2018 Update on ε_K with lattice QCD inputs

Jon Bailey, Yong-Chull Jang, Weonjong Lee[†], Sunkyu Lee, and Jaehoon Leem , Sungwoo Park
(LANL-SWME Collaboration)

Lattice Gauge Theory Research Center
Department of Physics and Astronomy
Seoul National University

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ε_K and $\hat{B}_K$, V_{cb} I

- Definition of ε_K

$$\varepsilon_K = \frac{A[K_L \to (\pi\pi)_{I=0}]}{A[K_S \to (\pi\pi)_{I=0}]}$$

- Master formula for ε_K in the Standard Model.

$$\varepsilon_K = \exp(i\theta)\,\sqrt{2}\,\sin(\theta)\left(C_\varepsilon\, X_{\mathsf{SD}}\, \hat{B}_K + \frac{\xi_0}{\sqrt{2}} + \xi_{\mathsf{LD}} \right) + \mathcal{O}(\omega\varepsilon') + \mathcal{O}(\xi_0\Gamma_2/\Gamma_1)$$

$$X_{\mathsf{SD}} = \mathrm{Im}\,\lambda_t \Big[\mathrm{Re}\,\lambda_c\, \eta_{cc}\, S_0(x_c) - \mathrm{Re}\,\lambda_t\, \eta_{tt}\, S_0(x_t) - (\mathrm{Re}\,\lambda_c - \mathrm{Re}\,\lambda_t)\, \eta_{ct}\, S_0(x_c, x_t) \Big]$$

ε_K and $\hat{B}_K$, V_{cb} II

$$\lambda_i = V_{is}^* V_{id}, \qquad x_i = m_i^2 / M_W^2, \qquad C_\varepsilon = \frac{G_F^2 F_K^2 m_K M_W^2}{6\sqrt{2}\,\pi^2 \Delta M_K}$$

$$\frac{\xi_0}{\sqrt{2}} = \frac{1}{\sqrt{2}} \frac{\mathrm{Im} A_0}{\mathrm{Re} A_0} = \text{Absorptive LD Effect} \approx -7\%$$

$$\xi_{\text{LD}} = \text{Dispersive LD Effect} \approx \pm 2\% \qquad \longrightarrow \text{systematic error}$$

- Inami-Lim functions:

$$S_0(x_i) = x_i \left[\frac{1}{4} + \frac{9}{4(1-x_i)} - \frac{3}{2(1-x_i)^2} - \frac{3x_i^2 \ln x_i}{2(1-x_i)^3} \right],$$

$$S_0(x_i, x_j) = \left\{ \frac{x_i x_j}{x_i - x_j} \left[\frac{1}{4} + \frac{3}{2(1-x_i)} - \frac{3}{4(1-x_i)^2} \right] \ln x_i - (i \leftrightarrow j) \right\} - \frac{3 x_i x_j}{4(1-x_i)(1-x_j)}$$

ε_K and $\hat{B}_K$, V_{cb} III

$$
\begin{aligned}
&S_0(x_t) && \longrightarrow +72.4\% \\
&S_0(x_c, x_t) && \longrightarrow +45.4\% \\
&S_0(x_c) && \longrightarrow -17.8\%
\end{aligned}
$$

- Dominant contribution ($\approx (72.4 + \alpha)\%$) comes with $|V_{cb}|^4$.

$$
\begin{aligned}
\mathrm{Im}\lambda_t \cdot \mathrm{Re}\lambda_t &= \bar{\eta}\lambda^2 |V_{cb}|^4 (1 - \bar{\rho}) \\
\mathrm{Re}\lambda_c &= -\lambda(1 - \frac{\lambda^2}{2}) + \mathcal{O}(\lambda^5) \\
\mathrm{Re}\lambda_t &= -(1 - \frac{\lambda^2}{2}) A^2 \lambda^5 (1 - \bar{\rho}) + \mathcal{O}(\lambda^7) \\
\mathrm{Im}\lambda_t &= \eta A^2 \lambda^5 + \mathcal{O}(\lambda^7) = -\mathrm{Im}\,\lambda_c
\end{aligned}
$$

ε_K and $\hat{B}_K$, V_{cb} IV

- Definition of $\hat{B}_K$ in standard model.

$$B_K = \frac{\langle \bar{K}_0 | [\bar{s}\gamma_\mu(1-\gamma_5)d][\bar{s}\gamma_\mu(1-\gamma_5)d] | K_0 \rangle}{\frac{8}{3}\langle \bar{K}_0 | \bar{s}\gamma_\mu\gamma_5 d | 0 \rangle \langle 0 | \bar{s}\gamma_\mu\gamma_5 d | K_0 \rangle}$$

$$\hat{B}_K = C(\mu) B_K(\mu), \qquad C(\mu) = \alpha_s(\mu)^{-\frac{\gamma_0}{2b_0}} [1 + \alpha_s(\mu) J_3]$$

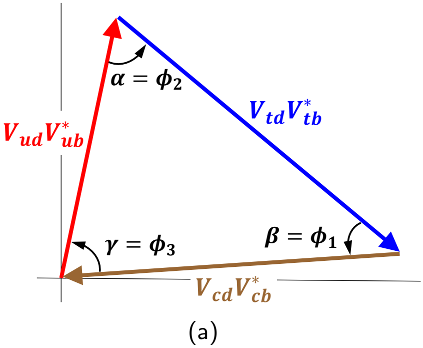
- Experiment:

$$\varepsilon_K = (2.228 \pm 0.011) \times 10^{-3} \times e^{i\phi_\varepsilon}$$

$$\phi_\varepsilon = 43.52(5)^\circ$$

ε_K with lattice QCD inputs

Unitarity Triangle $\rightarrow (\bar{\rho}, \bar{\eta})$



Global UT Fit and Angle-Only-Fit (AOF)

Global UT Fit

- Input: $|V_{ub}|/|V_{cb}|$, Δm_d, $\Delta m_s/\Delta m_d$, ε_K, and $\sin(2\beta)$.
- Determine the UT apex $(\bar{\rho}, \bar{\eta})$.
- Take λ from

$$|V_{us}| = \lambda + \mathcal{O}(\lambda^7)\,,$$

  which comes from K_{l3} and $K_{\mu 2}$.
- Disadvantage: unwanted correlation between $(\bar{\rho}, \bar{\eta})$ and ε_K.

AOF

- Input: $\sin(2\beta)$, $\cos(2\beta)$, $\sin(\gamma)$, $\cos(\gamma)$, $\sin(2\beta+\gamma)$, $\cos(2\beta+\gamma)$, and $\sin(2\alpha)$.
- Determine the UT apex $(\bar{\rho}, \bar{\eta})$.
- Take λ from $|V_{us}| = \lambda + \mathcal{O}(\lambda^7)\,,$ which comes from K_{l3} and $K_{\mu 2}$.
- Use $|V_{cb}|$ to determine A.

$$|V_{cb}| = A\lambda^2 + \mathcal{O}(\lambda^7)$$

- Advantage: NO correlation between $(\bar{\rho}, \bar{\eta})$ and ε_K.

Input Parameters: Wolfenstein Parameters

Angle-Only-Fit (AOF)

- ϵ_K, $\hat{B}_K$, and $|V_{cb}|$ are used as inputs to determine the UT angles in the global fit of `UTfit` and `CKMfitter`.
- Instead, we can use angle-only-fit result for the UT apex $(\bar{\rho}, \bar{\eta})$.
- Then, we can take λ independently from

$$|V_{us}| = \lambda + \mathcal{O}(\lambda^7)\,,$$

which comes from K_{l3} and $K_{\mu 2}$.

- Use $|V_{cb}|$ instead of A.

$$|V_{cb}| = A\lambda^2 + \mathcal{O}(\lambda^7)$$

λ	0.22509(29)	[1] CKMfitter
	0.22497(69)	[2] UTfit
	0.2248(6)	[3] $|V_{us}|$ (AOF)
$\bar{\rho}$	0.1598(76)	[1] CKMfitter
	0.153(13)	[2] UTfit
	0.146(22)	[4] UTfit (AOF)
$\bar{\eta}$	0.3499(63)	[1] CKMfitter
	0.343(11)	[2] UTfit
	0.333(16)	[4] UTfit (AOF)

Input Parameter: B_K

$\hat{B}_K$ in lattice QCD with $N_f = 2 + 1$.

Collaboration	Ref.	$\hat{B}_K$
SWME 15	[5]	$0.735(5)(36)$
RBC/UKQCD 14	[6]	$0.7499(24)(150)$
Laiho 11	[7]	$0.7628(38)(205)$
BMW 11	[8]	$0.7727(81)(84)$
FLAG 17	[9]	$0.7625(97)$

- RI-SMOM $\rightarrow$ $\overline{\text{MS}}$ matching at 2-loop : Kvedaraite Sandra [Thur 8:50]
- This will be useful to reduce the systematic error of $\hat{B}_K$ further.

Input Parameter: $|V_{cb}|$

$|V_{cb}|$ in units of 1.0×10^{-3}.

(a) Exclusive $|V_{cb}|$

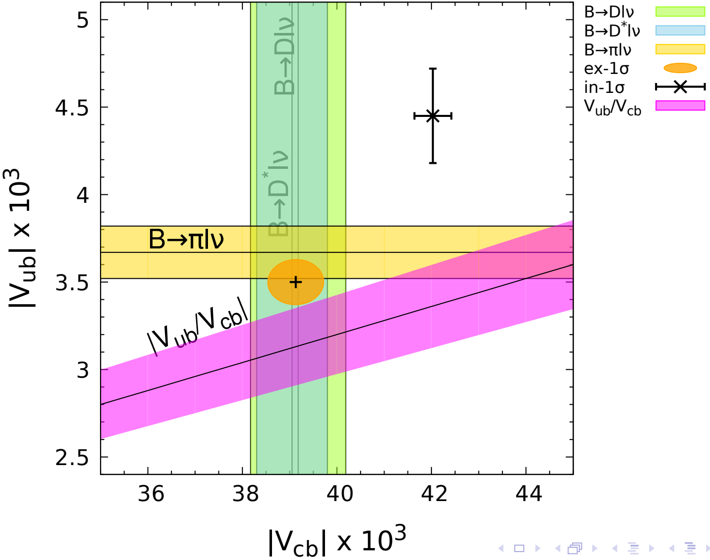
channel	value	Ref.
$B \to D^* \ell \bar{\nu}$	$39.05(47)(58)$	[10, 11]
$B \to D \ell \bar{\nu}$	$39.18(94)(36)$	[10, 12]
$\lvert V_{ub}\rvert/\lvert V_{cb}\rvert$	$0.080(4)(4)$	[10, 13]
ex-combined	$39.13(59)$	[10]

(b) Inclusive $|V_{cb}|$

channel	value	Ref.
kinetic scheme	$42.19(78)$	[10]
1S scheme	$41.98(45)$	[10]
in-combined	$42.03(39)$	this paper

- [10] $\leftrightarrow$ HFLAV (CLN)
- [11, 12] $\leftrightarrow$ FNAL/MILC
- [13] $\leftrightarrow$ W. Detmold, *et al.*

Current Status of $|V_{cb}|$ in 2018



Discrepancy between exclusive and inclusive $|V_{cb}|$

[14] $\leftrightarrow$ Bigi, Gambino, Schacht
[15] $\leftrightarrow$ Grinstein and Kobach

- Refs. [14, 15] proposed a potential solution to the problem.
- When experimentalists extract the $|V_{cb}|\mathcal{F}(1)$, they use the CLN method (Caprini, Lellouch, Neubert) [16].
- CLN is model-dependent (HQET and perturbation theory). CLN can NOT have precision better than 2%.
- At present, the trouble is that both the experiments and lattice QCD has high precision below the 2% level.
- Hence, they claimed that it is much better to use BGL [17] which is model independent and satisfies the unitarity conditions (both weak and strong versions).
- Details on CLN and BGL are in the backup slides.
- This is addressed in Takashi Kaneco's poster.

Input Parameter: ξ_0

Indirect Method

$$\xi_0 = \frac{\mathrm{Im}\, A_0}{\mathrm{Re}\, A_0}, \quad \xi_2 = \frac{\mathrm{Im}\, A_2}{\mathrm{Re}\, A_2}.$$

ξ_0	$-1.63(19) \times 10^{-4}$	RBC-UK-2015 [18]

- RBC-UKQCD calculated $\mathrm{Im}A_2$. $\mathrm{Im}A_2 \to \xi_2 \to \varepsilon'_K/\varepsilon_K \to \xi_0$

$$\mathrm{Re}\Big(\frac{\varepsilon'_K}{\varepsilon_K}\Big) = \frac{1}{\sqrt{2}|\varepsilon_K|}\omega\Big(\xi_2 - \xi_0\Big).$$

  Other inputs ω, ε_K and $\varepsilon'_K/\varepsilon_K$ are taken from the experimental values.

- Here, we choose an approximation of $\cos(\phi_{\epsilon'} - \phi_\epsilon) \approx 1$.
- $\phi_\epsilon = 43.52(5)$, $\phi_{\epsilon'} = 42.3(1.5)$
- Isopspin breaking effect: (at most 15% of ξ_0) $\to$ (1% in ε_K) $\to$ neglected!
- Update of RBC-UKQCD: Robert Mawhinney [Thur 11:00].

Input Parameter: ξ_0

Direct Method

- RBC-UKQCD calculated $\mathrm{Im} A_0$. $\mathrm{Im} A_0 \to \xi_0$.

$$\xi_0 = \frac{\mathrm{Im}\, A_0}{\mathrm{Re}\, A_0} = -0.57(49) \times 10^{-4}$$

  Other input $\mathrm{Re}\, A_0$ is taken from the experimental value.

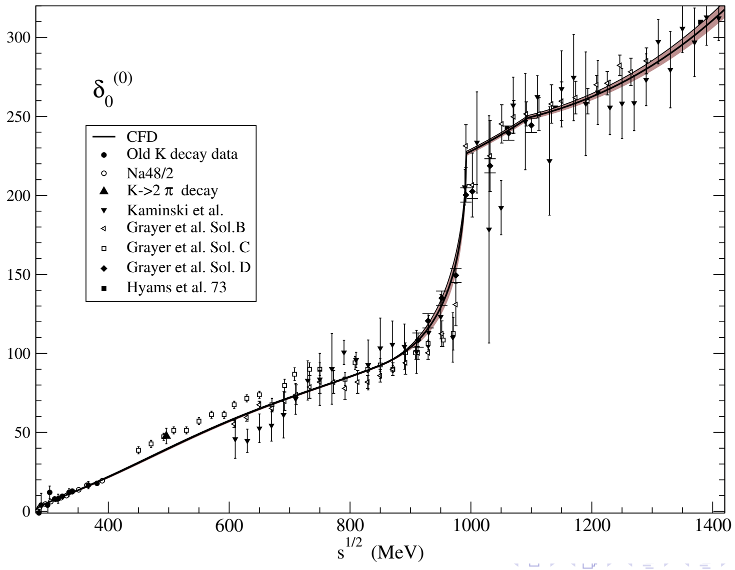
- RBC-UKQCD also calculated δ_0

$$\delta_0 = 23.8(49)(12)^\circ$$

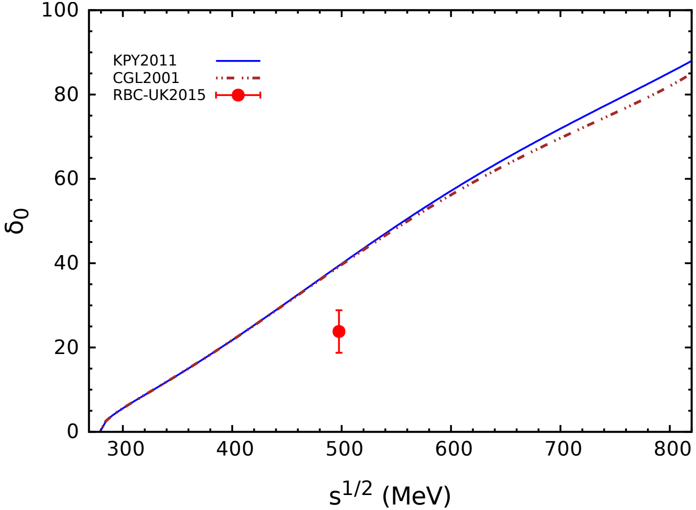
  This value is 3.0σ away from the experimental value: $\delta_0 = 39.1(6)^\circ$.

- It appears to me that this puzzle might be resolved in part by two state fitting: RBC-UKQCD, Wang Tianle [Thur 11:20].
- Here, we use the indirect method to determine ξ_0.

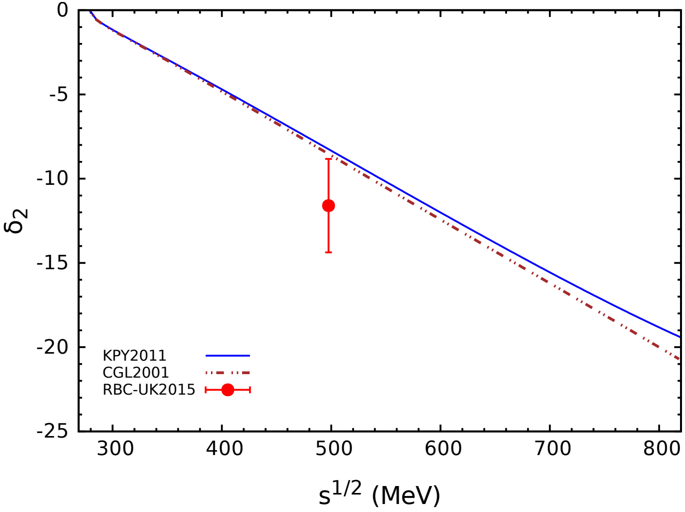
CFD analysis for δ_0: PRD83,074004 (2011)



Comparison of δ_0 between CFD and RBC-UKQCD



Comparison of δ_2 CFD and RBC-UKQCD



Input Parameter: ξ_0

Summary

Input Parameters: ξ_0

Method	Value	Reference
Indirect	$-1.63(19) \times 10^{-4}$	RBC-UK-2015 [18]
Direct	$-0.57(49) \times 10^{-4}$	RBC-UK-2015 [19]

Input Parameter: ξ_{LD}

$$\xi_{\text{LD}} = \frac{m'_{\text{LD}}}{\sqrt{2}\,\Delta M_K}$$

$$m'_{\text{LD}} = -\text{Im}\left[\mathcal{P}\sum_C \frac{\langle \overline{K}^0|H_{\text{w}}|C\rangle\langle C|H_{\text{w}}|K^0\rangle}{m_{K^0} - E_C}\right]$$

- NHC estimate [PRD 88, 014508] gives

$$\xi_{\text{LD}} = (0 \pm 1.6)\%$$

- BGI estimate [PLB 68, 309, 2010] gives

$$\xi_{\text{LD}} = -0.4(3) \times \frac{\xi_0}{\sqrt{2}}$$

- Precision measurement of lattice QCD is not available yet.

Input Parameter: charm quark mass m_c

- HPQCD [20] reported

$$m_c(m_c) = 1.2733(76)\,\text{GeV} \tag{1}$$

- FNAL/MILC/TUMQCD [21] reported another results for m_c:

$$m_c(m_c) = 1.273(10)\,\text{GeV} \tag{2}$$

- We use the HPQCD results here.

Input Parameter: top quark mass m_t

- Top quark mass m_t: the problem is that the experimentalists (CMS and ATLAS) produce only the pole mass of top quarks. But we need to know the scale invariant $\overline{\text{MS}}$ mass $m_t(m_t)$.
- The pole mass of top quarks: [PDG]

$$M_t = 173.5 \pm 1.1\,\text{GeV} \tag{3}$$

- The conversion formula is available at the four loop level:

$$\frac{m_t(\mu)}{M_t} = z(\mu) = \frac{Z_{\text{OS}}}{Z_{\overline{\text{MS}}}} \tag{4}$$

  where Z_{OS} is the renormalization factor in the on-shell scheme.
- The scale invariant $\overline{\text{MS}}$ top quark mass is [SWME]

$$m_t(m_t) = 163.65 \pm 1.05 \pm 0.17\,\text{GeV} \tag{5}$$

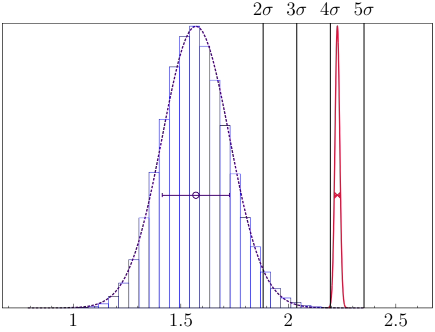
- We have not taken into account the renormalon ambiguity and corrections due to the three-loop fermion mass such as m_b and m_c.

Other Input Parameters

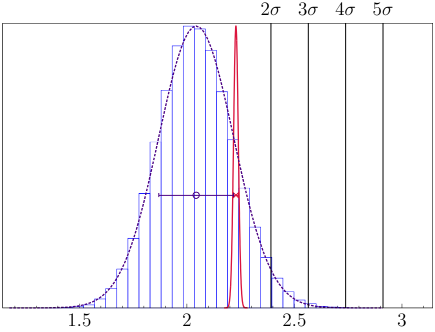
parameter	value	reference
G_F	$1.1663787(6) \times 10^{-5}$ GeV^{-2}	[3]
M_W	$80.385(15)$ GeV	[3]
θ	$43.52(5)^\circ$	[3]
m_{K^0}	$497.611(13)$ MeV	[3]
ΔM_K	$3.484(6) \times 10^{-12}$ MeV	[3]
F_K	$155.6(4)$ MeV	[3]
η_{cc}	$1.72(27)$	[22]
η_{tt}	$0.5765(65)$	[23]
η_{ct}	$0.496(47)$	[24]

ε_K from FLAG $\hat{B}_K$, AOF of $(\bar{\rho}, \bar{\eta})$, V_{us}

NHC estimate for ξ_{LD}



Exclusive V_{cb}



Inclusive V_{cb}

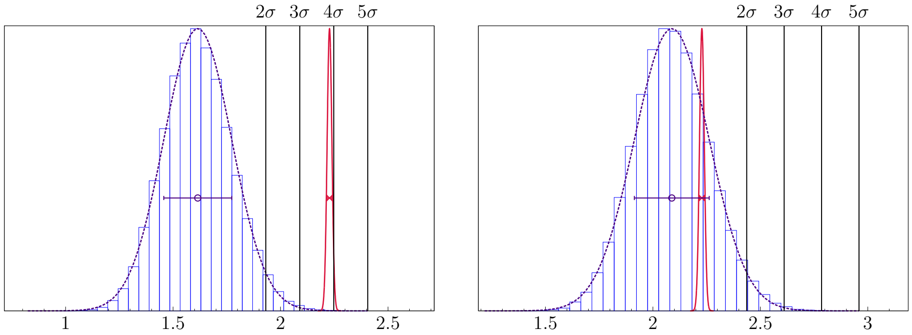
- With exclusive $|V_{cb}|$, it has 4.2σ tension.

$$|\varepsilon_K|^{\text{Exp}} = (2.228 \pm 0.011) \times 10^{-3}$$

$$|\varepsilon_K|^{\text{SM}}_{\text{excl}} = (1.570 \pm 0.156) \times 10^{-3}$$

ε_K from FLAG $\hat{B}_K$, AOF of $(\bar{\rho}, \bar{\eta})$, V_{us}

BGI estimate for ξ_{LD}



Exclusive V_{cb} — Inclusive V_{cb}

- With exclusive $|V_{cb}|$, it has 3.9σ tension.

$$|\varepsilon_K|^{\text{Exp}} = (2.228 \pm 0.011) \times 10^{-3}$$

$$|\varepsilon_K|^{\text{SM}}_{\text{excl}} = (1.615 \pm 0.158) \times 10^{-3}$$

Current Status of ε_K

- FLAG 2017 + PDG 2017: (in units of 1.0×10^{-3}, AOF)

$$|\varepsilon_K|^{\text{SM}}_{\text{excl}} = 1.570 \pm 0.156 \quad \text{for Exclusive } V_{cb} \text{ (Lattice QCD + CLN)}$$

$$|\varepsilon_K|^{\text{SM}}_{\text{incl}} = 2.043 \pm 0.174 \quad \text{for Inclusive } V_{cb} \text{ (Heavy Quark Expansion)}$$

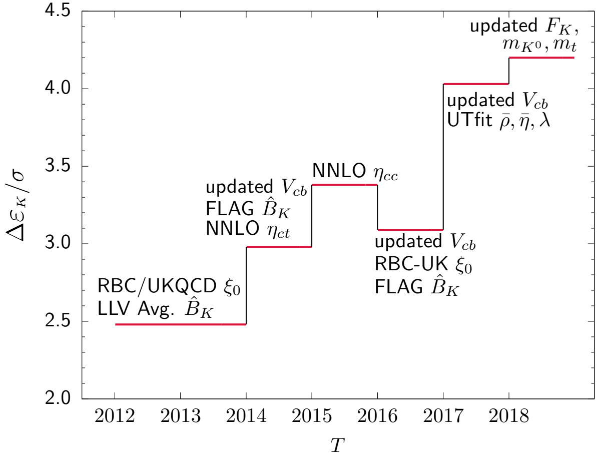
- Experiments:

$$|\varepsilon_K|^{\text{Exp}} = 2.228 \pm 0.011$$

- Hence, we observe $4.2(3)\,\sigma$ difference between the SM theory (Lattice QCD) and experiments.

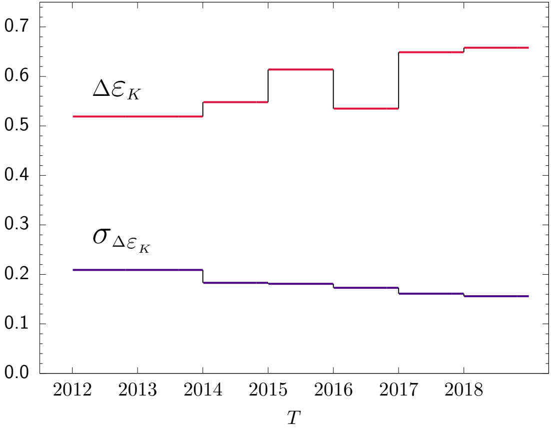
- What does this mean? $\longrightarrow$ Breakdown of SM ?

Time Evolution of $\Delta\varepsilon_K$ on the Lattice



- $\Delta\varepsilon_K \equiv |\varepsilon_K|^{\text{Exp}} - |\varepsilon_K|^{\text{SM}}_{\text{excl}}$

Time Evolution of Average and Error



- The average $\Delta\varepsilon_K$ has increased by 27% with some fluctuations.
- The error $\sigma_{\Delta\varepsilon_K}$ has decreased by 25% monotonically.

Error Budget of Exclusive ε_K

source	error (%)	memo
$\lvert V_{cb}\rvert$	31.3	Exclusive Combined
$\bar{\eta}$	26.7	AOF
η_{ct}	21.4	$c-t$ Box
η_{cc}	9.0	$c-c$ Box
$\bar{\rho}$	4.0	AOF
ξ_{LD}	2.6	Long-distance
$\hat{B}_K$	1.9	FLAG
η_{tt}	0.77	$c-c$ Box
ξ_0	0.70	$\text{Im}(A_0)/\text{Re}(A_0)$
m_t	0.66	top quark mass
$\vdots$	$\vdots$	

To Do List

- It is highly desirable if the HFLAV group may perform a comprehensive reanalysis over the entire sets of the experimental data for $\bar{B} \to D^* \ell \bar{\nu}$ using the BGL method and compare the results with those of CLN.
- It would be nice to reduce overall errors on $|V_{cb}|$: $1.4\% \to 0.8\%$.
[OK action project: LANL-SWME: Sungwoo Park, previous talk]
- It would be nice to monitor $\sigma(550)$ resonance in δ_0.
[my personal wish list]
- We need to reduce overall errors on ξ_0 and ξ_2. [RBC-UKQCD]
- We need to reduce overall errors on $\bar{\eta}$. [BELLE2]

Summary and Conclusion

Summary

1. We find that

$$\Delta\varepsilon_K^{\text{excl}} = 4.2(3)\sigma \qquad \text{(Lattice QCD)} \qquad (6)$$

$$\Delta\varepsilon_K^{\text{incl}} = 1.1\sigma \qquad \text{(HQE, QCD Sum Rules)} \qquad (7)$$

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2. It is too early to conclude that there might something wrong with the SM yet.

3. Let us wait for the next round reanalysis of the HFLAV group on the entire data sets of the $\bar{B} \to D^* \ell \bar{\nu}$ decays, using BGL.

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4. Meanwhile, it would be very helpful to reduce the errors for $|V_{cb}|$, ξ_0, ξ_2, and ξ_{LD}.

5. Please stay tuned for the update.

Thank God for your help !!!

CLN

CLN: Caprini, Lellouch, Neubert I

- Consider $\bar{B} \to D^* \ell \bar{\nu}$ decays.

$$\frac{d\Gamma(\bar{B} \to D^* \ell \bar{\nu})}{dw} = \frac{G_F^2 m_{D^*}^3}{48\pi^3} (m_B - m_{D^*})^2 \, \chi(w) \, \eta_{\text{EW}}^2 \, \mathcal{F}^2(w) \, |V_{cb}|^2$$

- Here, G_F is Fermi constant, η_{EW} is a small electroweak correction, and $\mathcal{F}(w)$ is the form factor.
- The kinematic factor $\chi(w)$ is

$$\chi(w) = \sqrt{w^2 - 1}(w+1)^2 \times Y(w)$$
$$Y(w) = \left[1 + \frac{4w}{w+1} \frac{1 - 2wr + r^2}{(1-r)^2}\right]$$

CLN: Caprini, Lellouch, Neubert II

- The form factor can be rewritten as follows,

$$\mathcal{F}^2(w) = h_{A_1}^2(w) \times \frac{1}{Y(w)} \times \left\{ 2\frac{1-2wr+r^2}{(1-r)^2} \left[1 + \frac{w-1}{w+1} R_1^2(w)\right] + \left[1 + \frac{w-1}{1-r}\Big(1 - R_2(w)\Big)\right]^2 \right\}$$

- So far the formalism is quite general.

CLN: Caprini, Lellouch, Neubert III

- CLN method [16]: ($\approx$ model-dependent approximation)

$$h_{A_1}(w) = h_{A_1}(1)\Big[1 - 8\rho^2 z + (53\rho^2 - 15)z^2 - (231\rho^2 - 91)z^3\Big] \quad (8)$$

$$R_1(w) = R_1(1) - 0.12(w-1) + 0.05(w-1)^2 \quad (9)$$

$$R_2(w) = R_2(1) + 0.11(w-1) - 0.06(w-1)^2 \quad (10)$$

where z is a conformal mapping variable:

$$z = \frac{\sqrt{w+1} - \sqrt{2}}{\sqrt{w+1} + \sqrt{2}} \quad (11)$$

- The trouble is that the slopes and curvatures of $R_1(w)$ and $R_2(w)$ are fixed by the HQET perturbation theory (zero-recoil expansion). The HQET results for the slopes and curvatures has about 10% uncertainty of order $\mathcal{O}(\Lambda^2/m_c^2)$ and $\mathcal{O}(\alpha_s\Lambda/m_c)$.

CLN: Caprini, Lellouch, Neubert IV

- Hence, CLN can NOT have precision better than 2% by construction.
- The trouble is that the experimental results have errors less than 2% and that the lattice QCD results for the form factors have such a high precision that the errors are below the 2% level.
- At any rate, the experimental group (HFLAV 2017) use CLN to fit the experimental data to determine four parameters: $\eta_{\text{EW}}\mathcal{F}(1)|V_{cb}|$, ρ^2, $R_1(1)$, $R_2(1)$.
- Lattice QCD determines $\mathcal{F}(1)$ very well.
- η_{EW} is very well known.
- Hence, we can determine exclusive $|V_{cb}|$ out of this.

BGL

BGL: Boyd, Grinstein, Lebed I

- BGL is model-independent.
- BGL is constructed on three building blocks:
  1. Dispersion relation
  2. Crossing symmetry
  3. Analytic continuation: analyticity
- Consider the 2-point function:

$$\begin{aligned}\Pi_J^{\mu\nu}(q) &= (q^\mu q^\nu - q^2 g^{\mu\nu})\Pi_J^T(q^2) + g^{\mu\nu}\Pi_J^L(q^2) \\ &\equiv i\int d^4x e^{iq\cdot x}\langle 0|T J^\mu(x)[J^\nu(0)]^\dagger|0\rangle \end{aligned} \tag{12}$$

- In general, $\Pi_J^{T,L}(q^2)$ is not finite.

BGL: Boyd, Grinstein, Lebed II

- Hence, we need to make one or two subtractions to obtain finite dispersion relations:

$$\chi_J^L(q^2) = \frac{\partial \Pi_J^L}{\partial q^2} = \frac{1}{\pi}\int_0^\infty dt \frac{\mathrm{Im}\,\Pi_J^L(t)}{(t-q^2)^2} \tag{13}$$

$$\chi_J^T(q^2) = \frac{\partial \Pi_J^T}{\partial q^2} = \frac{1}{\pi}\int_0^\infty dt \frac{\mathrm{Im}\,\Pi_J^T(t)}{(t-q^2)^2} \tag{14}$$

- Källen-Lehmann spectral decomposition:

$$\begin{aligned}&(q^\mu q^\nu - q^2 g^{\mu\nu})\mathrm{Im}\,\Pi_J^T(q^2) + g^{\mu\nu}\mathrm{Im}\,\Pi_J^L(q^2)\\ &= \frac{1}{2}\sum_X (2\pi)^4 \delta^4(q-p_X)\langle 0|J^\mu(0)|X\rangle\langle X|[J^\nu(0)]^\dagger|0\rangle\end{aligned} \tag{15}$$

BGL: Boyd, Grinstein, Lebed III

- Multiply $\xi_\mu \xi_\nu^*$ on both sides:

$$\left[(q^\mu q^\nu - q^2 g^{\mu\nu})\mathrm{Im}\,\Pi_J^T(q^2) + g^{\mu\nu}\mathrm{Im}\,\Pi_J^L(q^2)\right]\xi_\mu \xi_\nu^* \geq 0 \qquad (16)$$

  for any complex 4-vector ξ_μ.

- From this we can prove the positivity:

$$\mathrm{Im}\,\Pi_J^T(q^2) \geq 0 \qquad (17)$$

$$\mathrm{Im}\,\Pi_J^L(q^2) \geq 0 \qquad (18)$$

BGL: Boyd, Grinstein, Lebed IV

- Consider the two body state of $X = H_b(p_1)H_c(p_2)$.

$$\begin{aligned}\operatorname{Im}\Pi_J^{ii}(q^2) =& \frac{1}{2}\int \frac{d^3p_1 d^3p_2}{(2\pi)^3 4E_1E_2}\delta^4(q-p_1-p_2) \\ & \times \sum_{\text{pol}} \langle 0|J^i|H_b(p_1)H_c(p_2)\rangle\langle H_b(p_1)H_c(p_2)|[J^i]^\dagger|0\rangle \\ & + \cdots \end{aligned} \tag{19}$$

- Here, the ellipsis $(\cdots)$ represents strictly positive contributions from the higher resonances and multi-particle states.
- We may assume that $H_b = B, B^*$ meson states, and $H_c = D, D^*$ meson states.

BGL: Boyd, Grinstein, Lebed V

- Let us consider a simple example of $H_b = B$ and $H_c = D^*$.

$$\mathrm{Im}\,\Pi_J^{ii}(t) \geq k(t)|\mathcal{F}(t)|^2 \tag{20}$$

  where $t = q^2$, $k(t)$ is a calculable kinematic function arising from two-body phase space.

- Let us use the crossing symmetry and analytic continuation:

$$\langle 0|J^i|H_b(p_1)H_c(p_2)\rangle = \mathcal{F}(t) \qquad (t_+ \leq t < \infty) \tag{21}$$

$$\langle \bar{H}_b(-p_1)|J^i|H_c(p_2)\rangle = \mathcal{F}(t) \qquad (m_\ell^2 \leq t < t_-) \tag{22}$$

BGL: Boyd, Grinstein, Lebed VI

- Hadronic moments $\chi_J^{(n)}$:

$$\chi_J^{(n)} \equiv \frac{1}{\Gamma(n+3)} \left. \frac{\partial^{n+2}\Pi_J^{ii}}{\partial^{n+2}q^2} \right|_{q^2=0} = \frac{1}{\pi}\int_0^\infty dt \left. \frac{\operatorname{Im}\Pi_J^{ii}(t)}{(t-q^2)^{n+3}} \right|_{q^2=0} \quad (23)$$

- Hence, the inequality is

$$\chi_J^{(n)} \geq \frac{1}{\pi}\int_{t_+}^\infty dt \frac{k(t)|\mathcal{F}(t)|^2}{t^{n+3}} \quad (24)$$

$$\longrightarrow \quad \frac{1}{\pi}\int_{t_+}^\infty dt |h^{(n)}(t)F(t)|^2 \leq 1 \quad (25)$$

BGL: Boyd, Grinstein, Lebed VII

where

$$[h^{(n)}(t)]^2 = \frac{k(t)}{t^{n+3}\chi_J^{(n)}} \geq 0\,. \tag{26}$$

- Let us introduce the conformal mapping function:

$$z(t,t_s) = \frac{\sqrt{t_+ - t} - \sqrt{t_+ - t_s}}{\sqrt{t_+ - t} + \sqrt{t_+ - t_s}} \tag{27}$$

- The inequality can be rewritten as follows,

$$\frac{1}{\pi}\int_{t_+}^{\infty} dt \left|\frac{dz(t,t_0)}{dt}\right| |\phi(t,t_0)P(t)F(t)|^2 \leq 1\,, \tag{28}$$

BGL: Boyd, Grinstein, Lebed VIII

- Here, the outer function ϕ is

$$\phi(t, t_0) = \tilde{P}(t) \frac{h^{(n)}(t)}{\sqrt{\left| \frac{dz(t, t_0)}{dt} \right|}} \tag{29}$$

- Here, the factor $\tilde{P}(t)$ removes the sub-threshold poles and branch cuts in $h^{(n)}(t)$.

$$\tilde{P}(t) = \prod_{i=1}^{\tilde{N}} z(t, t_{s_i}) \prod_{j=1}^{\tilde{M}} \sqrt{z(t, t_{s_j})} \tag{30}$$

BGL: Boyd, Grinstein, Lebed IX

- The Blaschke factor $P(t)$ removes all the sub-threshold poles in $\mathcal{F}(t)$.

$$P(t) \equiv \prod_{i=1}^{N} \frac{z - z_{P_i}}{1 - z z^*_{P_i}} = \prod_{i=1}^{N} \frac{z - z_{P_i}}{1 - z z_{P_i}} \tag{31}$$

$$z_{P_i} \equiv z(t_{P_i}, t_-) = \frac{\sqrt{t_+ - t_{P_i}} - \sqrt{t_+ - t_-}}{\sqrt{t_+ - t_{P_i}} + \sqrt{t_+ - t_-}} \tag{32}$$

  where $t_{P_i} = M^2_{P_i}$ represents the pole positions of $F(t)$ below the threshold ($t_{P_i} < t_+$).
- $|\tilde{P}(t)| = 1$ and $|P(t)| = 1$ for $t_+ \leq t < \infty$.
- Hence, $\phi(t, t_0)P(t)\mathcal{F}(t)$ is analytic even in the sub-threshold region.

BGL: Boyd, Grinstein, Lebed X

- BGL method for the form factor parametrization:

$$F(t) = \frac{1}{\phi(t,t_0)P(t)} \sum_{n=0}^{\infty} a_n z^n(t,t_0) \tag{33}$$

- After the Fourier analysis, the inequality is

$$\sum_{n=0}^{\infty} |a_n|^2 \leq 1\,. \tag{34}$$

- This is called the unitarity conditions (the weak version).

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