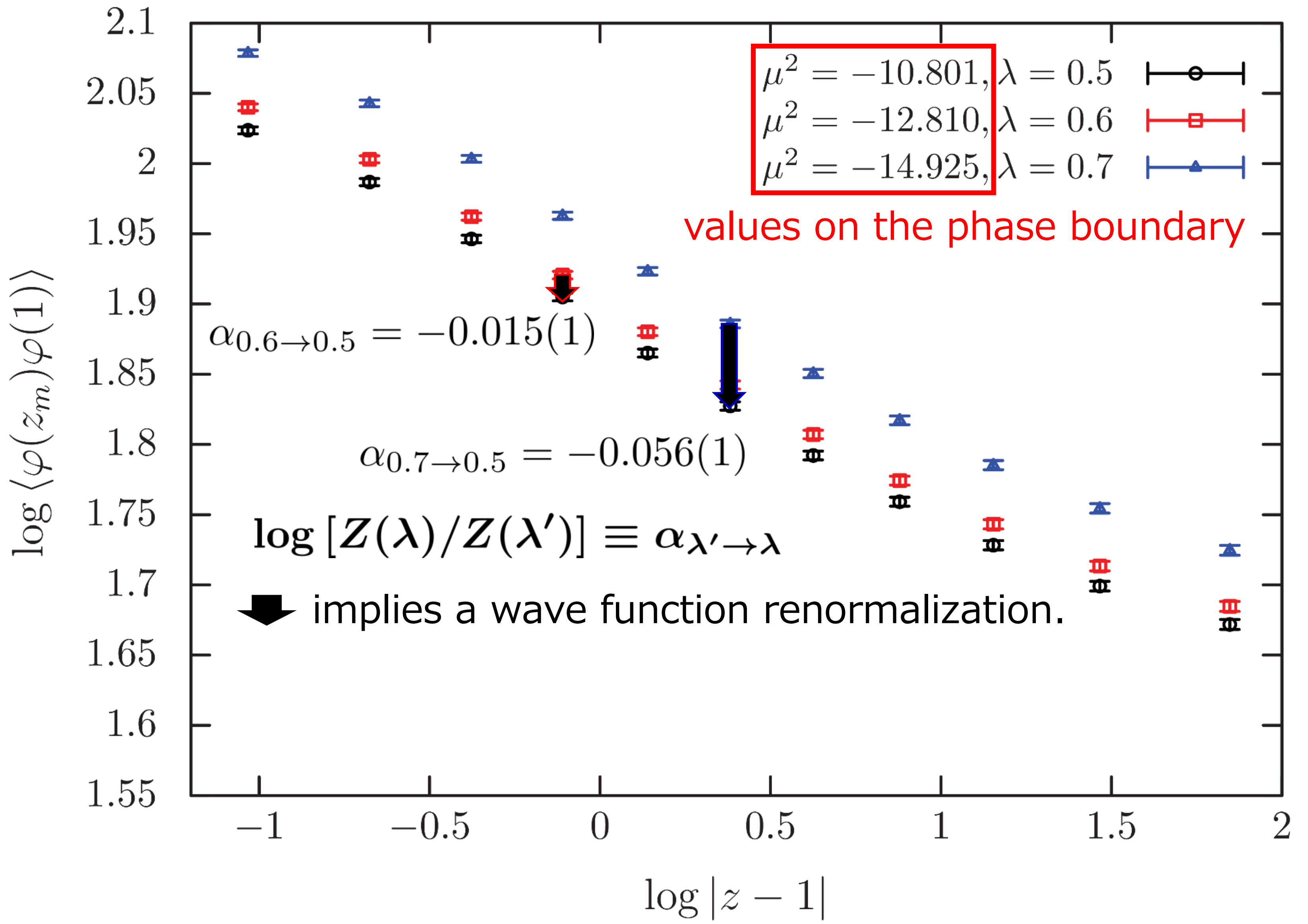
$$\Omega_4 = \left(\frac{\pi}{2}, -\frac{\pi}{3}\right) \to z = e^{i\frac{5\pi}{3}}$$

Fixed

In order to see a connection to a CFT, we use a log-log plot.

Logarithmic 2-point function

$$\log \langle \varphi(z_m)\varphi(1)\rangle = \log Z + \log \langle \varphi_r(z_m)\varphi_r(1)\rangle$$



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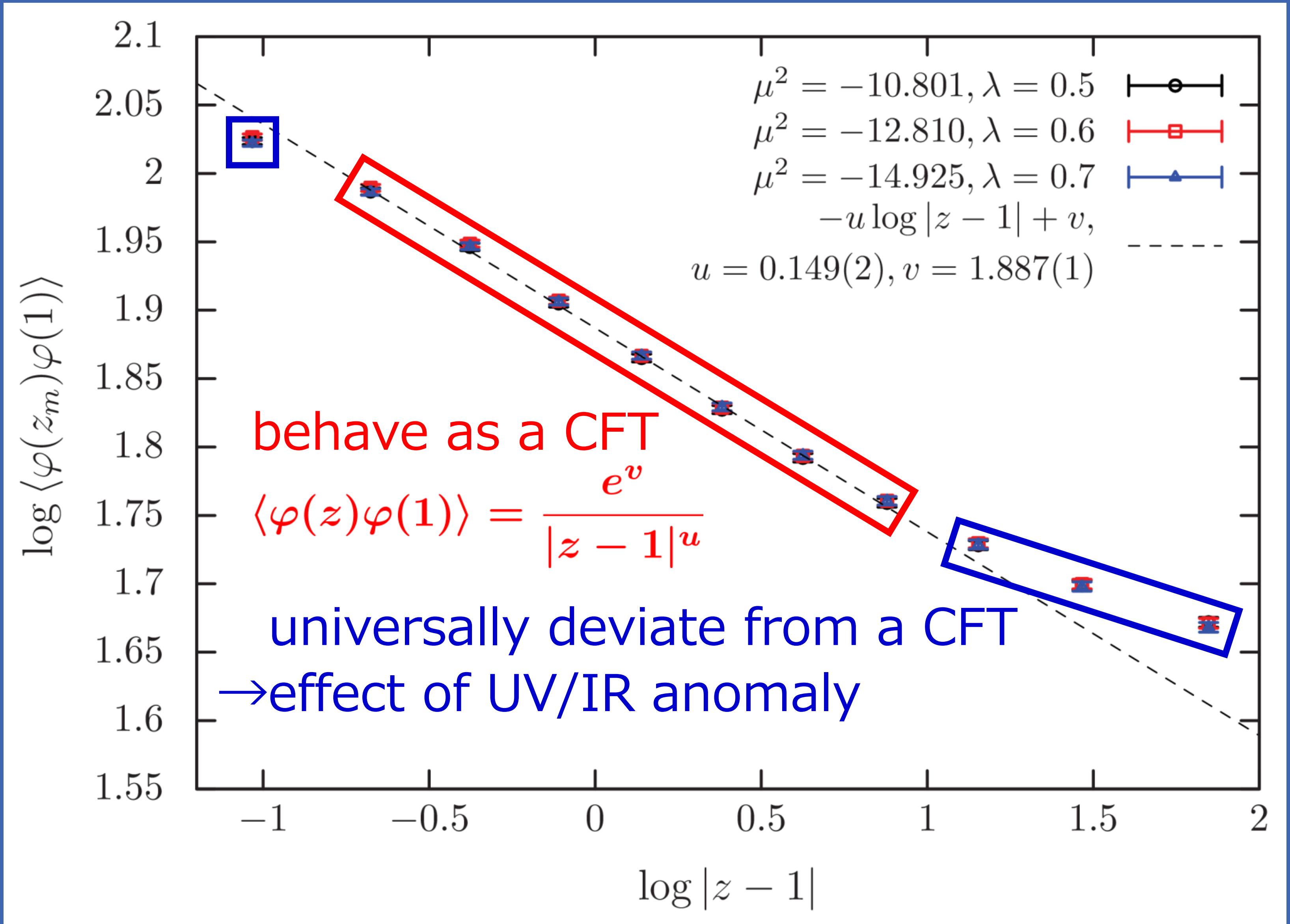
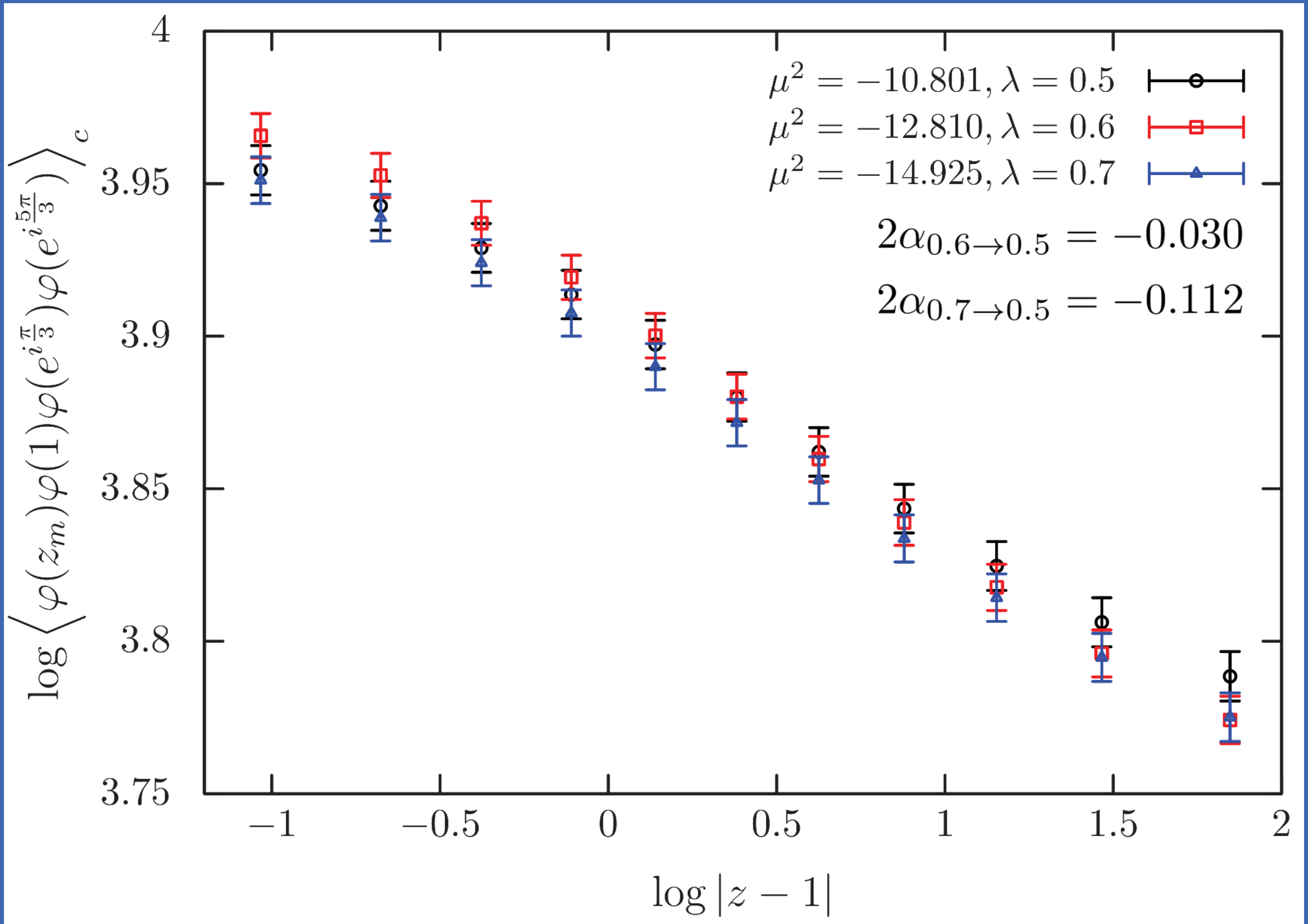
$$\log \langle \varphi(z_m)\varphi(1)\rangle = \log Z + \log \langle \varphi_r(z_m)\varphi_r(1)\rangle$$



Logarithmic connected 4-point function

$$\log\left\langle \varphi(z_m)\varphi(1)\varphi(e^{i\frac{\pi}{3}})\varphi(e^{i\frac{5\pi}{3}})\right\rangle_c = 2\log Z + \log\left\langle \varphi_r(z_m)\varphi_r(1)\varphi_r(e^{i\frac{\pi}{3}})\varphi_r(e^{i\frac{5\pi}{3}})\right\rangle_c$$



5. Conclusion and discussion

Conclusion

- We constructed the correlation functions in a scalar field theory on the fuzzy sphere by using the Berezin symbol. We calculated them by Monte Carlo simulation.
- We found that the non-trivial agreement of correlation functions at different $\boldsymbol{N}$ after tuning one parameter ($\boldsymbol{\mu^2}$ or $\boldsymbol{\lambda}$) and performing the wave function renormalization, which strongly suggests that correlation functions are independent of the cutoff $\boldsymbol{N}$, namely, the theory on the fuzzy sphere is renormalizable.
- We examined correlation functions on the phase boundary beyond which the $\boldsymbol{Z_2}$ symmetry is spontaneously broken. We found that correlation functions at different points on the boundary agree up to the wave function renormalization, which implies that the critical theory is universal.

At short distances, we observed 2-pt functions behave as a those in a CFT.

Discussion

◆The CFT that we observed at short distances seems to differ from the critical Ising model, because the value of u disagrees with 2Δ, where Δ is the scaling dimension of the spin operator $\mathcal{O}(z)$, 1/8.

$$\langle\varphi(z)\varphi(1)\rangle = \frac{e^v}{|z-1|^u},$$

our result: $u = 0.149$

$$\langle\mathcal{O}(z)\mathcal{O}(z')\rangle \sim \frac{1}{|z-z'|^{2\Delta}},$$

critical Ising model: $2\Delta = 1/4 = 0.25$

➡ This suggests that the universality classes of the scalar field theory on the fuzzy sphere are totally different from those of an ordinary theory.

➢Many people reported that there exists a novel phase in the theory on the fuzzy sphere that is called the non-uniformly ordered phase. We hope to elucidate the universality classes by studying renormalization in the whole phase diagram.

Backup

UV/IR mixing

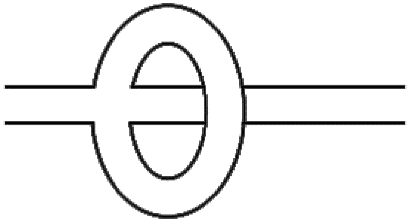
◆UV/IR mixing

Planar diagram

$$= -2\lambda \int \frac{d^2q}{(2\pi)^2} \frac{1}{q^2+\mu^2}$$

same as the ordinary field theory

Non-planar diagram



$$= -\lambda \int \frac{d^2q}{(2\pi)^2} \frac{e^{-i\theta(p_1q_2-p_2q_1)}}{q^2+\mu^2}$$

$$\sim \frac{\lambda}{2\pi}\left[\gamma + \log\left(\frac{\mu}{2}\sqrt{\theta^2p^2+\frac{1}{\Lambda^2}}\right)\right]$$

(Λ: UV cutoff)

$$\rightarrow \begin{cases} \boxed{\text{logarithmic div.}} \ (\theta = 0, \Lambda \rightarrow \infty) \\ \text{IR div.} \ (\theta \neq 0, \Lambda \rightarrow \infty, p \rightarrow 0) \end{cases}$$

ordinary field theory
UV div.

quadratic terms of action

For the Berezin symbol $\langle\Omega|\Phi|\Omega\rangle = \varphi(\Omega)$,
we obtain the following relations.

$$\langle\Omega|[\hat{L}_i, \Phi]|\Omega\rangle = \mathcal{L}_i\varphi(\Omega)\,, \quad \frac{1}{N}\mathrm{tr} \longleftrightarrow \int\frac{d\Omega}{4\pi} \quad \text{and}$$

$$\langle\Omega|\Phi_1|\Omega\rangle \star \langle\Omega|\Phi_2|\Omega\rangle \xrightarrow{N\to\infty} \langle\Omega|\Phi_1|\Omega\rangle\ \langle\Omega|\Phi_2|\Omega\rangle$$

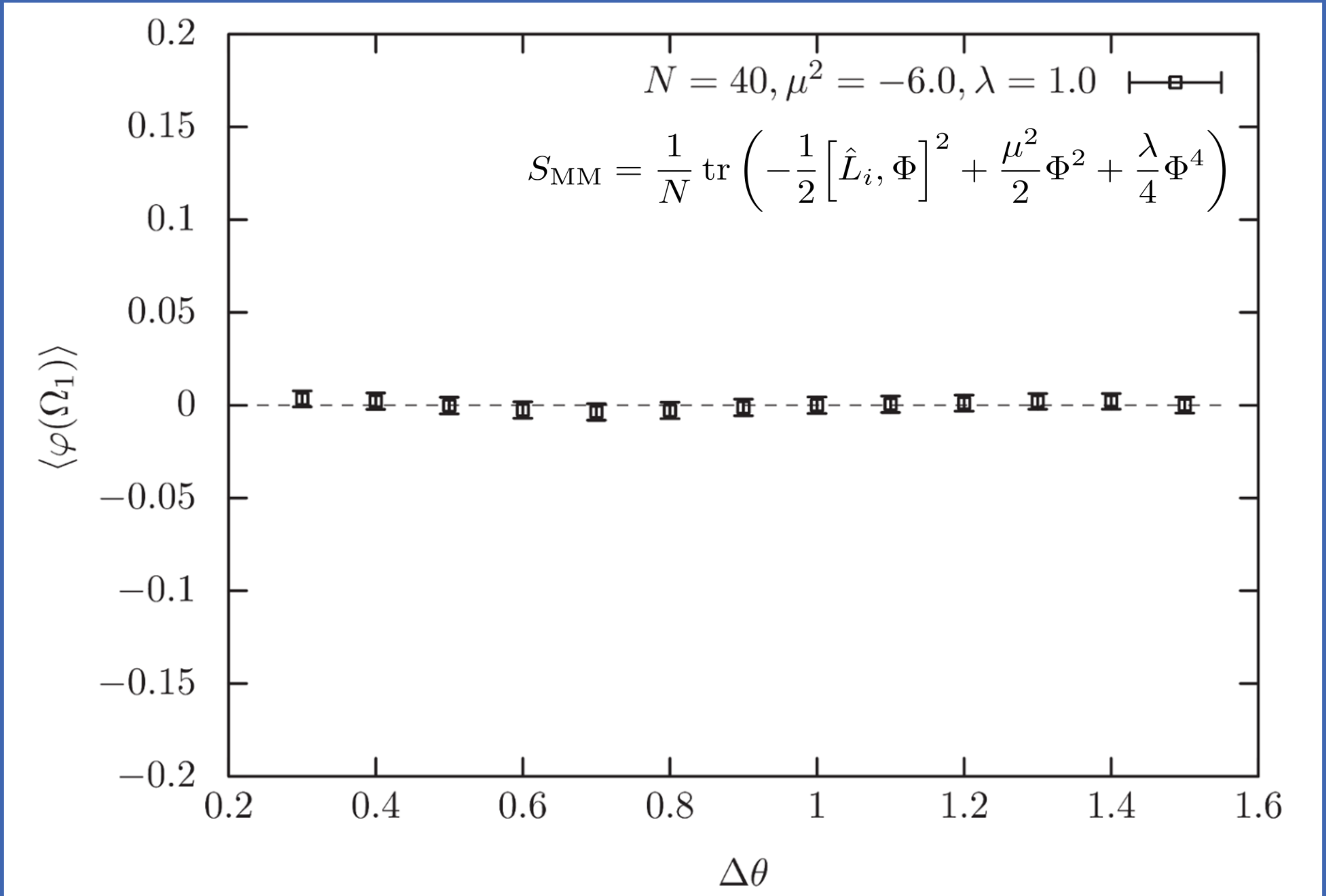
ordinary product

If we set $\varphi(\Omega) = \phi(\Omega)$, the quadratic terms of

$$S_{\mathrm{MM}} = \frac{1}{N}\mathrm{tr}\left(-\frac{1}{2}\left[\hat{L}_i, \Phi\right]^2 + \frac{\mu^2}{2}\Phi^2 + \frac{\lambda}{4}\Phi^4\right) \quad \text{and} \quad S_{\mathrm{C}} = R^2\int\frac{d\Omega}{4\pi}\left(-\frac{1}{2R^2}[\mathcal{L}_i\phi(\Omega)]^2 + \frac{\mu^2}{2}\phi(\Omega)^2 + \frac{\lambda}{4}\phi(\Omega)^4\right)$$

agree with each other. The quartic terms agree at tree level,
but including the quantum correction, they do not agree.

1-point function



- Correlation functions calculated by Monte Carlo simulation

  1-point function: $\langle \varphi(\Omega_1) \rangle = 0$,

  2-point function: $\langle \varphi(\Omega_i)\varphi(\Omega_j) \rangle = \langle \varphi(\Omega_i)\varphi(\Omega_j) \rangle_c$,

  4-point function: $\langle \varphi(\Omega_1)\varphi(\Omega_2)\varphi(\Omega_3)\varphi(\Omega_4) \rangle$,

  where c stands for the connected part.

- Connected 4-point function

$$
\begin{aligned}
&\langle \varphi(\Omega_1)\varphi(\Omega_2)\varphi(\Omega_3)\varphi(\Omega_4) \rangle_c \\
&= \langle \varphi(\Omega_1)\varphi(\Omega_2)\varphi(\Omega_3)\varphi(\Omega_4) \rangle - \langle \varphi(\Omega_1)\varphi(\Omega_2) \rangle \langle \varphi(\Omega_3)\varphi(\Omega_4) \rangle \\
&\quad - \langle \varphi(\Omega_1)\varphi(\Omega_3) \rangle \langle \varphi(\Omega_2)\varphi(\Omega_4) \rangle - \langle \varphi(\Omega_1)\varphi(\Omega_4) \rangle \langle \varphi(\Omega_2)\varphi(\Omega_3) \rangle
\end{aligned}
$$