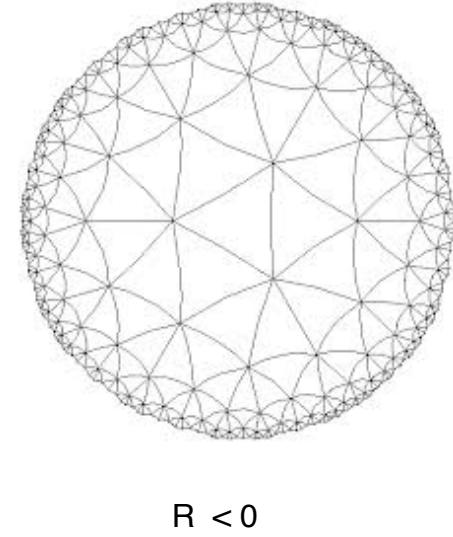
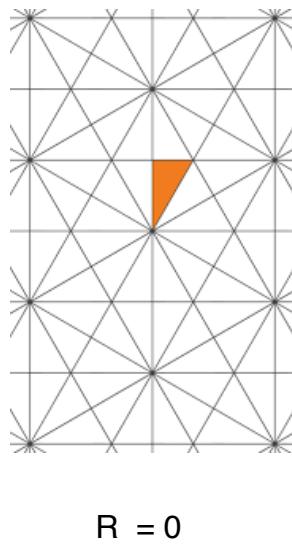
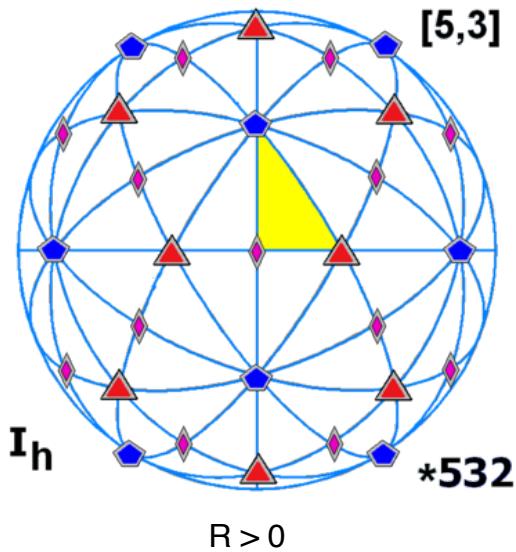


Prospects for LATTICE QFT on Curved RIEMANN MANIFOLDS



Rich Brower, Lattice 2018
with G. Fleming, A. Gasbarro, T. Raben, C-I Tan, E. Weinberg

See Details in 2 publications:

<https://arxiv.org/abs/1610.08587> Dirac Fermions on Simplicial Manifold

<https://arxiv.org/abs/1803.08512> Phi 4th on Riemannian Manifold

BIG PICTURE

GENERAL PROBLEM

All flat space Renormalizable QFT are Renormalizable on Smooth Riemann Manifold (see M. Luscher, H. Osborn & Literature in 1990's)

Find a rigorous non-perturbative Simplicial Lattice Definition of QFT on any target smooth Riemann Manifold

$$\{\mathbb{R}^d, \delta_{\mu\nu}\} \implies \{\mathcal{M}, g_{\mu\nu}\}$$

SPHERES AND CYLINDERS ARE NICE*

SPECIAL MAXIMALLY SYMMETRIC SPACES

- Conformal Field Theory are more easily studied on **Sphere, Cylinders (Radial Quantization) and Hyperbolic Spaces** (Gauge/Gravity Duality)

$$\mathbb{S}^d$$

$$\mathbb{R} \times S^{d-1}$$

$$\mathbb{A}d\mathbb{S}^{d+1}$$

$$\mathbb{R}^d \rightarrow \mathbb{S}^d$$

$$ds_{flat}^2 = \sum_{\mu=1}^d dx^\mu dx^\mu = e^{2\sigma(x)} d\Omega_d^2 \xrightarrow{Weyl} d\Omega_d^2 .$$

$$\mathbb{R}^d \rightarrow \mathbb{R} \times S^{d-1}$$

$$ds_{flat}^2 = \sum_{\mu=1}^d dx^\mu dx^\mu = e^{2\tau} (d\tau^2 + d\Omega_d^2) \xrightarrow{Weyl} (d\tau^2 + d\Omega_d^2) .$$

$$\mathbb{R}^{d+1} \rightarrow \mathbb{A}d\mathbb{S}^{d+1}$$

$$ds_{flat}^2 = \sum_{\mu=1}^{d+1} dx^\mu dx^\mu \xrightarrow{Weyl} z^{-2} (dz^2 + d\vec{x} \cdot d\vec{x})$$

Lattice Radial Quantization & BSM

$$\mathbb{R} \times \mathbb{T}^3$$

vs

$$\mathbb{R} \times \mathbb{S}^3$$



$$H = P_0 \text{ in } t \implies D \text{ in } \tau = \log(r)$$

Potential advantage:

$$1 < t < aL \implies 1 < \tau = \log(r) < L$$

Goal for BSM theories? Begin with exact CFT in the IR and study spectral flow due to adiabatic “mass” deformation of Dimensions to Masses as the Dilatation reverts to the Hamiltonian.

Constructing the Classical Simplicial Action

$$S = \frac{1}{2} \int_{\mathcal{M}} d^d x \sqrt{g(x)} [g^{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x) + m^2 \phi^2(x) + \lambda \phi^4(x)]$$

Regge Calculus discretized
Manifold $g_{\mu\nu}(x)$

Finite Element discretized
field $\phi(x)$

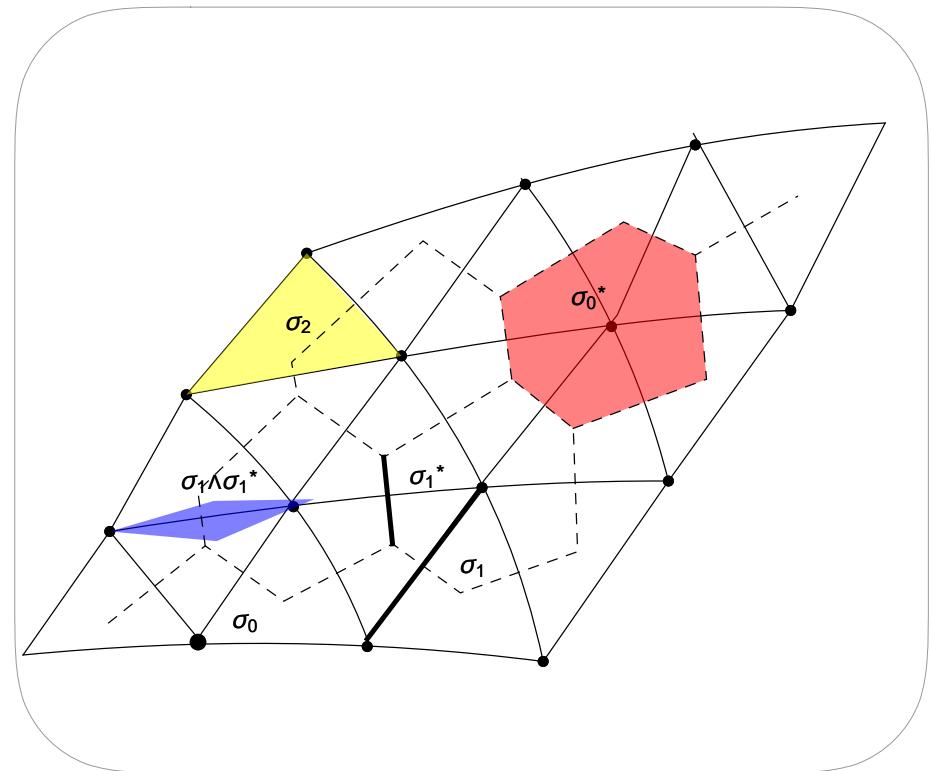


Classical Simplicial Action

$$S_\sigma[\phi] = \frac{1}{2} \sum_{\langle i,j \rangle} V_{ij} \frac{(\phi_i - \phi_j)^2}{l_{ij}^2} + \frac{1}{2} \sqrt{g_i} [m^2 \phi_i^2 + \lambda \phi_i^4]$$

REGGE: Piecewise linear metric

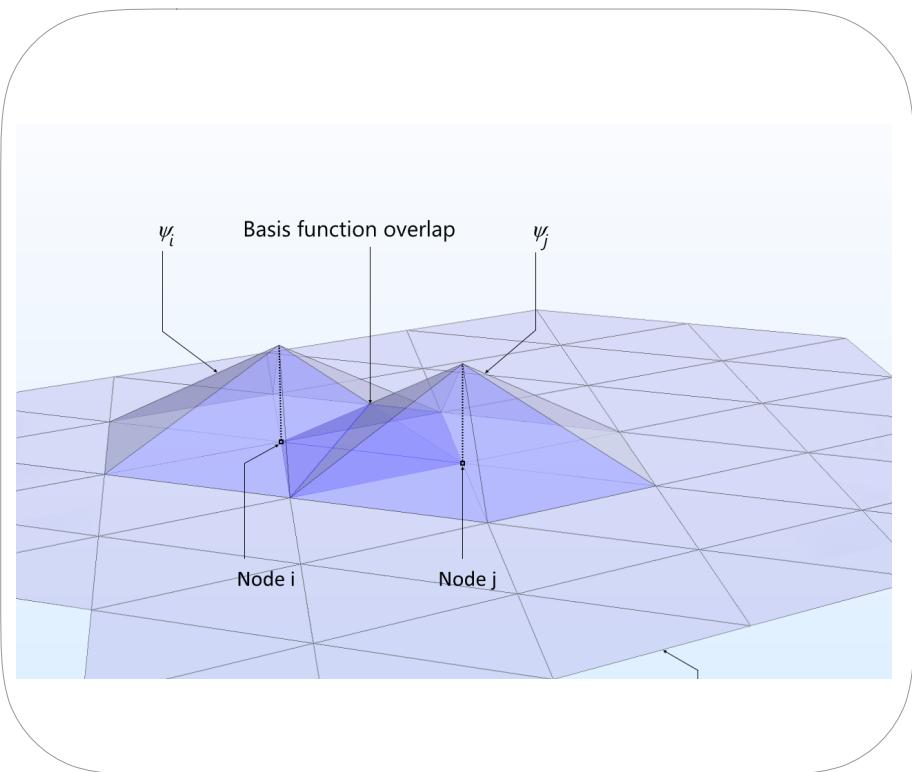
$$(\mathcal{M}, g_{\mu\nu}(x)) \leftrightarrow (\mathcal{M}_\sigma, g_\sigma = \{l_{ij}\})$$



Simplicial Complex/Delaunay Dual Complex +
Regge flat metric on each Simplex

FEM: Piecewise linear fields

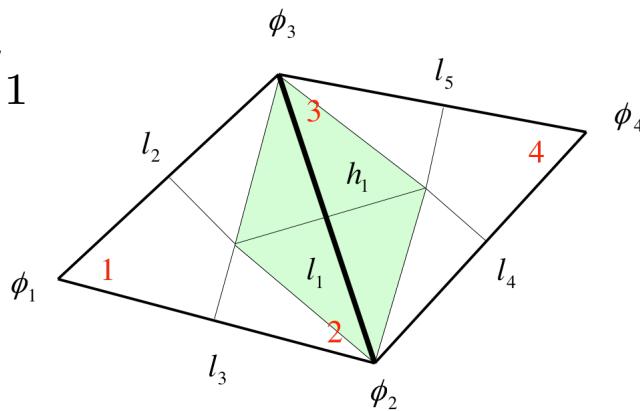
$$\phi(x) \leftrightarrow \phi = \sum_i \phi_i W_i(\xi)$$



Actually fancier methods: Discrete Exterior Calculus
(scalar), Spin connection (Fermion), Wilson links
(gauge) , etc.

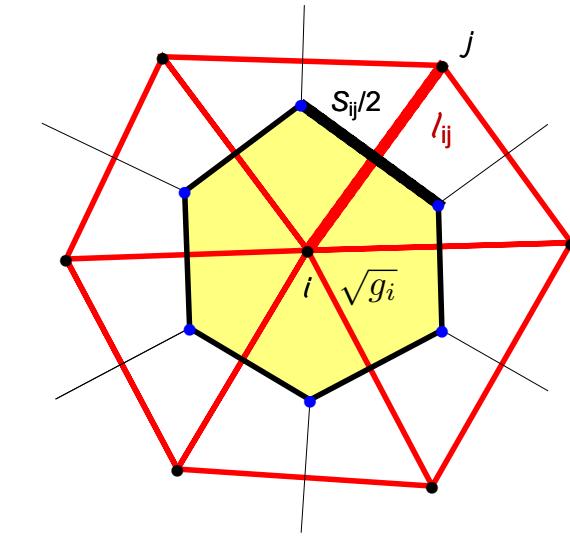
DISCRETE BETRAMI-LAPLACE OPERATOR

$$A_{23} = h_1 l_1$$



FEM: $A_{23} \frac{(\phi_2 - \phi_3)^2}{l_{12}^2}$

\implies



$$\langle \sigma_n | d\omega \rangle = \langle \partial \sigma_n | \omega \rangle$$

(General dimension Discrete Exterior Calculus (DEC))

$$*d * d\phi_i = * \frac{1}{|\sigma_0^*(i)|} \int_{\sigma_0^*} d[*(\phi_i - \phi_j)/l_{ij}] = \frac{1}{\sqrt{g_i}} \sum_{j \in \langle i, j \rangle} \frac{V_{ij}}{l_{ij}} \frac{\phi_i - \phi_j}{l_{ij}}$$

DEC implement discrete exterior derivative, Hodge $*$ to Dual lattice and Stokes Theorem etc
 (see also classic papers by Christ, Friedberg and Lee. NP 1982)

SUMMARY OF SIMPLICIAL FIELDS

J = 0

$$S_{scalar} = \frac{1}{2} \sum_{\langle i,j \rangle} \frac{V_{ij}}{l_{ij}^2} (\phi_i - \phi_j)^2 ,$$

$$l_{ij}^2 = |\sigma_1(ij)|^2$$

J = 1/2

$$S_{Wilson} = \frac{1}{2} \sum_{\langle i,j \rangle} \frac{V_{ij}}{l_{ij}} (\bar{\psi}_i \hat{e}_a^{j(i)} \gamma^a \Omega_{ij} \psi_j - \bar{\psi}_j \Omega_{ji} \hat{e}_a^{i(j)} \gamma^a \psi_i)$$

J = 1

$$S_{gauge} = \frac{1}{2g^2 N_c} \sum_{\triangle_{ijk}} \frac{V_{ijk}}{A_{ijk}^2} Tr[2 - U_{\triangle_{ijk}} - U_{\triangle_{ijk}}^\dagger]$$

FFdual

$$\epsilon^{ijkl} Tr[U_{\triangle_{0ij}} U_{\triangle_{0kl}}] \simeq V_{ijkl} \epsilon^{\mu\nu\rho\sigma} Tr[F_{\mu\nu}(0) F_{\rho\sigma}(0)]$$

$$U_{\triangle_{ijk}} = U_{ij} U_{jk} U_{ki} \quad A_{ijk} = |\sigma_2(ijk)| \quad V_{ijk} = |\sigma_2(ijk) \wedge \sigma_2^*(ijk)|$$

$$U_{0ij} = U_{0i} U_{ij} U_{j0} \quad , \quad U_{0ij}^\dagger = U_{0j} U_{ji} U_{i0} \quad V_{ij} = |\sigma_1(ij) \wedge \sigma_1^*(ij)|$$

Dirac ON SIMPLICIAL MANIFOLD

$$S = \frac{1}{2} \int d^D x \sqrt{g} \bar{\psi} [\mathbf{e}^\mu (\partial_\mu - \frac{i}{4} \boldsymbol{\omega}_\mu(x)) + m] \psi(x)$$

$\mathbf{e}^\mu(x) \equiv e_a^\mu(x) \gamma^a$ Verbein & Spin connection*

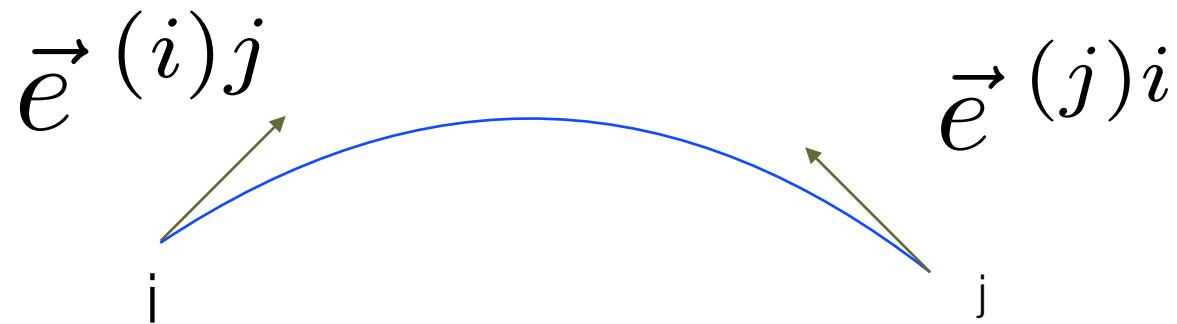
$\boldsymbol{\omega}_\mu(x) \equiv \omega_\mu^{ab}(x) \sigma_{ab}$, $\sigma_{ab} = i[\gamma_a, \gamma_b]/2$

- (1) New spin structure “knows” about intrinsic geometry
- (2) Need to avoid simplex curvature singularities at sites.
- (3) Spinors rotations (Lorentz group) is double of O(D).

* Must satisfy the Tetrad Hypothesis

$$\omega_\mu^{ab} = \frac{1}{2} e^\nu{}^{[a} (e_{\nu,\mu}{}^{b]} - e_{\mu,\nu}{}^{b]} + e^{b]\sigma} e_\mu^c e_{\nu c,\sigma}).$$

$$S_{naive} = \frac{1}{2} \sum_{\langle i,j \rangle} \frac{V_{ij}}{l_{ij}} [\bar{\psi}_i \vec{e}^{(i)j} \cdot \vec{\gamma} \Omega_{ij} \psi_j - \bar{\psi}_j \Omega_{ji} \vec{e}^{(i)j} \cdot \vec{\gamma} \psi_i] + \frac{1}{2} m V_i \bar{\psi}_i \psi_i$$



Simplicial Tetrad Hypothesis

$$e_a^{(i)j} \gamma^a \Omega_{ij} + \Omega_{ij} e_a^{(j)i} \gamma^a = 0$$

Gauge Invariance under Spin(D) transformations

$$\psi_i \rightarrow \Lambda_i \psi \quad , \quad \bar{\psi}_j \rightarrow \bar{\psi}_j \Lambda_j^\dagger \quad , \quad \mathbf{e}^{(i)j} \rightarrow \Lambda_i \mathbf{e}^{(i)j} \Lambda_i^\dagger \quad , \quad \Omega_{ij} \rightarrow \Lambda_i \Omega_{ij} \Lambda_j^\dagger$$

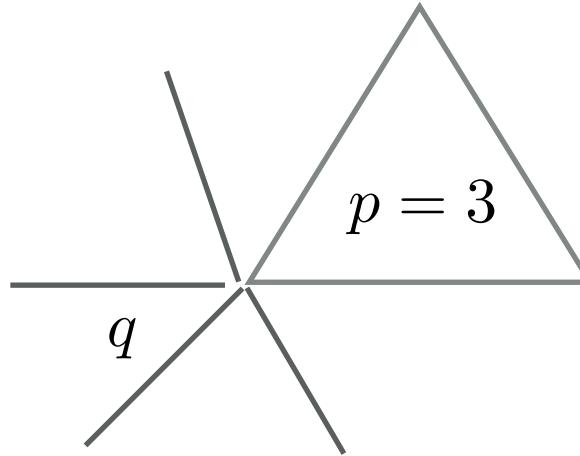
DO NOT USE PIECEWISE LINEAR REGGE CALCULUS MANIFOLD!

MAXIMALLY SYMMETRIC TRANGULATIONS.

- *Symmetry*: Preserve maximal subgroup of isometries — very aid testing and build correlators.
- *Classical Convergence*: Shape Regular refinement to maximize “spectral fidelity” accelerated convergence and simplify quantum counter terms.
- *Efficient Data Parallel Code*: To refine with graphs that with regular geometries to enable fast data parallel code

EQUILATERAL TRIANGULATION

Triangle case



Preserves Discrete
Subgroup of Isometries

$$\frac{1}{p} + \frac{1}{q} > 1/2$$

de Sitter \mathbb{S}^2

vertex $q = 3, 4, 5$

$$\frac{1}{p} + \frac{1}{q} = 1/2$$

flat \mathbb{T}^2

vertex $q = 6$

$$\frac{1}{p} + \frac{1}{q} < 1/2$$

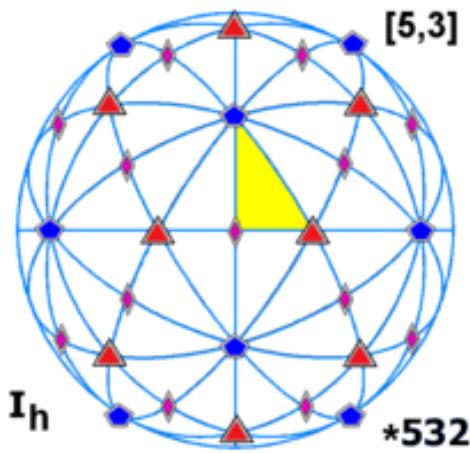
Hyperbolic AdS^2

vertex $q = 7, 8, 9, \dots$

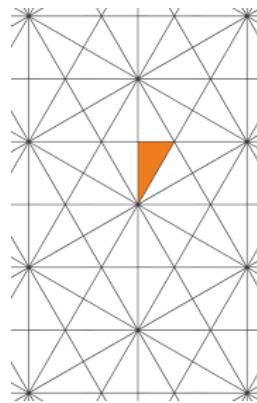
DISCRETE ISOMETRIES & THE TRIANGLE GROUP

$$\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} \quad \left\{ \begin{array}{ll} > \pi & \text{Postive curvature} \\ = \pi & \text{Zero curvature} \\ < \pi & \text{Negative Curvature} \end{array} \right.$$

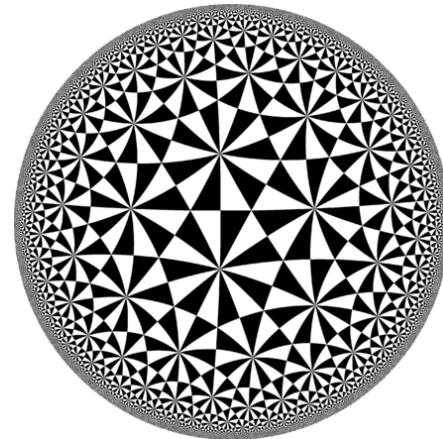
Tesselate by x, y, z refection
give p,q,r rotations: S = xy, T = yz, U = zx



(2, 3, 5)
120 element
Icosahedral in $O(3)$



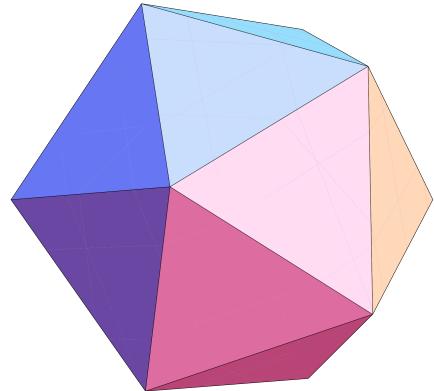
(2, 3, 6)
Triangle Lattice
on Euclidean \mathbb{R}^2



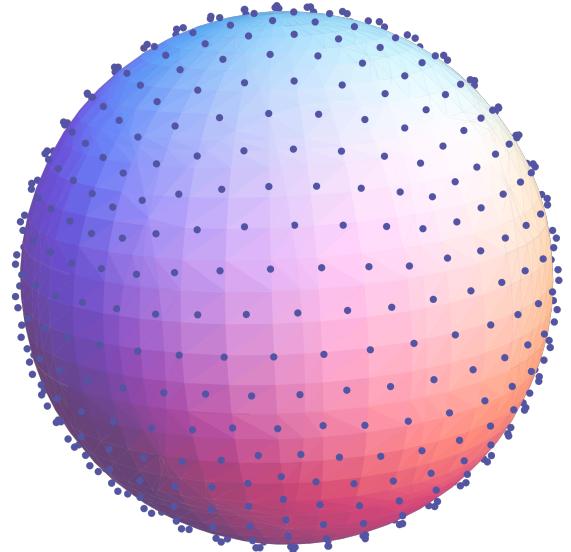
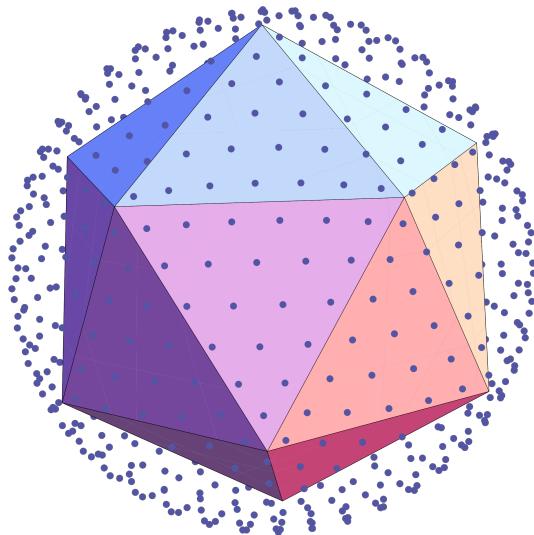
(2, 3, 7)
Subgroup of Modular
Group on \mathbb{H}^2

Start with maximum regular Tesselation

$s = 1$



$s = 8$



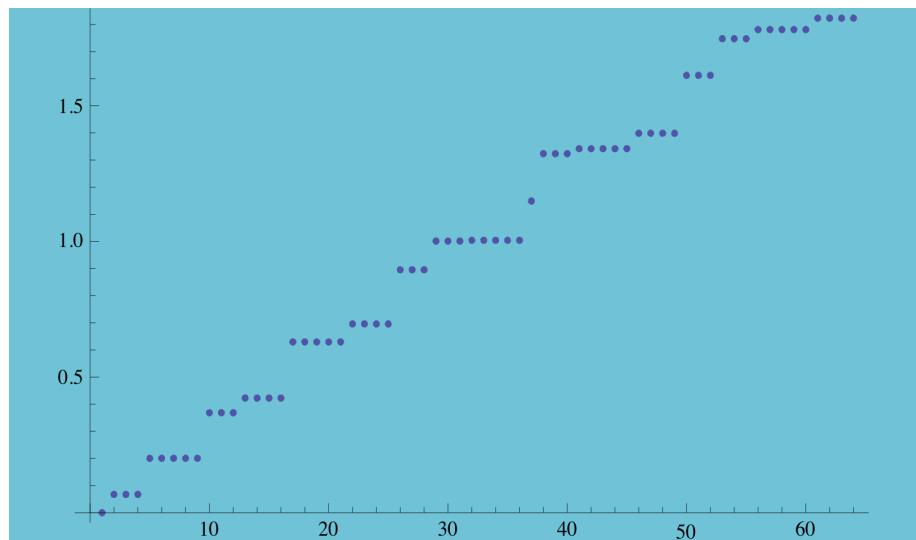
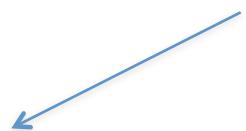
$I = 0$ (A), 1 (T1) , 2 (H) are irreducible 120 Icosahedral subgroup of $O(3)$

$1/p + 1/q > 1/2$ for regular positive curvature tessellation

FEM FIXES SPECTRAL DEFECTS OF LAPLACIAN ON SPHERE

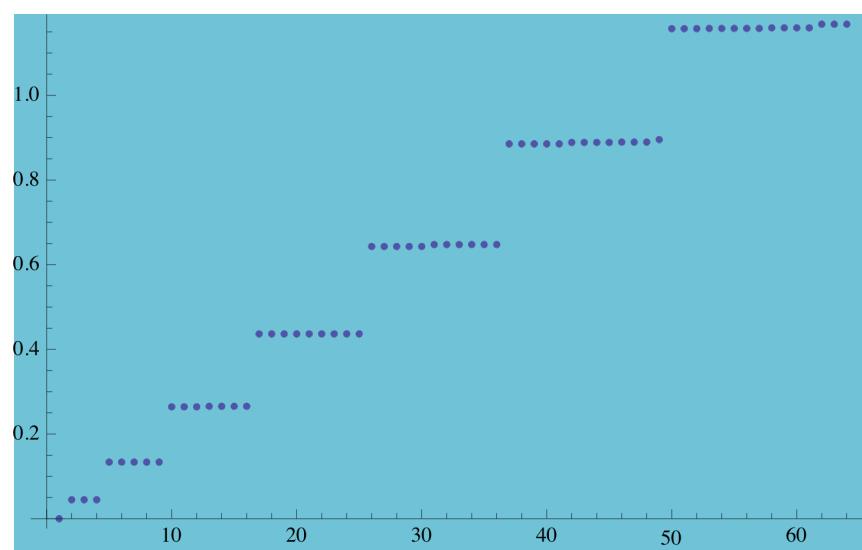
For $s = 8$ first $(l+1)^*(l + 1) = 64$ eigenvalues

BEFORE FEM



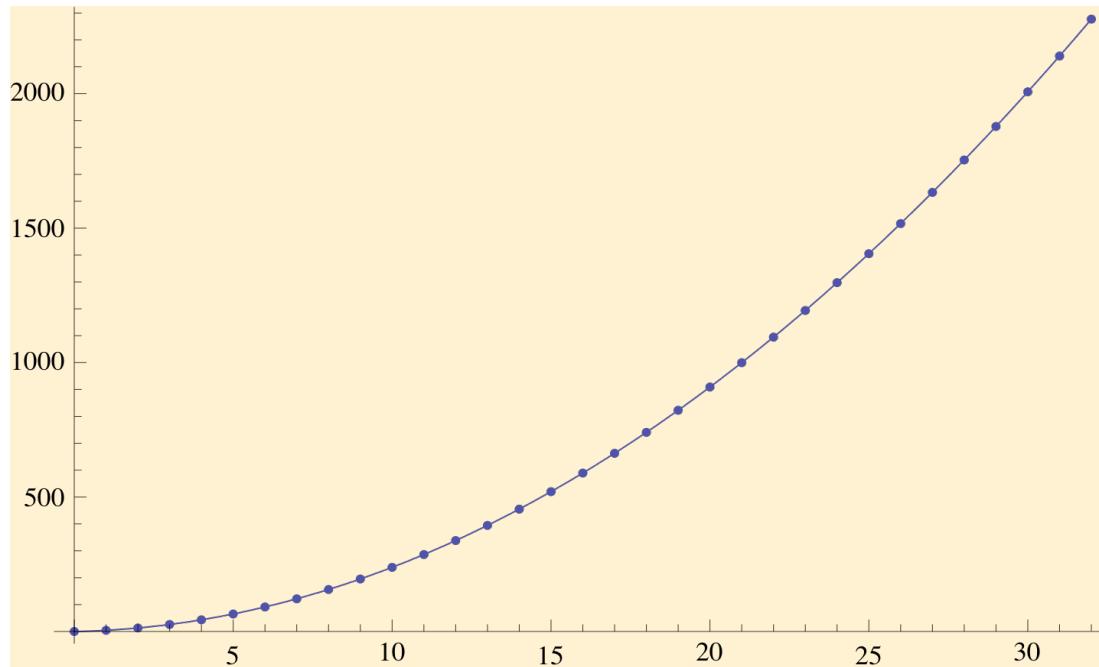
$|l, m$

AFTER FEM



$|l, m$

SPECTRAL FIDELITY ON \mathbb{S}^2



$S = 128$

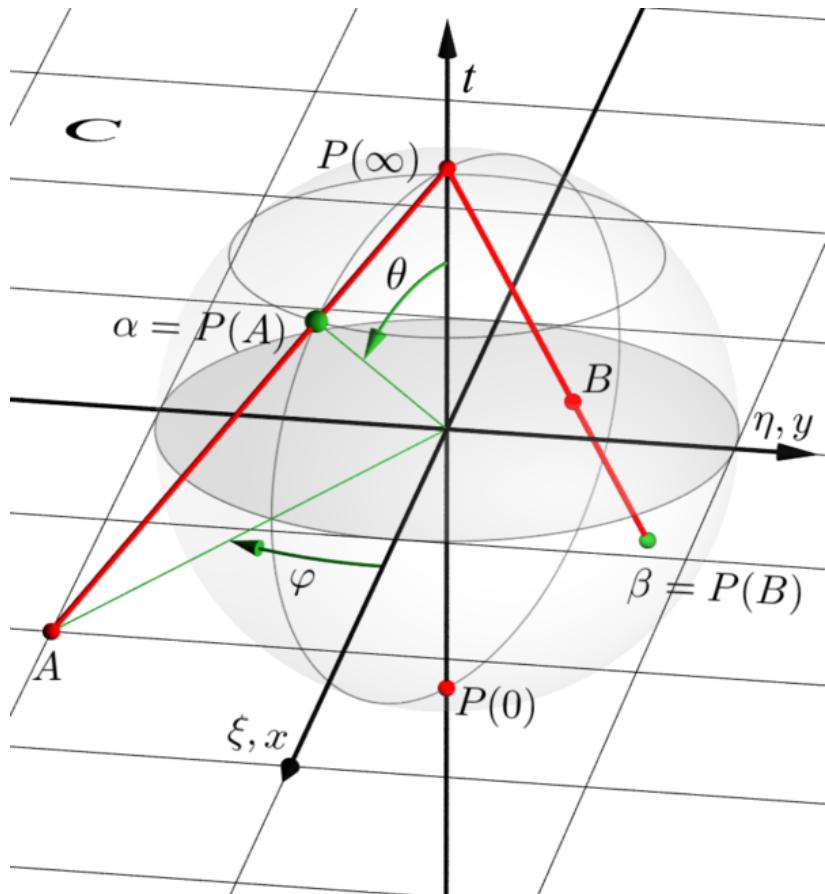
Fit



$$\begin{aligned} & l + 1.00012 l^2 \\ & - 13.428110^{-6} l^3 - 5.5724410^{-6} l^4 \end{aligned}$$

TEST 2D ISING/PHI 4TH ON THE RIEMANN SPHERE

Stereographic project of Complex Plane:



Conformally Invariant
Cross Ratios are “Preserved”

$$\xi = \tan(\theta/2)e^{-i\phi} = \frac{x + iy}{1 + z}$$

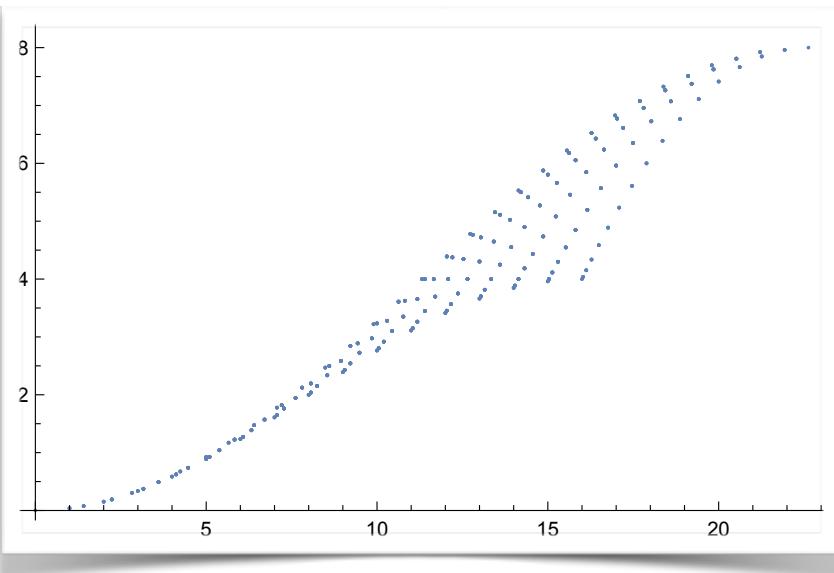
$$\xi = \xi_1 + i\xi_2$$

$$\vec{r} = (x, y, z) \quad \vec{r} \cdot \vec{r} = 1$$

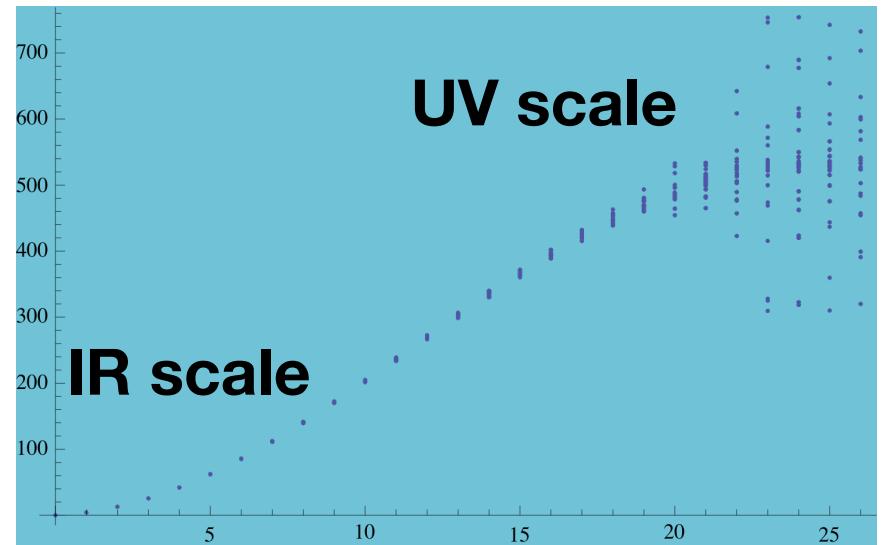
$$|\vec{r}_1 - \vec{r}_2| = 2 - 2 \cos(\theta_{12})$$

$$u = \frac{|\xi_1 - \xi_2||\xi_3 - \xi_4|}{|\xi_1 - \xi_3||\xi_1 - \xi_4|} = \frac{|\vec{r}_1 - \vec{r}_2||\vec{r}_1 - \vec{r}_2|}{|\vec{r}_1 - \vec{r}_2||\vec{r}_1 - \vec{r}_2|}$$

Restoring Isometries for ON A SIMPLICIAL COMPLEX

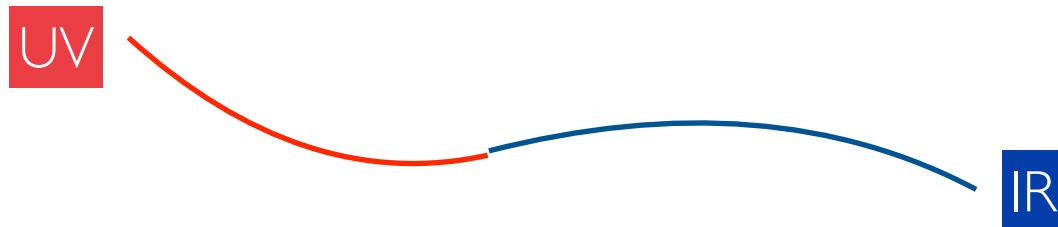


Hypercubic Lattice

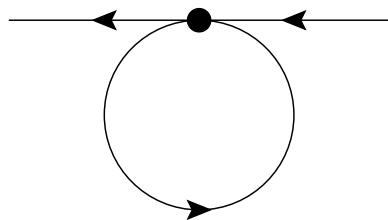


Simplicial Sphere

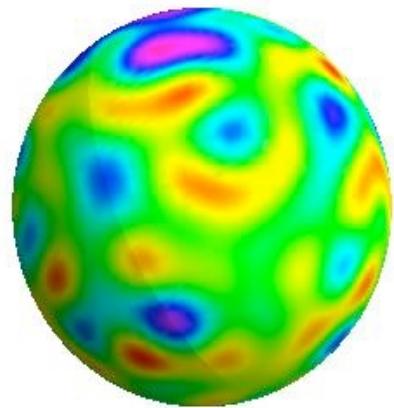
Is Renormalized perturbation theory at the UV fixed point enough to uniquely/correctly define the IR Quantum Field Theory?



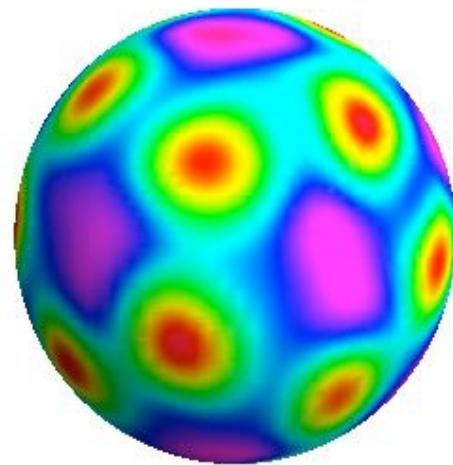
UV DIVERGENCE BREAKS ROTATIONS



$$\delta m^2 = \lambda \langle \phi(x) \phi(x) \rangle \rightarrow \frac{1}{K_{xx}}$$



one configuration



average of config.

One LOOP Counter Term

$$\Delta m_i^2 = 6\lambda [K^{-1}]_{ii} \simeq \frac{\sqrt{3}}{8\pi} \lambda \log(1/m_0^2 a_i^2) = \frac{\sqrt{3}}{8\pi} \lambda \log(N_s) + \frac{\sqrt{3}}{8\pi} \lambda \log(a^2/a_i^2)$$

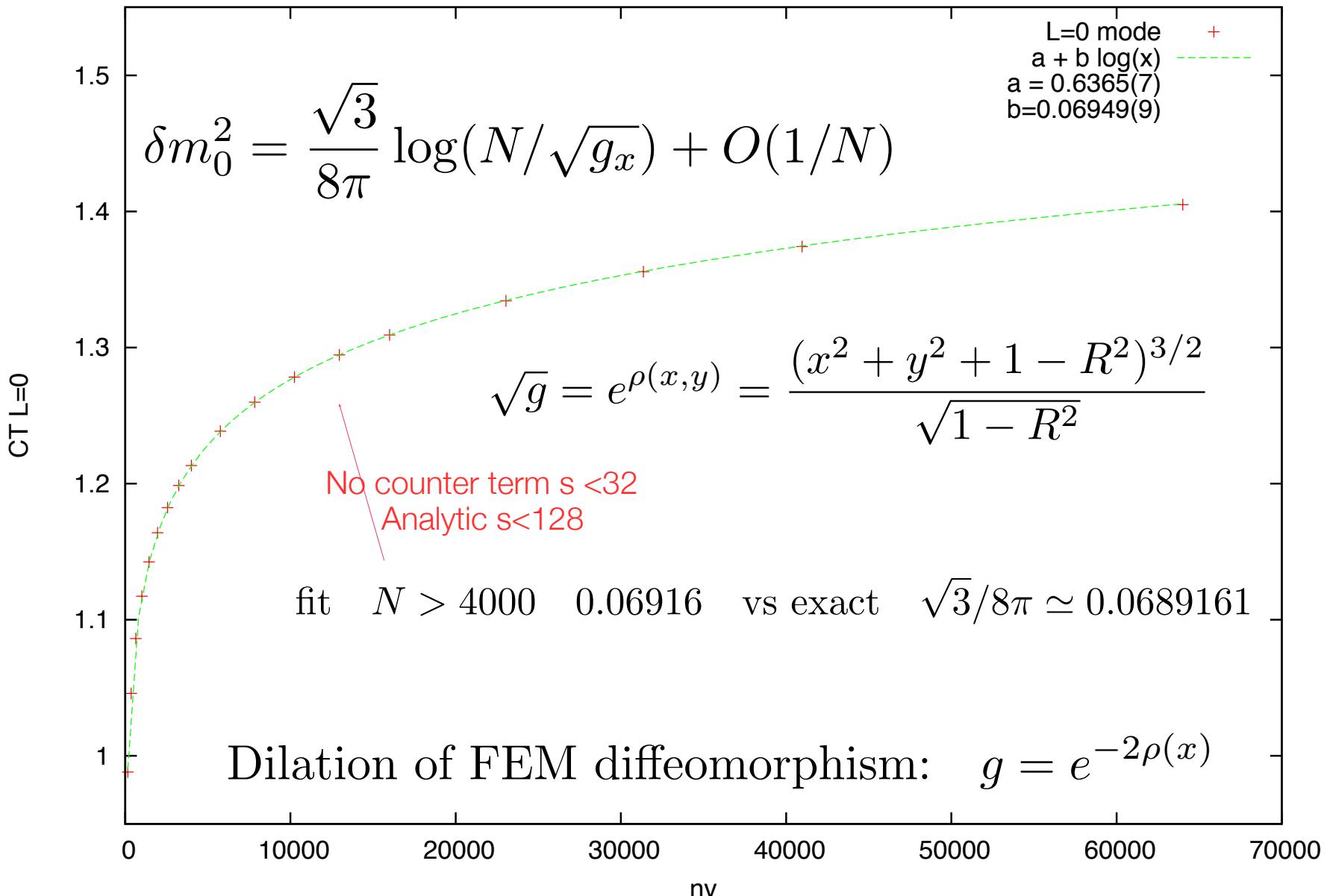
Exact Continuum
Divergence



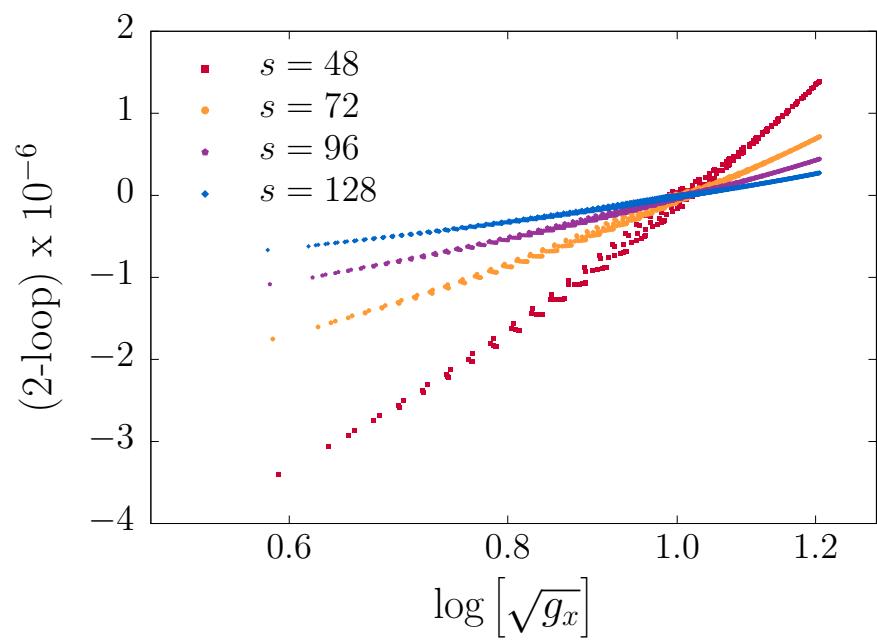
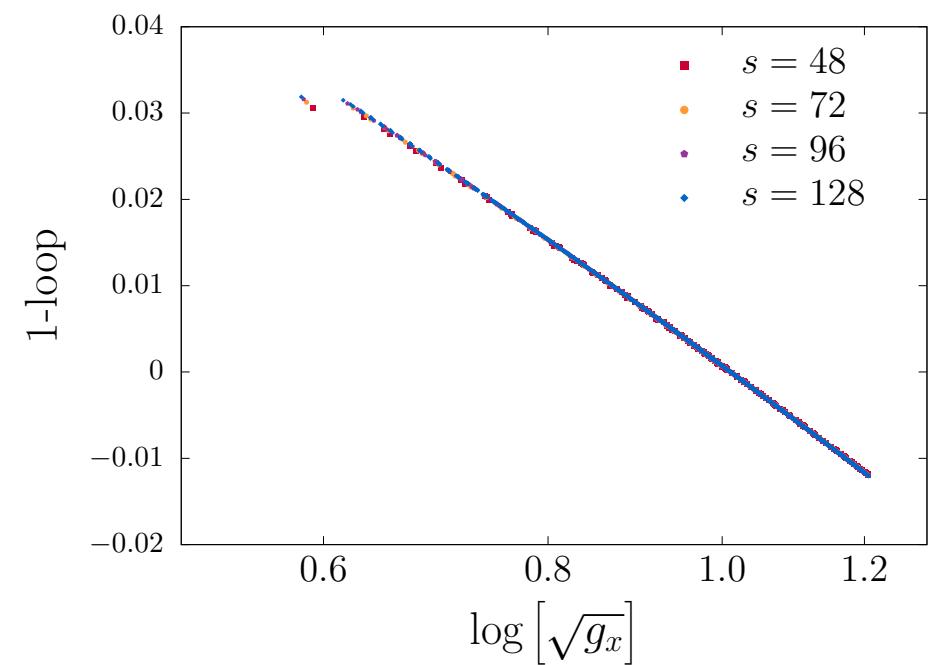
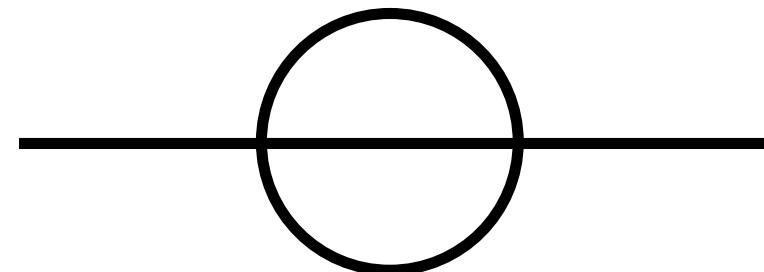
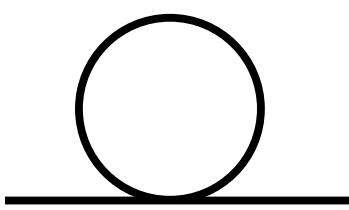
Local Cut-off
Scheme Dependence

$$\delta\mu_i^2 = -6\lambda([K^{-1}]_{ii} - \frac{1}{N_s} \sum_{j=1}^{N_s} [K^{-1}]_{jj})$$

MODEL OF COUNTER TERM



One Loop Counter Term vs Two Loop Convergence



RG Proof Of UNIVERSAL UV Logs

$$G_{xx}(m) \simeq c_x \log(1/m^2 a_x^2) + O(a^2 m^2)$$

$$\implies \gamma(m_0^2) = -m_0 \frac{\partial}{\partial m_0} G_{xx}(m_0^2) \simeq 2c_x + O(m_0^2)$$

$$m_0 = am \rightarrow 0$$

FEM Spectral Fidelity

$$\begin{aligned} G_{xy}(m^2) &= \sum_n \frac{\phi_n^*(x)\phi_n(x)}{E_n^{(0)} + m^2} && \text{IR: Region.} \\ &\simeq \frac{\sqrt{3}}{8\pi} \sum_{l=0}^{L_0} \frac{(2l+1)P_l(r_x \cdot r_y)}{l(l+1) + \mu_0^2} + \sum_{n=(L_0+1)^2}^N \frac{\phi_n^*(x)\phi_n(y)}{E_n^{(0)} + m^2} && \text{UV} \end{aligned}$$

In insensitive to UV defects

$$\begin{aligned} \gamma(m_0^2) &= -m_0 \frac{\partial}{\partial m_0} G_{xx}(m_0^2) \\ &\simeq \frac{\sqrt{3}}{8\pi} \int_0^{\Lambda_0^2} dE^{(0)} \frac{m_0^2}{(E^{(0)} + m_0^2)^2} = \frac{\sqrt{3}}{8\pi} \frac{1}{1 + m_0^2/\Lambda_0^2} \end{aligned}$$

Now Binder Cumulant Converges

$$U_{2n}(\mu^2, \lambda, s) = U_{2n,cr} + a_{2n}(\lambda)[\mu^2 - \mu_{cr}^2]s^{1/\nu} + b_{2n}(\lambda)s^{-\omega}$$

FIT $U_{4,cr} = 0.85020(58)(90)$

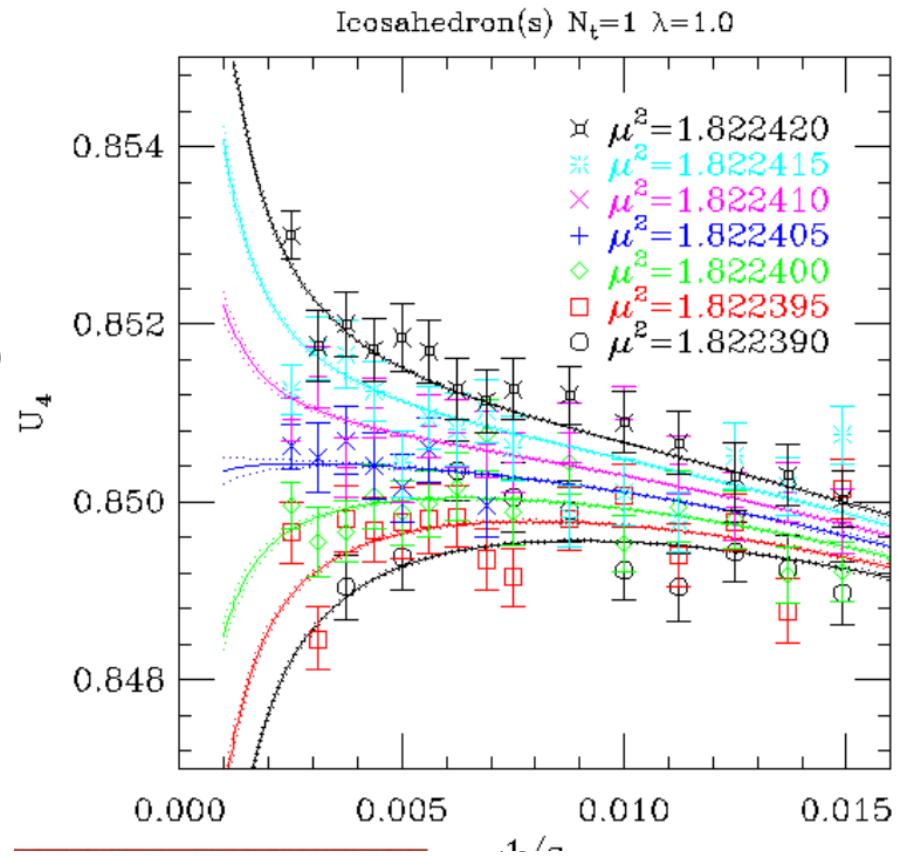
THEORY $U_4^* = 0.8510207(63)$

FIT $U_{6,cr} = 0.77193(37)(90)$

THEORY $U_6^* = 0.773144(21)$

$$\mu_{cr}^2 = 1.82240070(34)$$

$$dof = 1701 , \chi^2/dof = 1.026$$



Simultaneous fit for s up 800:
E.G. 6,400,002 Sites on Sphere

EXACT CORRELATOR FOR C = 1/2 CFT ON 2D SPHERE

2 pt function

$$\langle \phi(x_1)\phi(x_2) \rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \rightarrow \frac{1}{|1 - \cos\theta_{12}|^\Delta} \quad \Delta = \eta/2 = 1/8$$

4 pt function

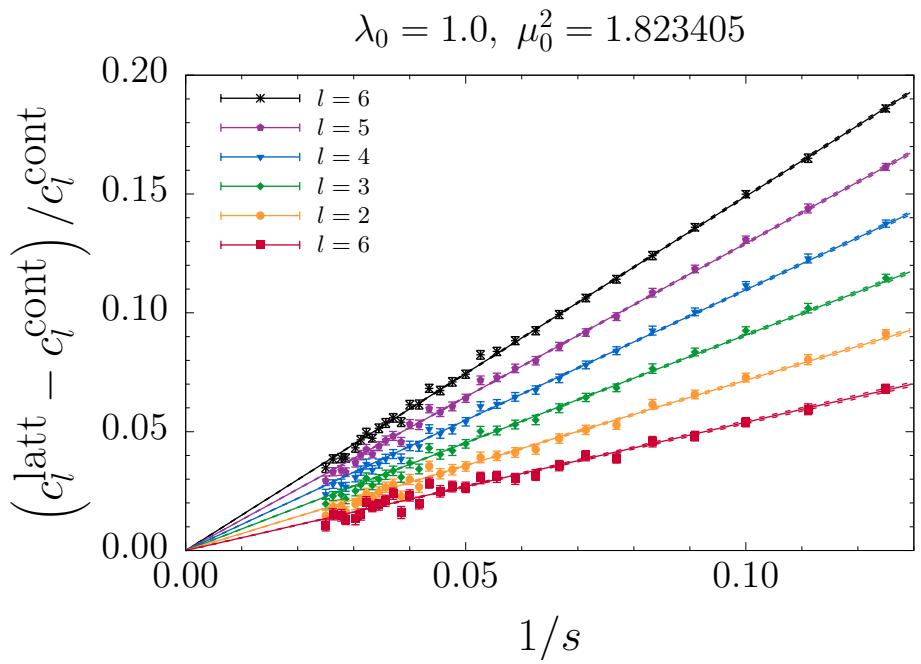
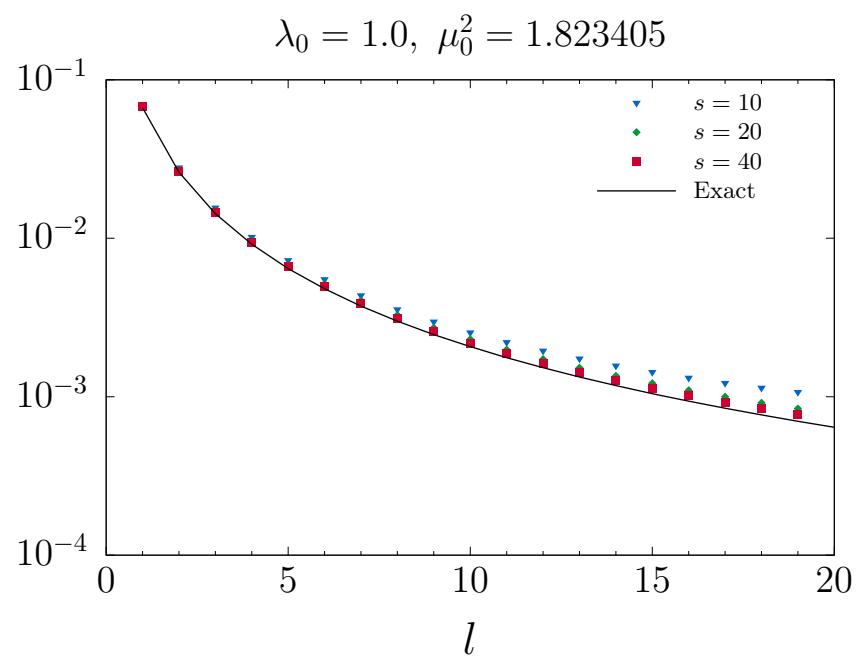
$$g(u, v) = \frac{\langle \phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \rangle}{\langle \phi(x_1)\phi(x_3) \rangle \langle \phi(x_2)\phi(x_4) \rangle} \\ = \frac{1}{2|z|^{1/4}|1-z|^{1/4}} [|1 + \sqrt{1-z}| + |1 - \sqrt{1-z}|]$$

$$u = \frac{r_{12}^2 r_{34}^2}{r_{13}^2 x_{24}^2} \quad , \quad v = \frac{r_{14}^2 r_{32}^2}{r_{13}^2 r_{24}^2} \quad u = |z|^2 \quad v = |1-z|^2$$

Critical Binder Cumulant

$$M = \sum_x \phi(x)$$

$$U_4^* = \frac{3}{2} \left[1 - \frac{\langle M^4 \rangle}{3 \langle M^2 \rangle \langle M^2 \rangle} \right] = 0.85102$$



$$\int_{-1}^1 dz \left(\frac{2}{1-z}\right)^{1/8} P_l(z)$$

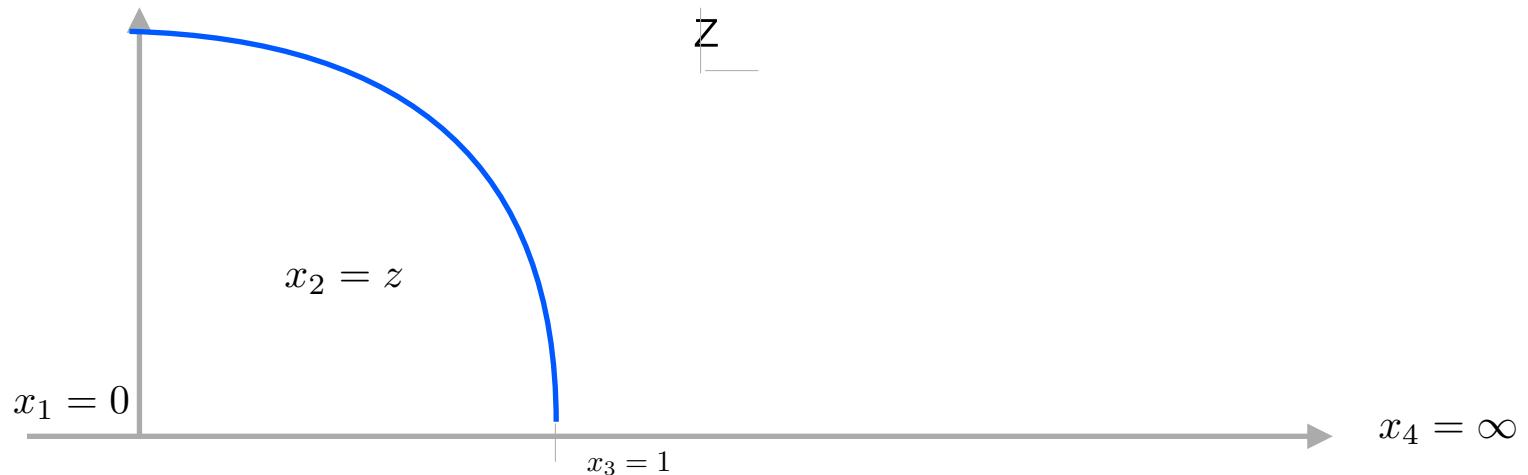
$$\implies \frac{16}{7}, \frac{16}{35}, \frac{48}{161}, \frac{816}{3565}, \frac{12240}{64883}, \frac{493680}{3049501}, \frac{33456}{234577}, \frac{55760}{435643}, \frac{3602096}{30930653}, \frac{20129360}{187963199}, \frac{541373840}{5450932771} \dots$$

$$\Delta_\sigma = \eta/2 = 1/8 \simeq 0.128$$

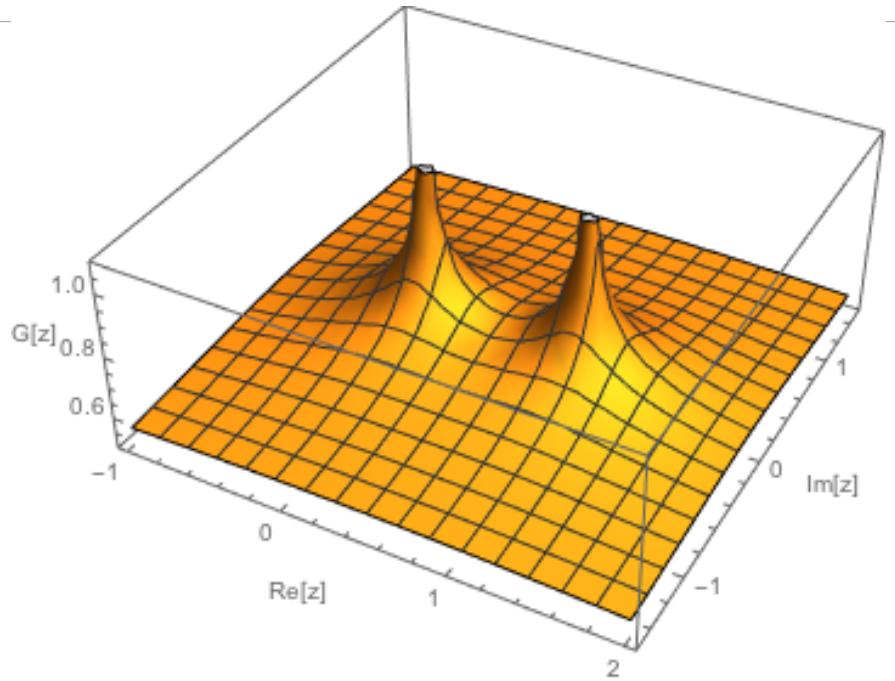
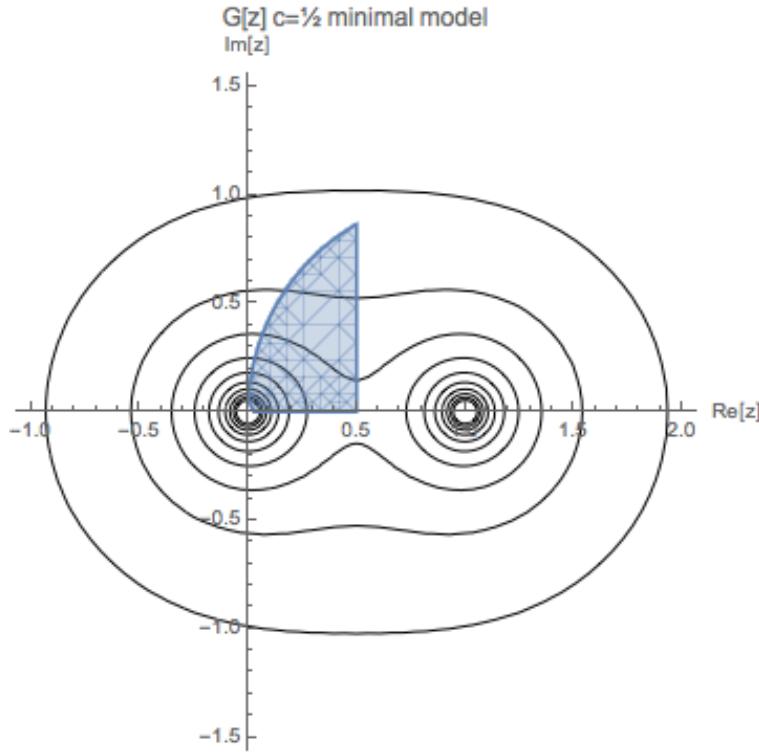
EXACT FOUR POINT FUNCTION

$$\begin{aligned} g(u, v) &= \frac{\langle \phi_1 \phi_2 \phi_3 \phi_4 \rangle}{\langle \phi_1 \phi_3 \rangle \langle \phi_2 \phi_4 \rangle} \\ &= \frac{1}{\sqrt{2}|z|^{1/4}|1-z|^{1/4}} [|1 + \sqrt{1-z}| + |1 - \sqrt{1-z}|] \end{aligned}$$

Crossing Sym: $|1 + \sqrt{1-z}| + |1 - \sqrt{1-z}| = \sqrt{2 + 2\sqrt{(1-z)(1-\bar{z})} + 2\sqrt{z\bar{z}}}$



OPE Expansion: $\phi \times \phi = \mathbf{1} + \phi^2$ or $\sigma \times \sigma = \mathbf{1} -$



$$G_s(r, \theta) \propto 1 + \lambda_\epsilon^2 g_{\epsilon,0}(r, \theta) + \lambda_T^2 g_{T,2}(r, \theta)$$

$$\lambda_T^2 = \frac{\Delta_\sigma^2 d^2 |z|^{d-2}}{C_T(d-1)^2} \rightarrow \frac{1}{16C_T} \quad \text{for } d = 2 , \quad g_{T,2}(z) = -3 \left(1 + \frac{1}{z} \left(1 - \frac{z}{2} \right) \log(1-z) \right) + \text{c.c.}$$

Fit TO OPE EXPANSION

μ^2	s	$r_{\min} \leq r \leq r_{\max}$	norm	Δ_ϵ	λ_ϵ^2	c
1.82241	9	$0.25 \leq r \leq 0.75$	0.2900	1.075	0.2536	0.4668
1.82241	9	$0.30 \leq r \leq 0.70$	0.2901	1.075	0.2533	0.4704
1.82241	9	$0.35 \leq r \leq 0.65$	0.2902	1.077	0.2533	0.4738
1.82241	9	$0.40 \leq r \leq 0.60$	0.2902	1.016	0.2427	0.4747
1.82241	18	$0.25 \leq r \leq 0.75$	0.2051	1.068	0.2563	0.4866
1.82241	18	$0.30 \leq r \leq 0.70$	0.2051	1.056	0.2544	0.4878
1.82241	18	$0.35 \leq r \leq 0.65$	0.2051	1.050	0.2535	0.4904
1.82241	18	$0.40 \leq r \leq 0.60$	0.2051	1.046	0.2526	0.4884
1.82241	36	$0.25 \leq r \leq 0.75$	0.1457	1.031	0.2528	0.4926
1.82241	36	$0.30 \leq r \leq 0.70$	0.1458	1.026	0.2519	0.4932
1.82241	36	$0.35 \leq r \leq 0.65$	0.1458	1.018	0.2508	0.4931
1.82241	36	$0.40 \leq r \leq 0.60$	0.1458	1.007	0.2486	0.4933

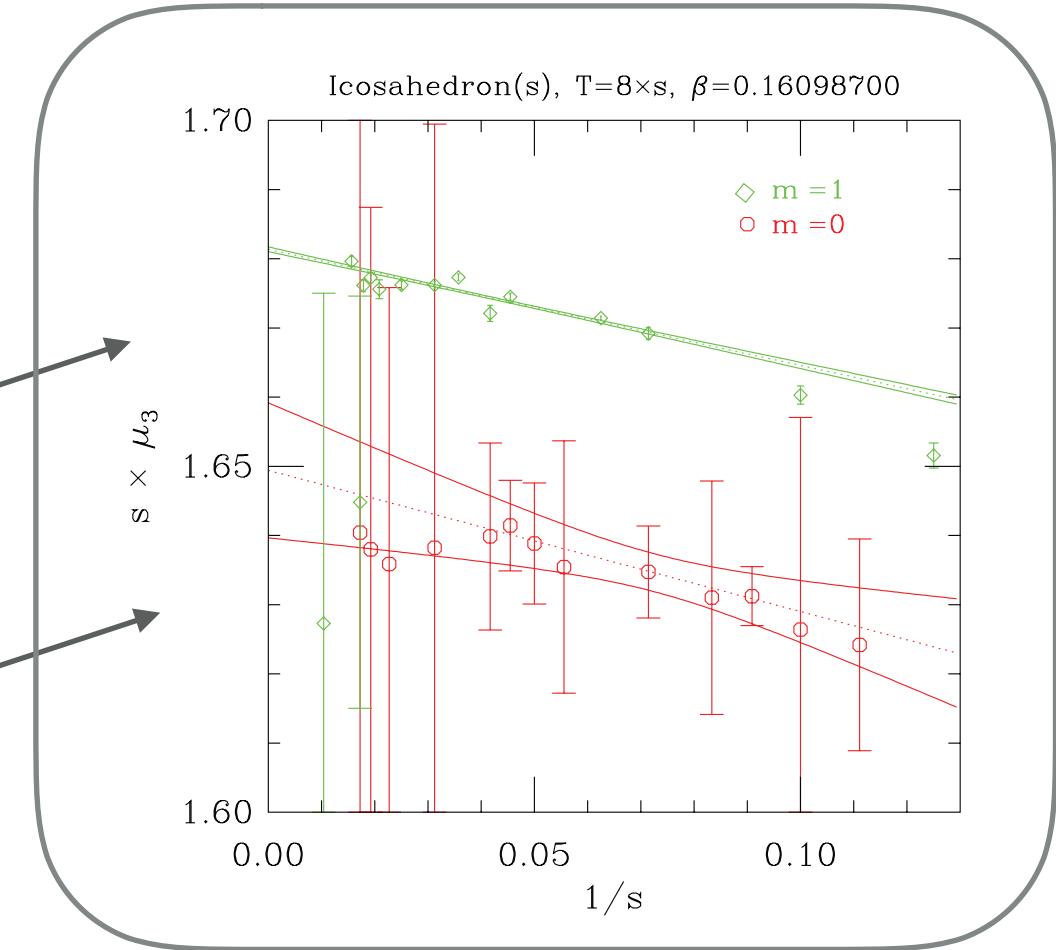
BACK TO THE FUTURE*

Quantum Finite Elements for 3D Phi 4-th in Radial Quantization

*Failure to recover $O(4, 1)$ of $l = 3$
with no FEM (Ising spins on
Triangulated Icosahedron)*

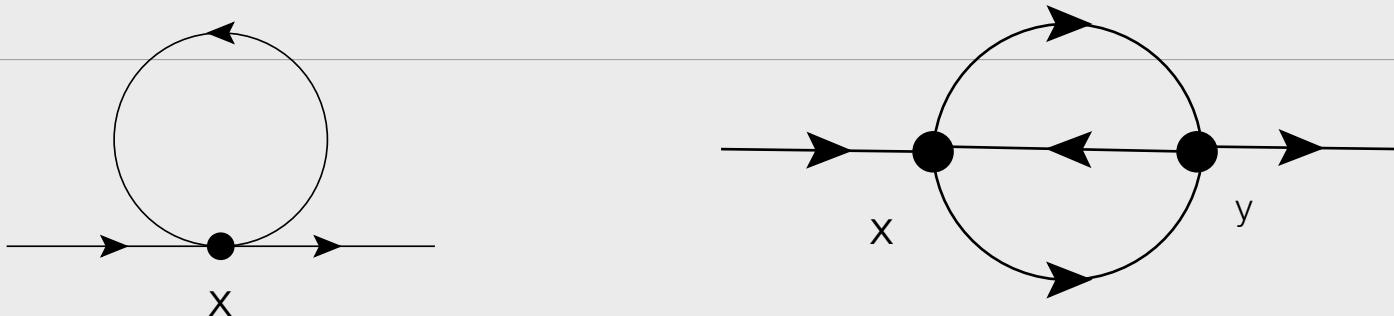
G rep

T2 rep

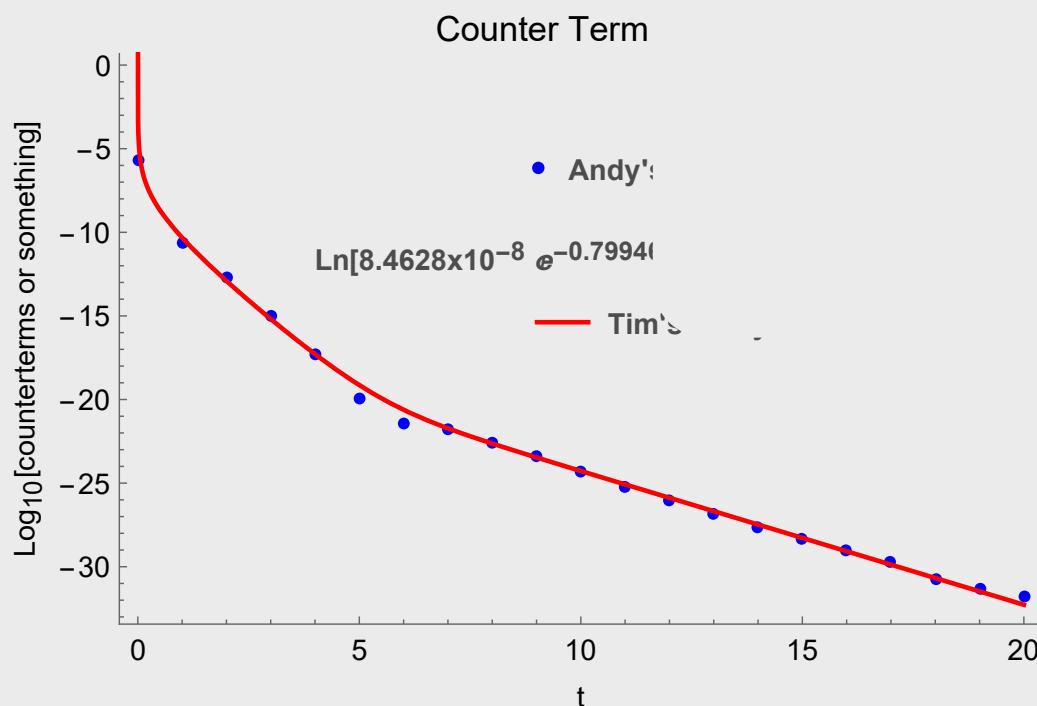


* R.C.Brower, G.T.Fleming and H. Neuberger, *Lattice Radial Quantization: 3D Ising*, Phys.Lett.B 721, 299 (2013)

Counter term in 3D



$$\delta\mu_{CT}^2(x, y) \sim \lambda_0 c_x \delta_{xy} + \lambda_0^2 e^{-6|x-y|/a}$$

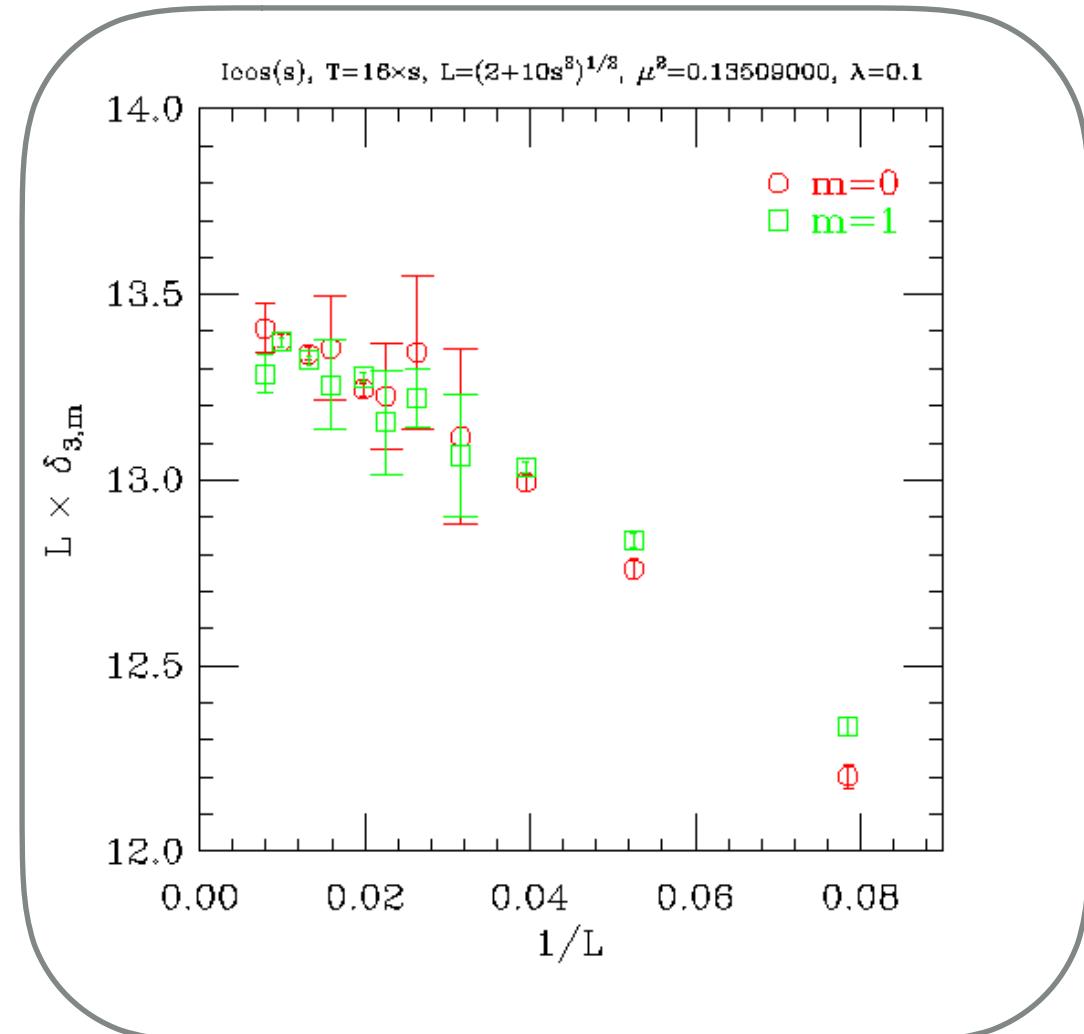


PRELINARY DATA ON QFE $L = 3$ Spectrum

Hope to show recovery $O(4,1)$ for $l = 3$?

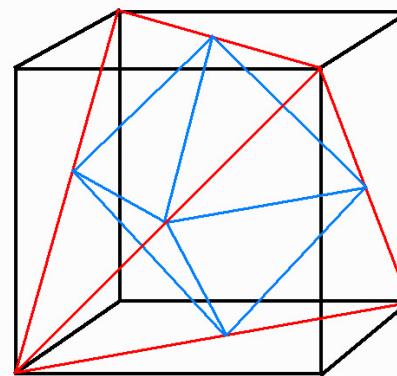
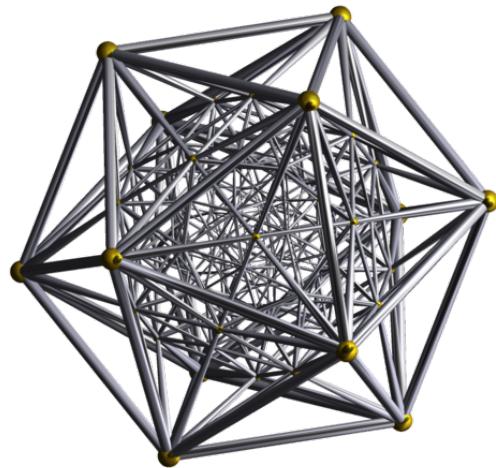
G rep
T2 rep

NEED MUCH MORE DATA!



3 Spheres and 4D Radial Simpicial Lattices

$$\mathbb{S}^3 \implies \mathbb{R} \times \mathbb{S}^3$$



Aristotle's 2% Error!

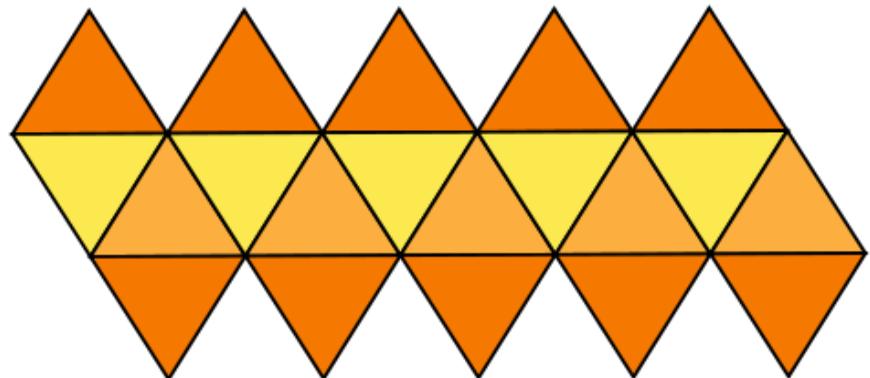
Fast Code Domains of
Regular 3D Grids on Refinement

$$(2\pi - 5 \operatorname{ArcCos}[1/3])/(2\pi) = 0.0204336$$

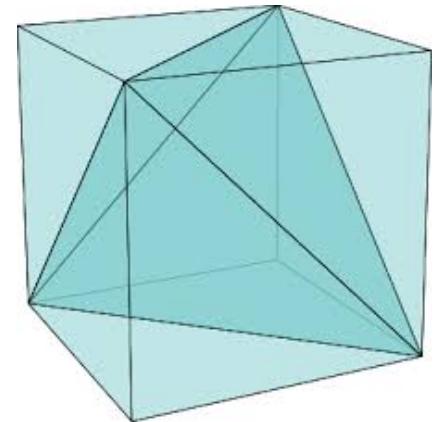
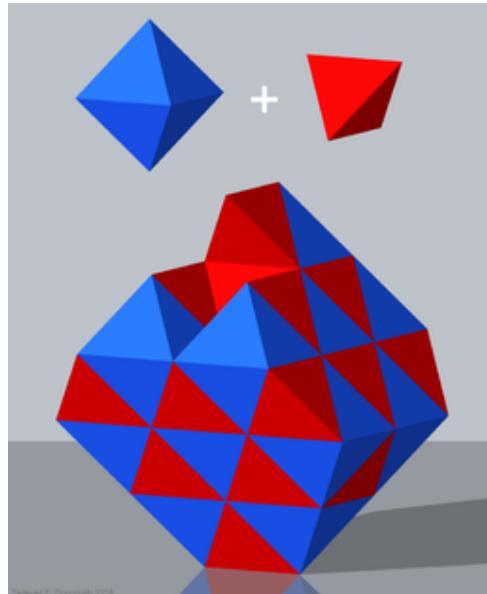
The full [symmetry group](#) of the 600-cell is the [Weyl group](#) of H_4 . This is a [group](#) of order 14400. It consists of 7200 [rotations](#) and 7200 rotation-reflections. The rotations form an [invariant subgroup](#) of the full symmetry group.

GPU DATA PARALLEL CODE: REGULAR DOMAIN REFINEMENTS

\mathbb{S}^2 Refinement \implies
5 Regular Cartesian
Triangle Graphs

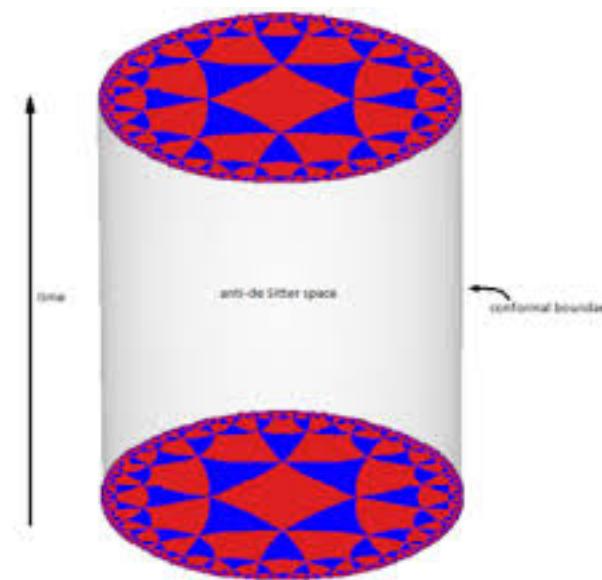
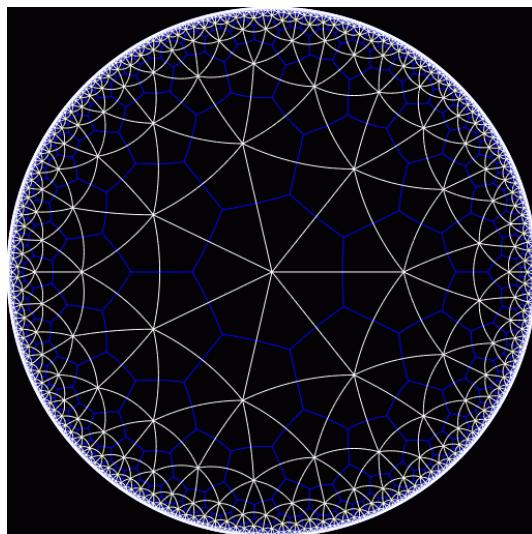


\mathbb{S}^3 Refinement \implies
Tetrahedral-octahedral
honeycomb refinement



Hyperbolic (e.g. Poincare Disk) and Global AdS

$$\mathbb{H}^d \rightarrow \mathbb{A}dS^d$$



$$1/p + 1/q < 1/2$$

Triangle Group Tesselation:
Preserve Finite subgroup of the Modular Group

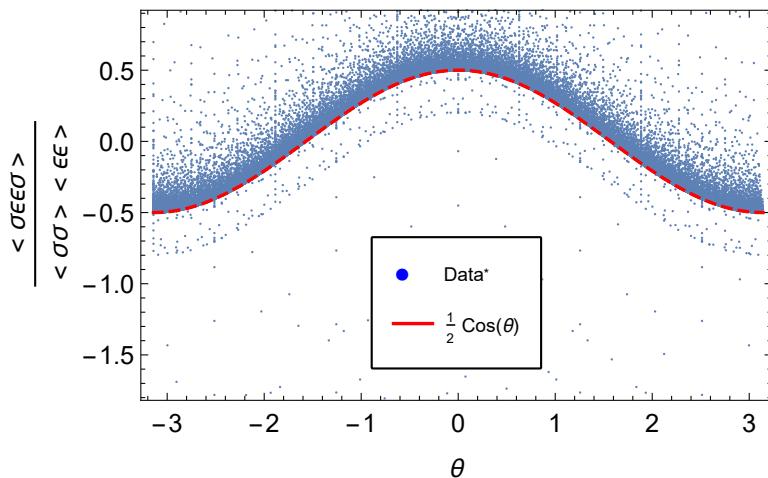
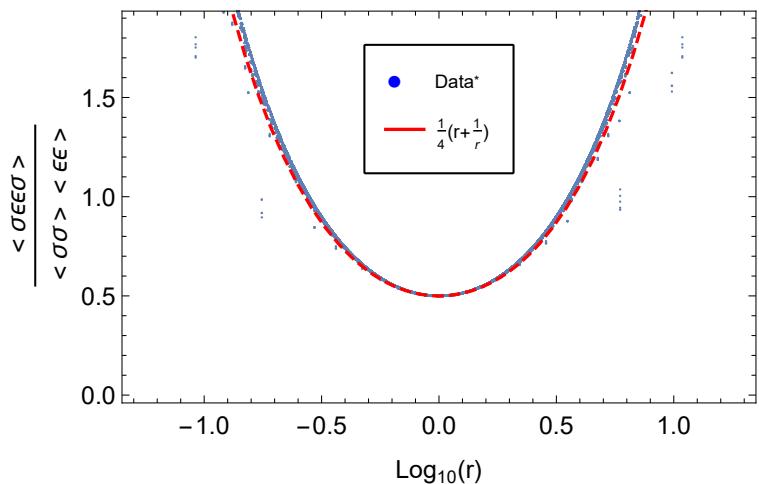
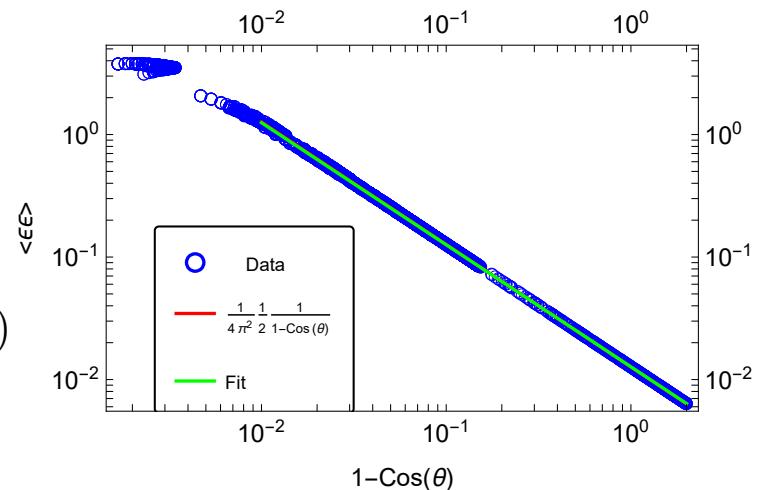
These Hyperbolic Tesselation are “Tensor Networks” :
What can we do them as lattice Field Theories?

EXTRA SLIDES!

FREE MAJORANA FERMIONS ON S₂

$$\langle \psi(z_1) \bar{\psi}(z_1) \bar{\psi}(z_1) \psi(z_2) \rangle = \left[\frac{1}{\partial} \right]_{z_1, z_2} \left[\frac{1}{\bar{\partial}} \right]_{z_1, z_2} = \frac{1}{4\pi^2} \frac{1}{|z_1 - z_2|^2}$$

$$\frac{\langle \sigma(0) \epsilon(z_2) \epsilon(z_3) \sigma(\infty) \rangle}{\langle \epsilon(z_2) \epsilon(z_3) \rangle} = \frac{1}{4} |\sqrt{z_1/z_2} + \sqrt{z_2/z_1}|^2 = \frac{1}{4} (r + 1/r + 2 \cos \theta)$$



Using Binder Cumulants

$$U_4 = \frac{3}{2} \left(1 - \frac{m_4}{3 m_2^2} \right)$$

$$U_6 = \frac{15}{8} \left(1 + \frac{m_6}{30 m_2^3} - \frac{m_4}{2 m_2^2} \right)$$

$$U_8 = \frac{315}{136} \left(1 - \frac{m_8}{630 m_2^4} + \frac{2 m_6}{45 m_2^3} + \frac{m_4^2}{18 m_2^4} - \frac{2 m_4}{3 m_2^2} \right)$$

$$U_{10} = \frac{2835}{992} \left(1 + \frac{m_{10}}{22680 m_2^5} - \frac{m_8}{504 m_2^4} - \frac{m_6 m_4}{108 m_2^5} + \frac{m_6}{18 m_2^3} + \frac{5 m_4^2}{36 m_2^4} - \frac{5 m_4}{6 m_2^2} \right)$$

$$U_{12} = \frac{155925}{44224} \left(1 - \frac{m_{12}}{1247400 m_2^6} + \frac{m_{10}}{18900 m_2^5} + \frac{m_8 m_4}{2520 m_2^6} - \frac{m_8}{420 m_2^4} \right. \\ \left. + \frac{m_6^2}{2700 m_2^6} - \frac{m_6 m_4}{45 m_2^5} + \frac{m_6}{15 m_2^3} - \frac{m_4^3}{108 m_2^6} + \frac{m_4^2}{4 m_2^4} - \frac{m_4}{m_2^2} \right)$$

$$m_n = \langle \phi^n \rangle$$

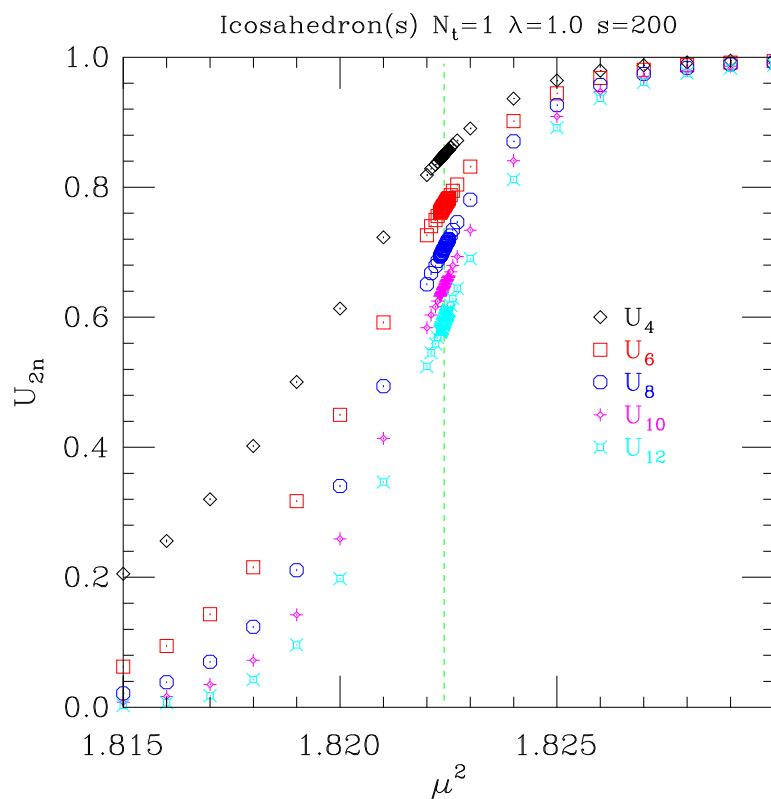
In infinite volume

$U_{2n}=0$ in disordered phase

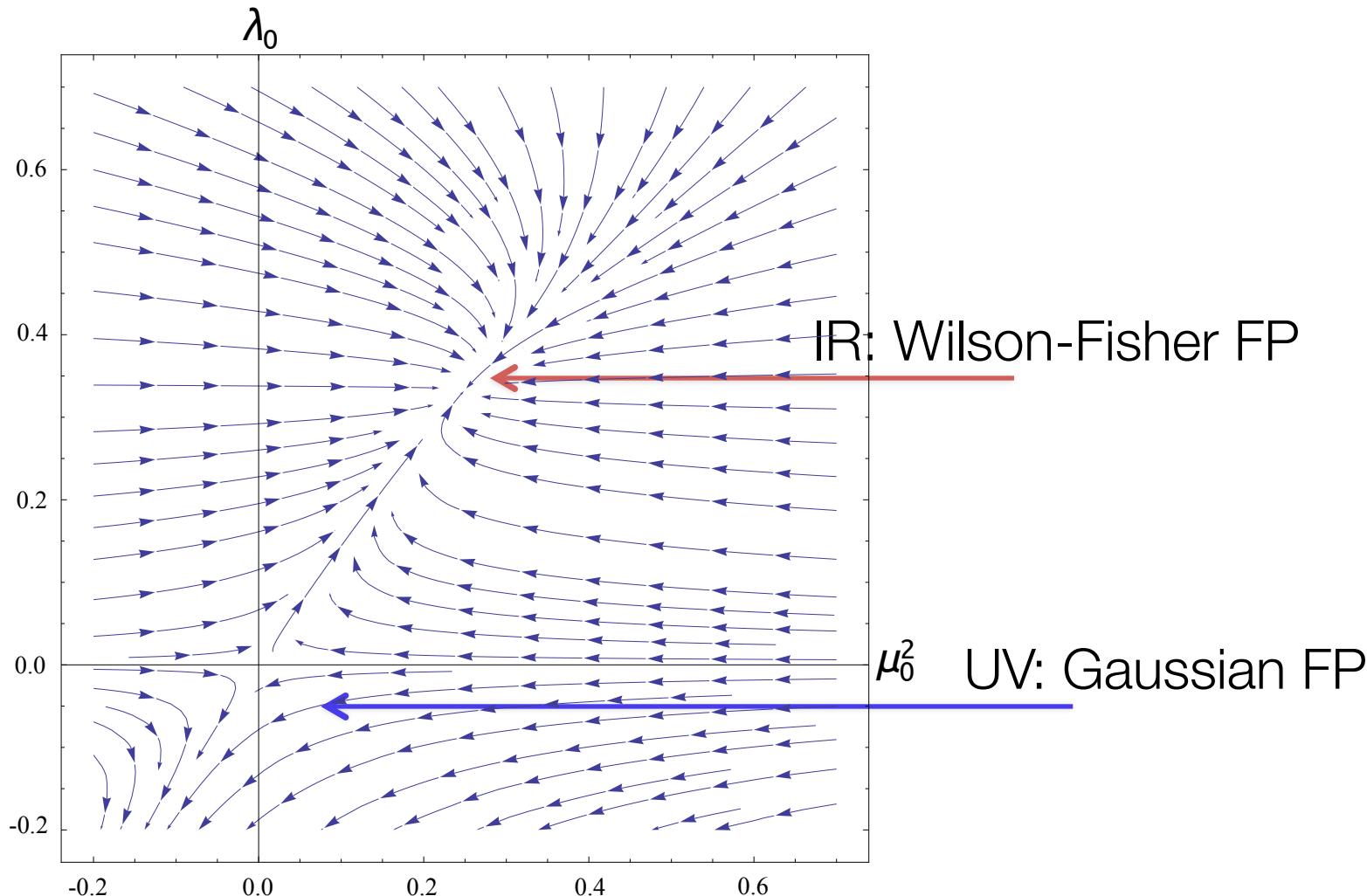
$U_{2n}=1$ in ordered phase

$0 < U_{2n} < 1$ on critical surface

- $U_{2n,cr}$ are universal quantities.
- Deng and Blöte (2003): $U_{4,cr}=0.851001$
- Higher critical cumulants computable using conformal $2n$ -point functions:
Luther and Peschel (1975)
Dotsenko and Fateev (1984)



TEST CFT: PHI 4TH AT WILSON-FISHER FIXED POINT IN 2D & 3D.



$$\mathcal{L}(x) = -\frac{1}{2}(\nabla\phi)^2 + \lambda(\phi^2 - \mu^2/2\lambda)^2$$