

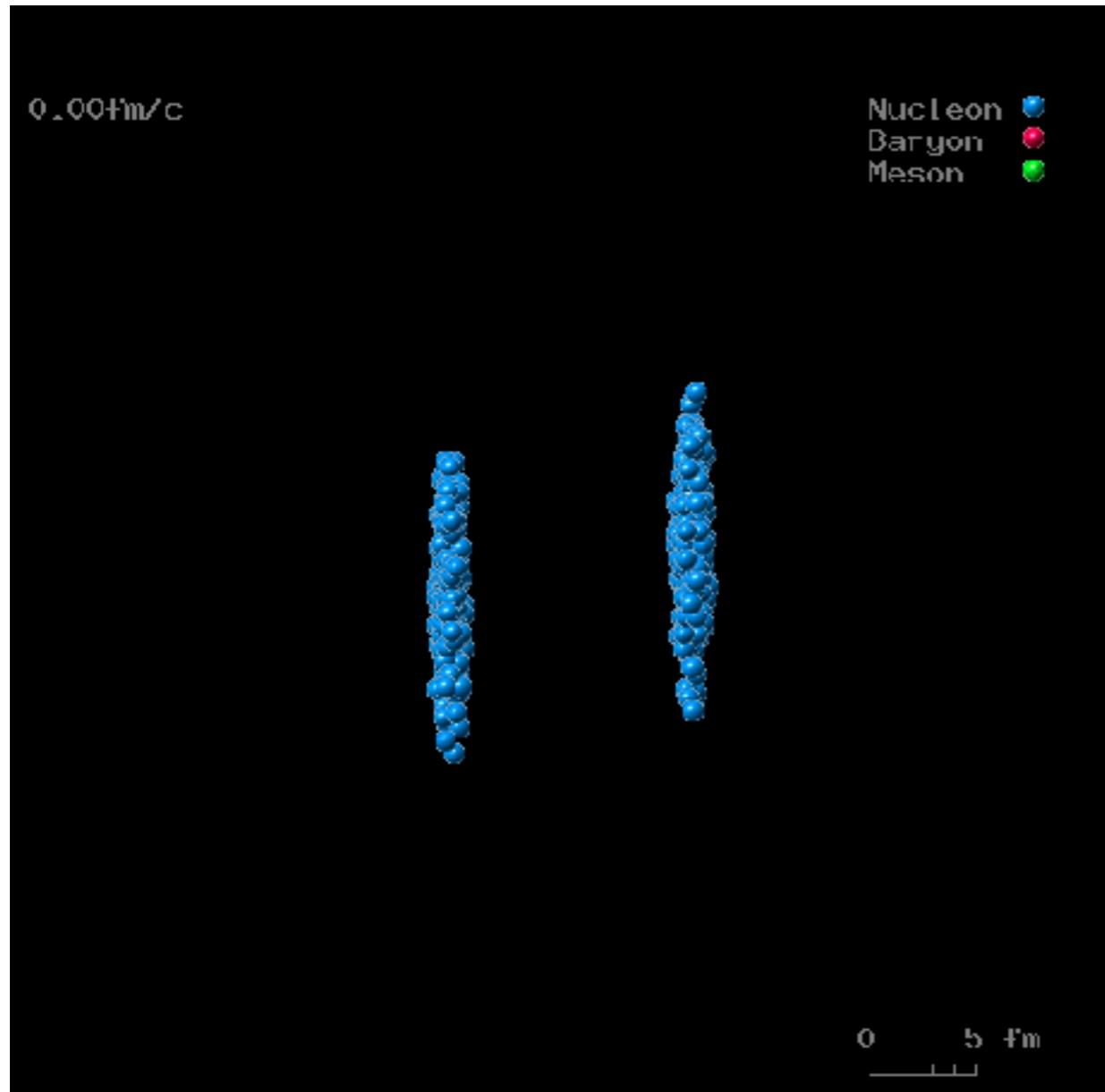
Study of energy-momentum tensor correlation function in $N_f=2+1$ full QCD for QGP viscosities

Yusuke Taniguchi
for
WHOT QCD collaboration

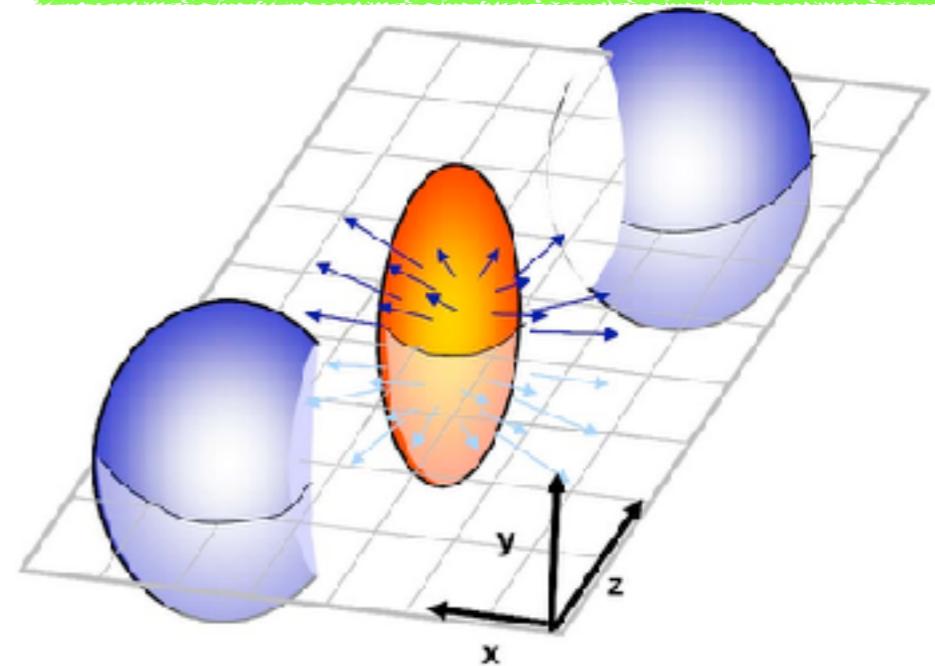
A. Baba, S. Ejiri, K. Kanaya, M. Kitazawa, T. Shimojo,
A. Suzuki, H. Suzuki, Y.T, T. Umeda

QGP viscosity is fascinating!

- BNL Relativistic Heavy Ion Collision (2001)



Discovery of elliptic flow



- non central collision
- non black body radiation
- systematic deviation in mom.
- signal of collective motion

QGP viscosity is fascinating!

Collective motion in QGP

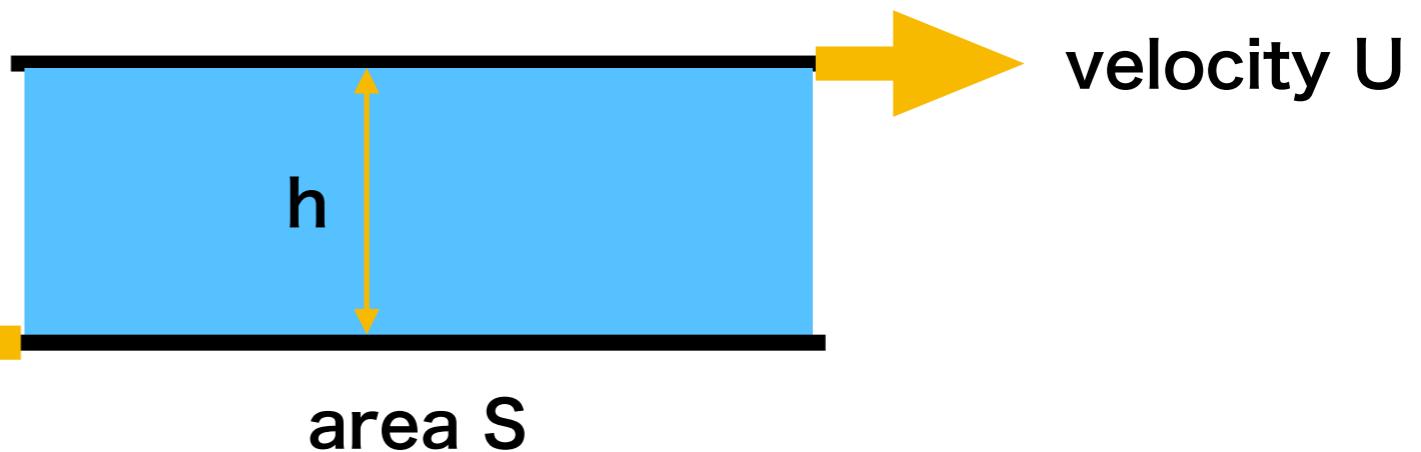
- Strong coupling phenomenon
 - If Boltzmann equation is applied **Molnar, Gyulassy(2002)**
→ scattering cross section=50x(perturbative one)
- Strongly coupled collective motion=hydrodynamics
- If hydrodynamics is applied **Teany(2003)**
→ Viscosity if really small $\frac{\eta}{s} \sim 0.04$
- Prediction from AdS/CFT **Policastro, Son, Starinets(2001)**
$$\frac{\eta}{s} \sim \frac{1}{4\pi} \sim 0.08$$
 Physically smallest value for viscosity?
Kovtun, Son, Starinets(2004)

How to calculate viscosity in field theory?

- Shear viscosity (Wikipedia)

$$\tau = \frac{F}{S} = \eta \frac{U}{h}$$

shear stress F



generalization $f_{ij} = \eta \frac{\partial U_i}{\partial x_j}$

- Bulk viscosity (Wikipedia)

$$f_{ii} = \zeta \frac{\partial U_j}{\partial x_j}$$

How to calculate viscosity in field theory?

Shear stress in field theory = Energy-momentum tensor

EM tensor=conserved current $T_{\mu\nu} = \frac{\delta\mathcal{L}}{\delta\partial^\mu\phi}\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}$

Free dust picture $T^{\mu\nu} = \rho u^\mu u^\nu$

Physical meaning of component

$$T^{00} = \gamma\rho u^0 = \epsilon \quad \text{energy density}$$

$$T^{0i} = \gamma\rho u^i = \pi^i \quad \text{momentum density}$$

$$T^{11} = \pi^1 v^1 \quad \text{momentum density escaping into x-direction}$$

variation of momentum = force  pressure

$$T^{12} = \pi^1 v^2 \quad \text{momentum density escaping into y-direction}$$

 shear stress f^{12}

How to calculate viscosity in field theory?

We want to calculate $\langle T^{12} \rangle_\beta = \eta \partial^1 u^2$

In thermal equilibrium $\langle T^{12} \rangle_\beta = 0$

Hydrodynamics is a non-equilibrium phenomenon

Locally non-equilibrium phenomenon in global equilibrium

- Use non-equilibrium statistical mechanics
- • Non-equilibrium fluctuation in real time formalism
 - introduce fluctuation as an external source
 - study its relaxation process

How to calculate viscosity in field theory?

Real time formalism

$$\langle \Delta \hat{T}_{ij}(t, \vec{x}) \rangle_{\text{neq}} = \text{Tr} \left(\hat{\rho}_{\text{neq}} \Delta \hat{T}_{ij}(t, \vec{x}) \right)$$

time dependent operator

$$\Delta \hat{T}_{ij} = \hat{T}_{ij} - \langle \hat{T}_{ij} \rangle$$

$$\hat{\rho}_{\text{neq}} = \frac{\exp \left(-\beta \hat{H} + \beta \int d^3x \hat{O}(\vec{x}) J(\vec{x}) \right)}{\text{Tr} \exp \left(-\beta \hat{H} + \beta \int d^3x \hat{O}(\vec{x}) J(\vec{x}) \right)}$$

tiny external source for fluctuation

$$\exp \left(-\beta \hat{H} + \beta \int d^3x \hat{O}(\vec{x}) J(\vec{x}) \right) = \exp \left(-\beta \hat{H} \right) T_\tau \exp \left(\int_0^\beta d\tau \int d^3x \hat{O}(-i\tau, \vec{x}) J(\vec{x}) \right)$$

$$\hat{O}(-i\tau, \vec{x}) = e^{\tau \hat{H}} \hat{O}(\vec{x}) e^{-\tau \hat{H}}$$

How to calculate viscosity in field theory?

Real time formalism

$$\langle \Delta \hat{T}_{ij}(t, \vec{x}) \rangle_{\text{neq}} = \text{Tr} \left(\hat{\rho}_{\text{neq}} \Delta \hat{T}_{ij}(t, \vec{x}) \right)$$
$$\hat{\rho}_{\text{neq}} = \frac{\exp \left(-\beta \hat{H} + \beta \int d^3x \hat{O}(\vec{x}) J(\vec{x}) \right)}{\text{Tr} \exp \left(-\beta \hat{H} + \beta \int d^3x \hat{O}(\vec{x}) J(\vec{x}) \right)}$$

Linear term in external source

$$\langle \Delta \hat{T}_{ij}(t, \vec{x}) \rangle_{\text{neq}} = \int_0^\beta d\tau \int d^3x' \langle \Delta \hat{O}(-i\tau, \vec{x}') \Delta \hat{T}_{ij}(t, \vec{x}) \rangle_\beta J(\vec{x}')$$

with Boltzmann weight $\hat{\rho} = \frac{\exp(-\beta \hat{H})}{\text{Tr} \exp(-\beta \hat{H})}$

How to calculate viscosity in field theory?

velocity as an external source $J(\vec{x}) = u_l(\vec{x})$

conserved charge density coupled to the source $T_{0l}(\vec{x})$

- Conservation law of energy-momentum tensor
- time reversal symmetry (in real time)
- translation invariance in real time

$$\langle \Delta \hat{T}_{ij}(t, \vec{x}) \rangle_{\text{neq}} = - \int_0^t ds \int d^3x' \int_0^\beta d\tau \langle \Delta \hat{T}_{ij}(s - i\tau, \vec{x}) \Delta \hat{T}_{kl}(0, \vec{x}') \rangle_\beta \partial_k u_l(\vec{x}')$$

Kubo's response function

Shear viscosity

$$\eta = - \int_0^\infty dt \int d^3x' \int_0^\beta d\tau \langle \Delta \hat{T}_{12}(t - i\tau, \vec{x}) \Delta \hat{T}_{12}(0, \vec{0}) \rangle_\beta$$

Spectral function

Kubo's function is not available on lattice

$$\int_0^\beta d\tau \langle \Delta \hat{T}_{ij}(t - i\tau, \vec{x}) \Delta \hat{T}_{kl}(0, \vec{0}) \rangle_\beta$$

Transpose to Euclidean correlation function with analytic continuation

Spectral function connects two correlators

$$\rho(k) \equiv \int d^4x e^{-ikx} \left\langle [\hat{\phi}(x), \hat{\phi}(0)] \right\rangle_\beta = G^P(k) - G^N(k)$$

$$G^P(x) = \langle \hat{\phi}(x) \hat{\phi}(0) \rangle_\beta = \int \frac{d^4k}{(2\pi)^4} e^{ikx} G^P(k)$$

This is related to Matsubara Green's function

$$G^N(x) = \langle \hat{\phi}(0) \hat{\phi}(x) \rangle_\beta = G^P(t - i\beta, \vec{x})$$

$$G^P(k) = \frac{e^{\beta k_0}}{e^{\beta k_0} - 1} \rho(k)$$

Spectral function

Matsubara Green's function (in momentum space)

$$\begin{aligned} G_\beta(\tau) &= \int d^3x \langle \hat{\phi}(-i\tau, \vec{x}) \hat{\phi}(0, \vec{0}) \rangle_\beta \\ &= \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} e^{-ik_0(-i\tau)} G^P(k_0) = \int_{-\infty}^{\infty} \frac{dk_0}{2\pi} e^{-k_0\tau} \frac{e^{\beta k_0}}{e^{\beta k_0} - 1} \rho(k_0) \\ &= \int_0^{\infty} \frac{dk_0}{2\pi} \frac{\cosh k_0 \left(\tau - \frac{\beta}{2} \right)}{\sinh k_0 \frac{\beta}{2}} \rho(k_0, \vec{0}) \end{aligned}$$

Spectral function

Kubo's response function

$$\begin{aligned}\Delta_K(k) &= \int_0^\beta d\tau \int dt e^{ik_0 t} \int d^3x e^{-i\vec{k}\vec{x}} \langle \hat{\phi}(t - i\tau, \vec{x}) \hat{\phi}(0, \vec{0}) \rangle_\beta \\ &= \int_0^\beta d\tau \int dt' e^{ik_0(t' + i\tau)} \int d^3x e^{-i\vec{k}\vec{x}} \langle \hat{\phi}(t', \vec{x}) \hat{\phi}(0, \vec{0}) \rangle_\beta \\ &= \int_0^\beta d\tau e^{-k_0\tau} G^P(k) = \frac{\rho(k)}{k_0}\end{aligned}$$

shear viscosity

$$\begin{aligned}\eta &= \int_0^\infty dt \int d^3x' \int_0^\beta d\tau \langle \Delta \hat{T}_{12}(t - i\tau, \vec{x}) \Delta \hat{T}_{12}(0, \vec{0}) \rangle_\beta \\ &= \lim_{k_0 \rightarrow 0} \frac{\rho(k_0, \vec{0})}{2k_0}\end{aligned}$$

Severe problem on lattice

How to calculate spectral function on lattice?

$$\int d^3x \langle \hat{\phi}(-i\tau, \vec{x}) \hat{\phi}(0, \vec{0}) \rangle_\beta = \int_0^\infty \frac{dk_0}{2\pi} \frac{\cosh k_0 \left(\tau - \frac{\beta}{2}\right)}{\sinh k_0 \frac{\beta}{2}} \rho(k_0, \vec{0})$$

Discretized finite points for τ  ill-defined inverse problem

Successive trials

- Model fit of spectral function
- Maximal entropy method
- Backus-Gilbert method
- Machine learning (Heng-Tong Ding. Fri.)

Model fit of spectral function

Karsch, Wyld (1987)

Nakamura, Sakai (2005)

Meyer (2007,2009)

ITEP Lattice group (2017)

Breit-Wigner ansatz

$$\frac{\rho(\omega)}{\omega} = \frac{F}{1 + b^2(\omega - \omega_0)^2} + \frac{F}{1 + b^2(\omega + \omega_0)^2}$$

hydro model

hard thermal loop ansatz

perturbative

$$\frac{\rho(\omega)}{\omega} = \frac{2\eta}{1 + b^2\omega^2} + \theta(\omega - \omega_0) \frac{A\omega^3}{\tanh \frac{\omega}{4T}}$$

fit correlation functions on lattice

Whats' new?

- Define energy-momentum tensor on lattice using Gradient Flow

Non-perturbative renormalization scheme on lattice

Narayanan-Neuberger(2006), Lüscher(2010), Lüscher-Weisz(2011)

H. Suzuki (2013), Makino-Suzuki (2014)

- First application to Nf=2+1 QCD
- Clover fermion with NP csw + Iwasaki gauge action
- $a \sim 0.07$ [fm], heavy ud quark $\frac{m_\pi}{m_\rho} \sim 0.6$
- $T = 174 - 464$ MeV

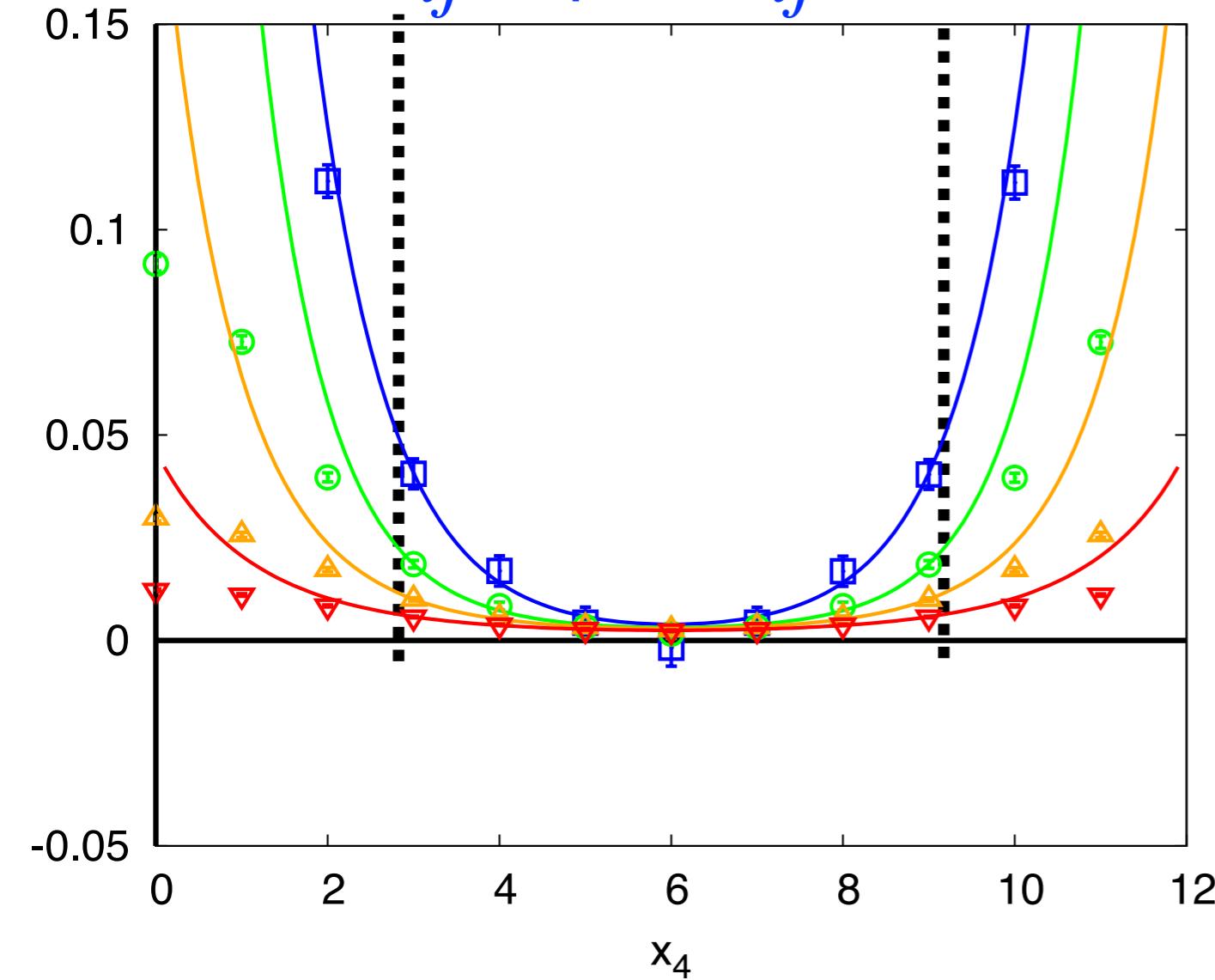
WHOT-QCD, Phys. Rev. D 85, 094508 (2012), PRD 96, 014509 (2017)

Shear viscosity

Breit-Wigner ansatz

$T=232 \text{ MeV } (N_t=12)$

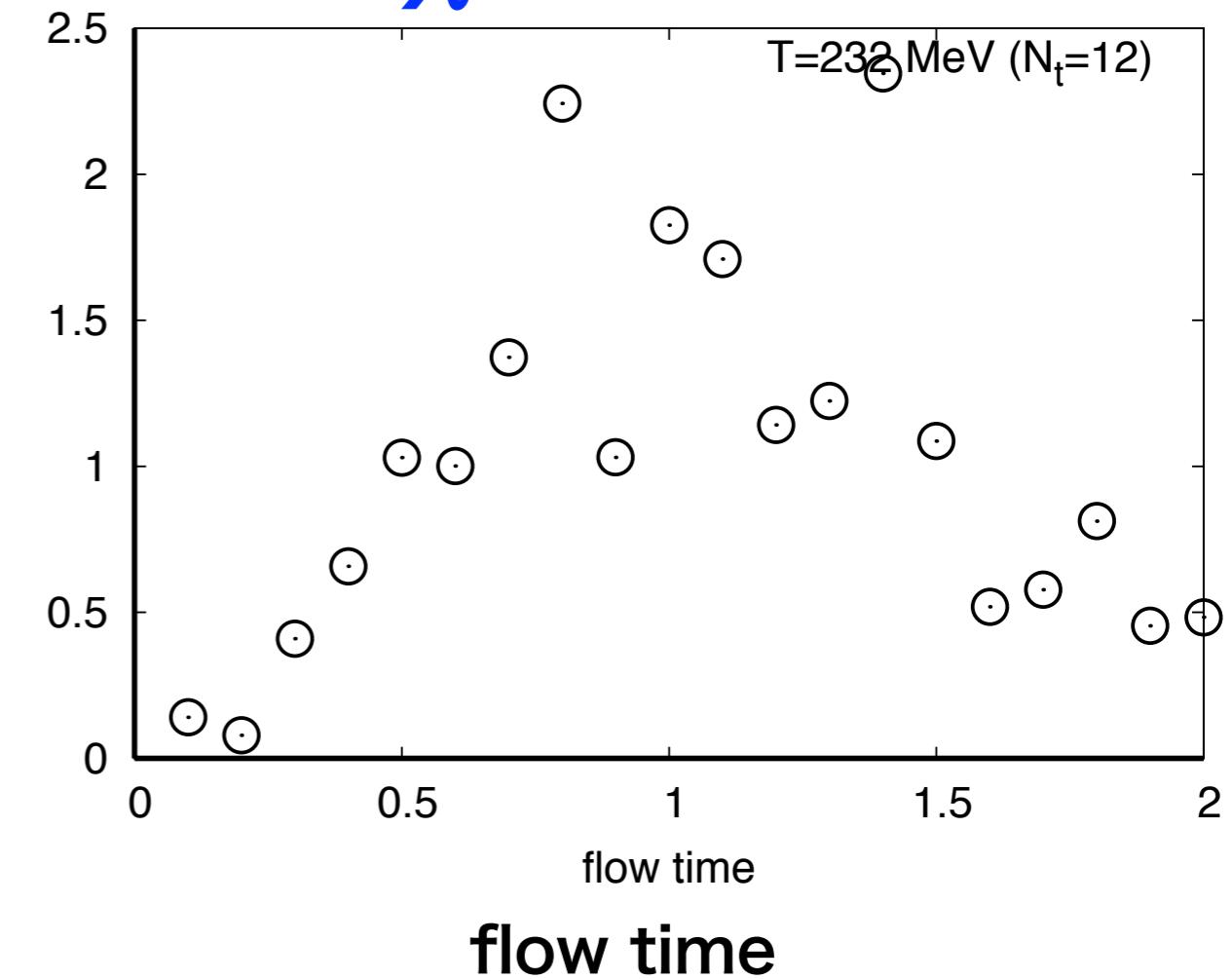
$$\langle \Delta T_{ij}(x_4) \Delta T_{ij}(0) \rangle$$



flow time=0.5
flow time=1.0

flow time=1.5
flow time=2.0

$$\chi^2/\text{dof}$$



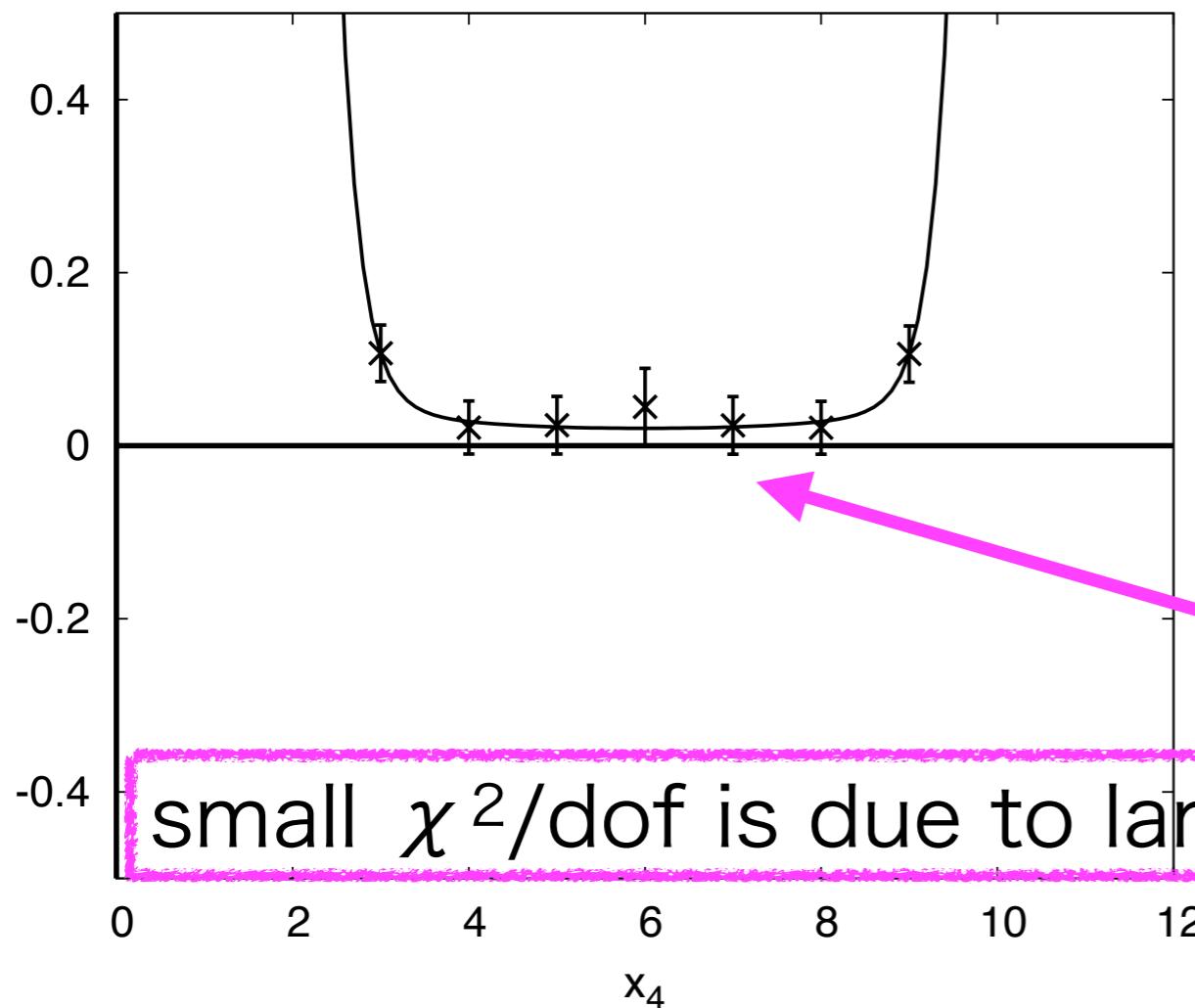
$$\frac{\rho(\omega)}{\omega} = \frac{F}{1 + b^2(\omega - \omega_0)^2} + \frac{F}{1 + b^2(\omega + \omega_0)^2}$$

Shear viscosity

Breit-Wigner ansatz

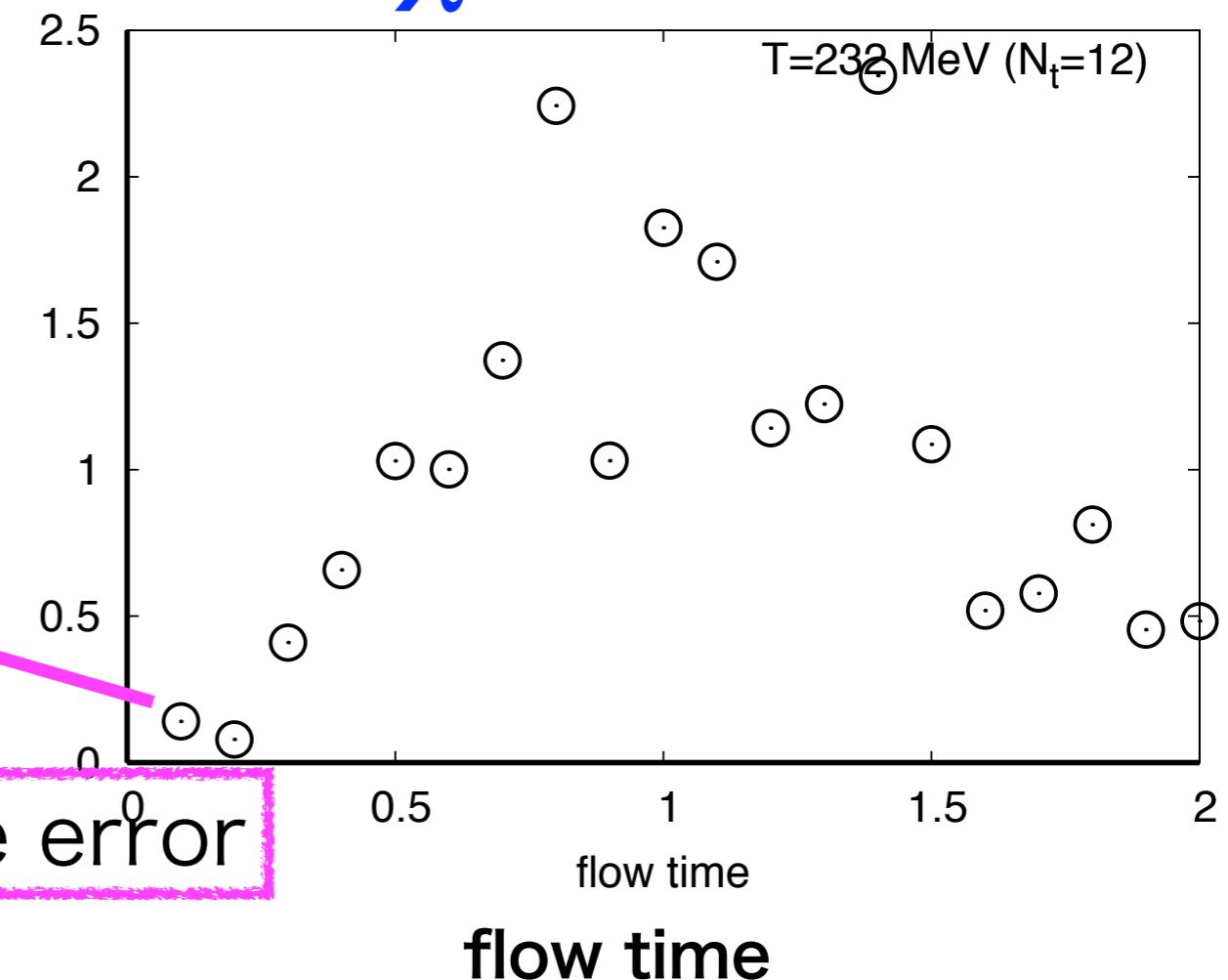
$T=232 \text{ MeV } (N_t=12)$

$$\langle \Delta T_{ij}(x_4) \Delta T_{ij}(0) \rangle$$



flow time=0.1 \rightarrow

$$\chi^2/\text{dof}$$

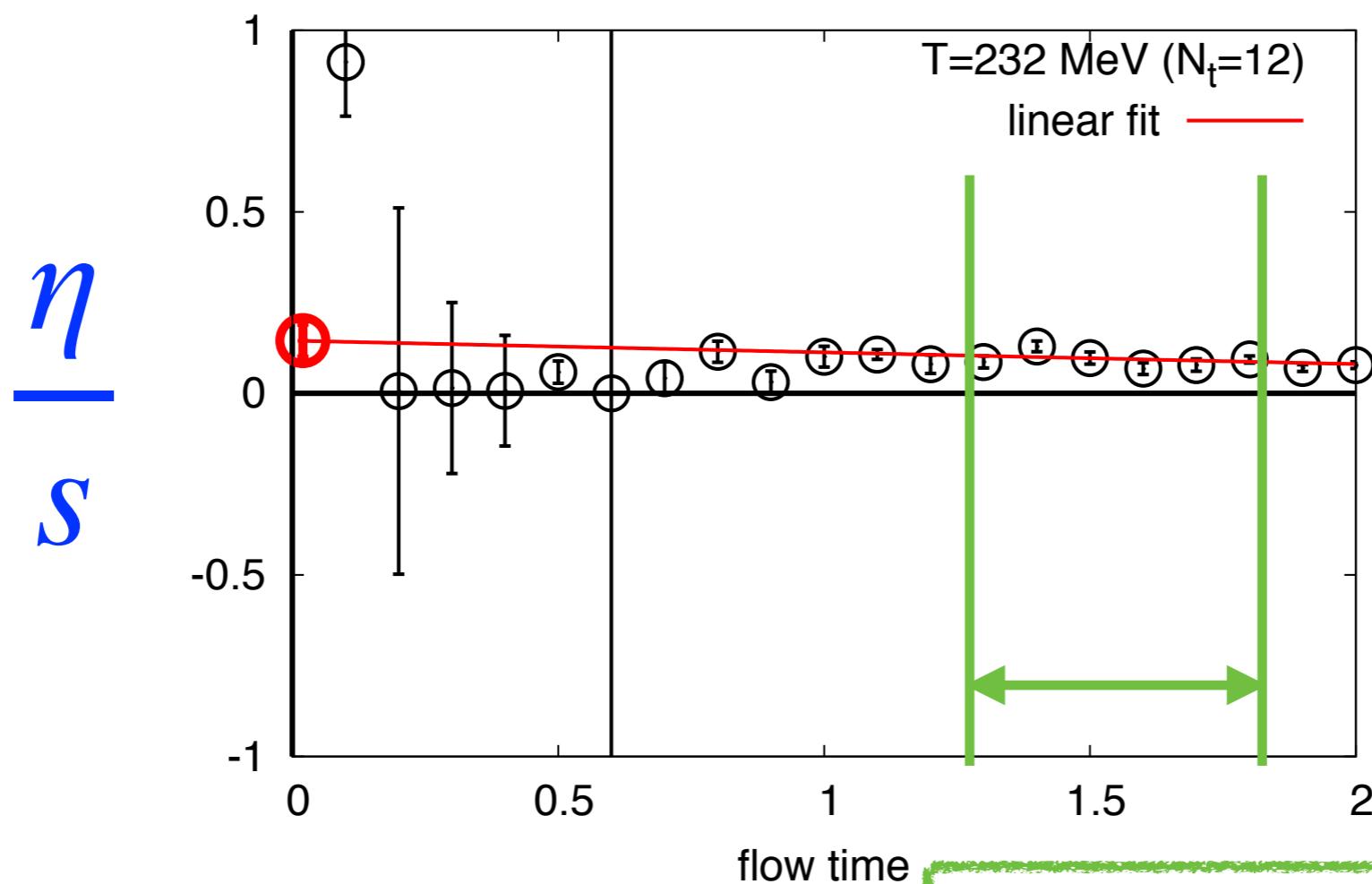


$$\frac{\rho(\omega)}{\omega} = \frac{F}{1 + b^2(\omega - \omega_0)^2} + \frac{F}{1 + b^2(\omega + \omega_0)^2}$$

Shear viscosity

Breit-Wigner ansatz

T=232 MeV (N_t=12)

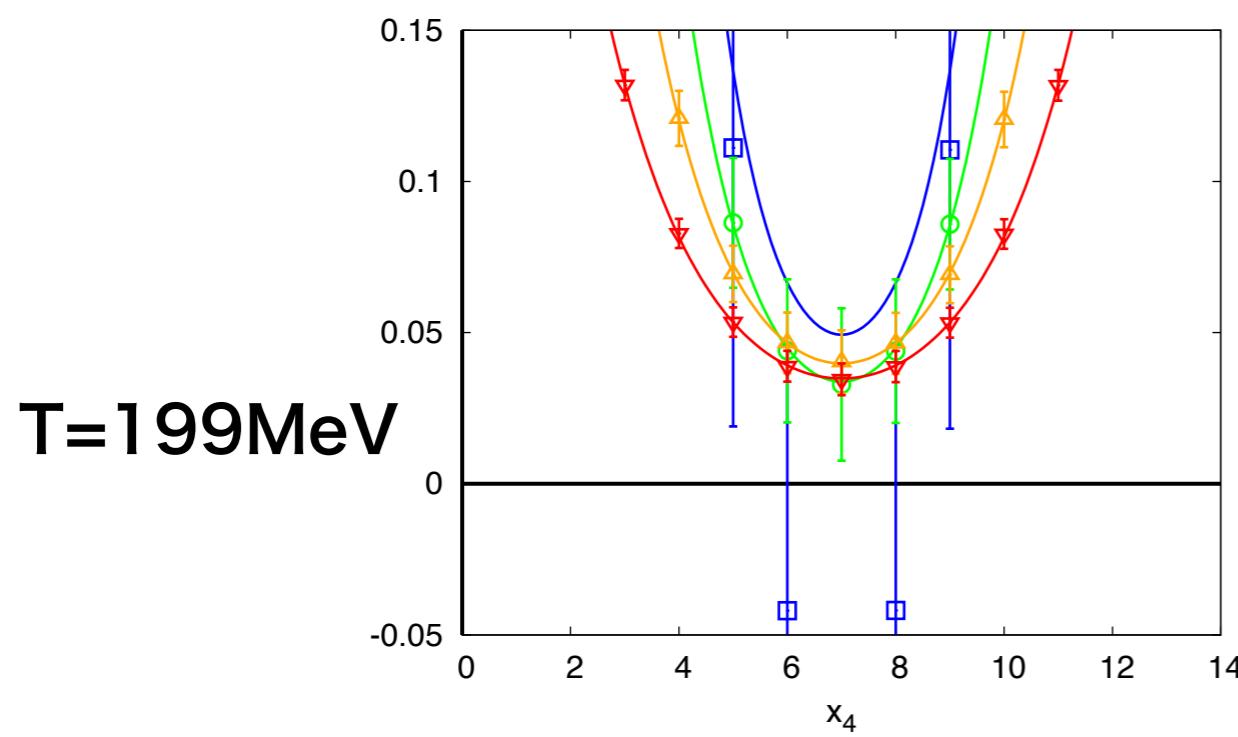
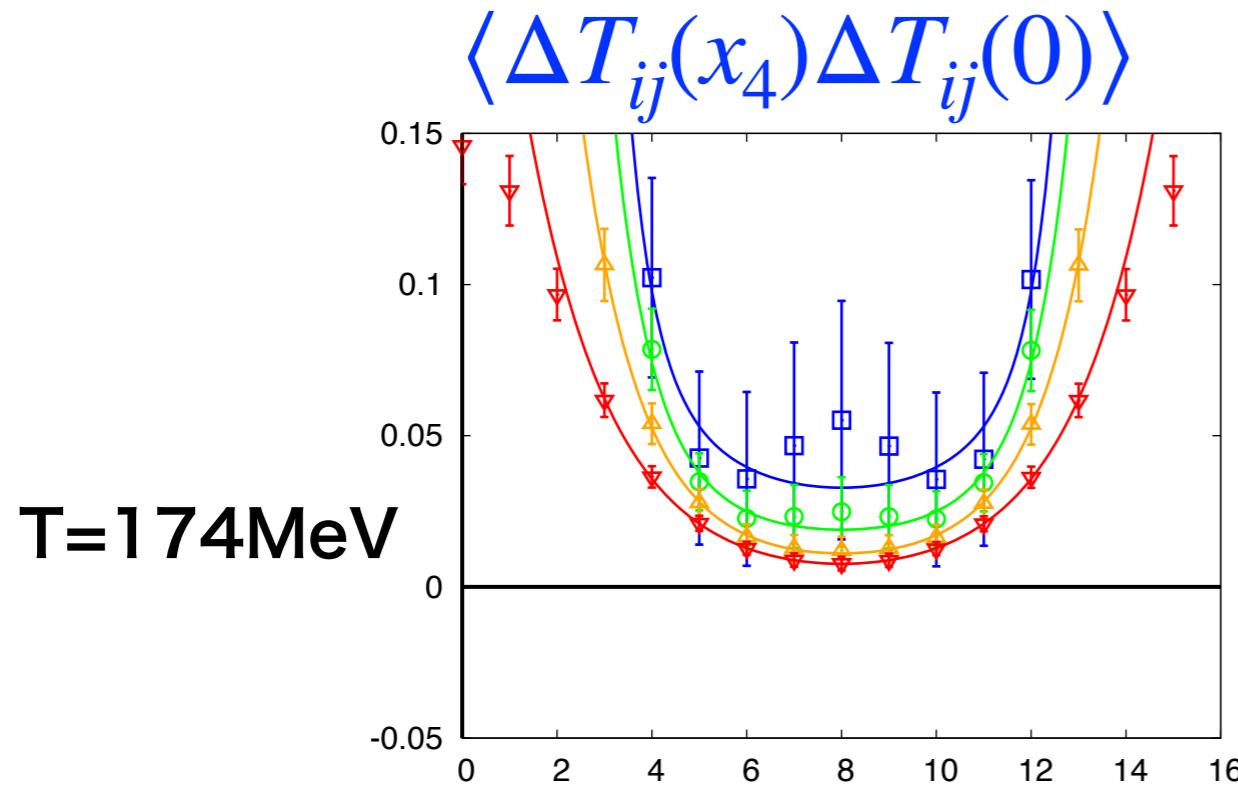


$$\eta = \frac{F}{1 + b^2 \omega_0^2}$$

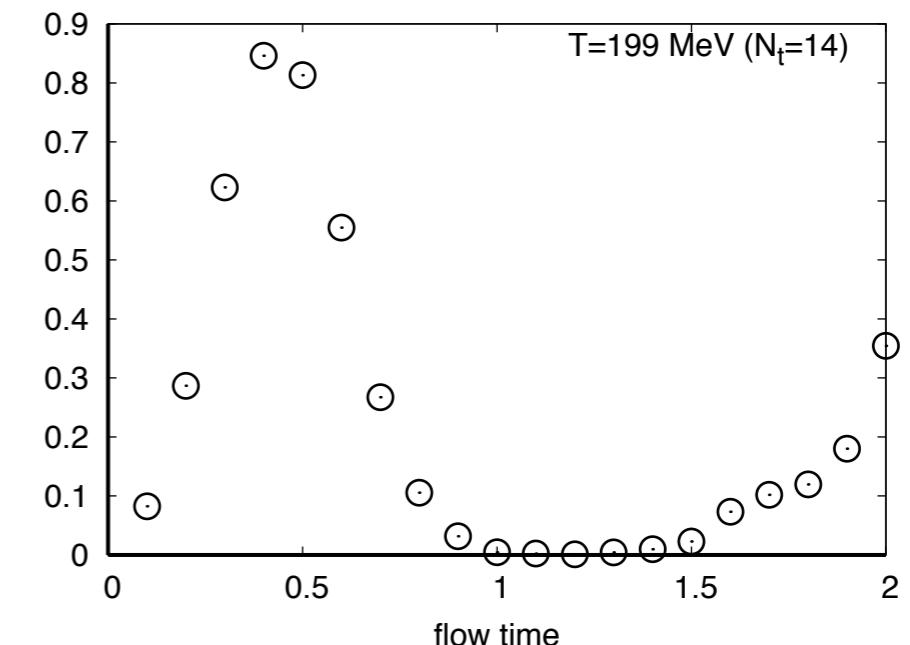
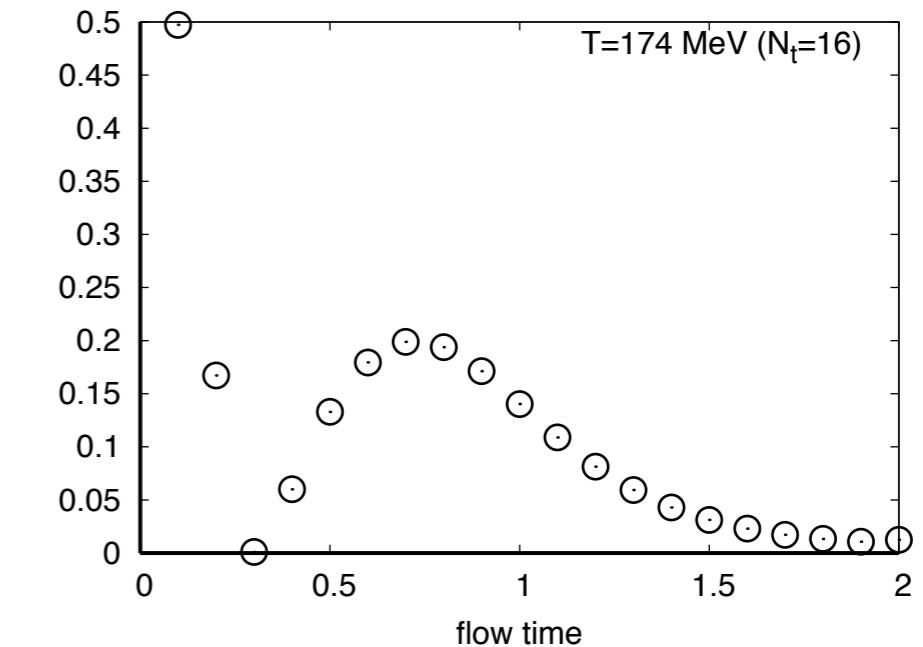
$$\frac{\eta}{s} = 0.145(51)$$

Shear viscosity

Breit-Wigner ansatz



χ^2/dof

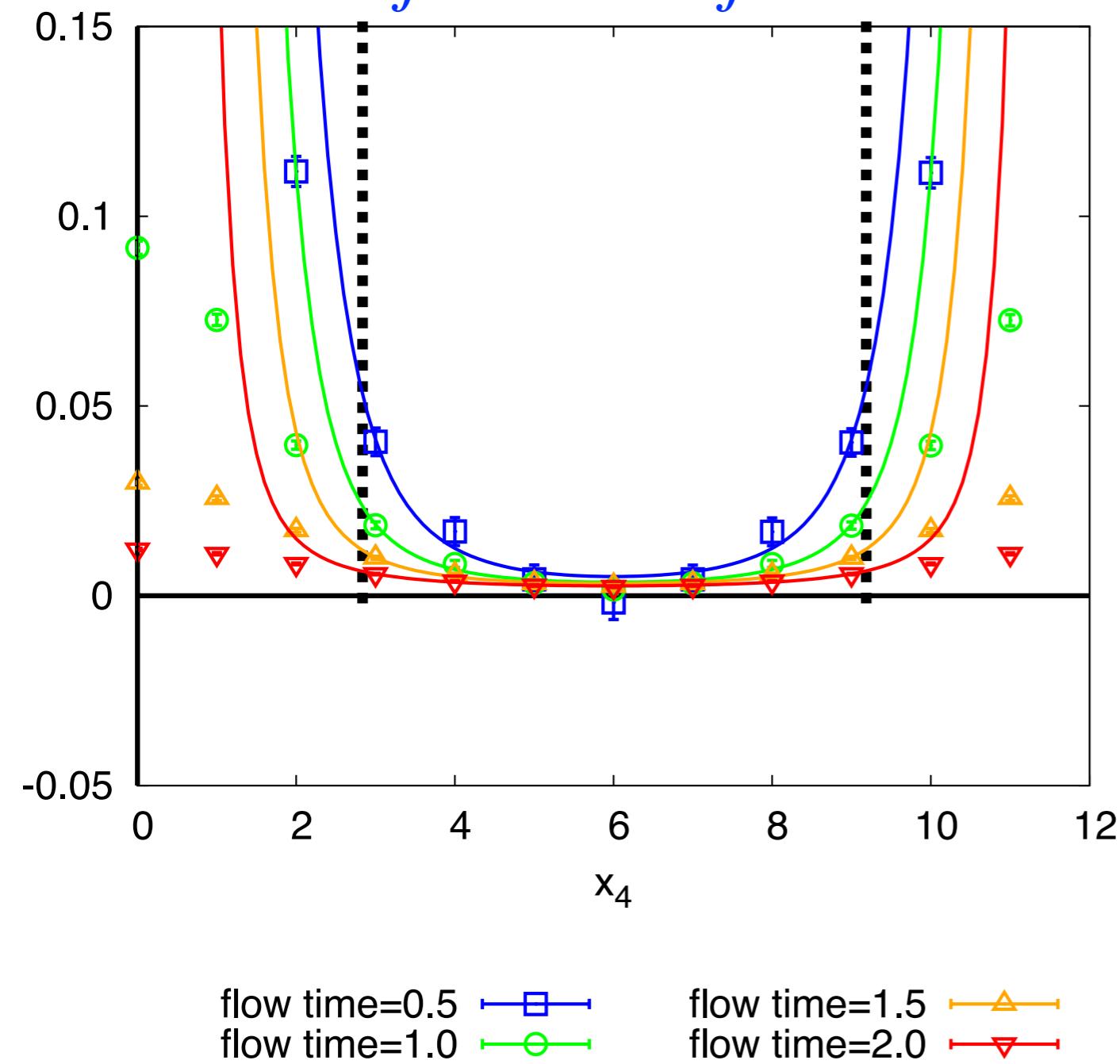


Shear viscosity

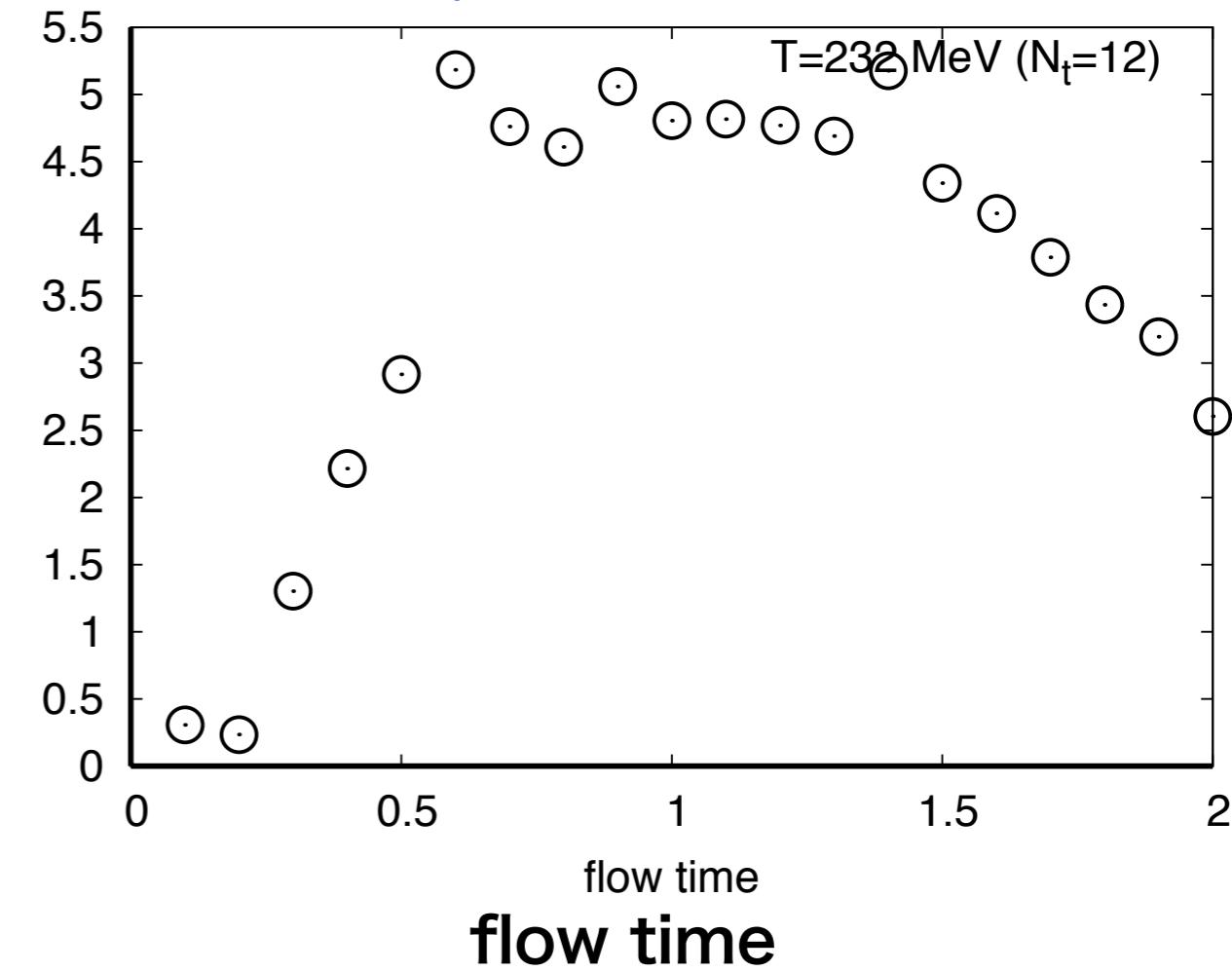
Hard thermal loop ansatz

T=232 MeV (N_t=12)

$$\langle \Delta T_{ij}(x_4) \Delta T_{ij}(0) \rangle$$



$$\chi^2/\text{dof}$$

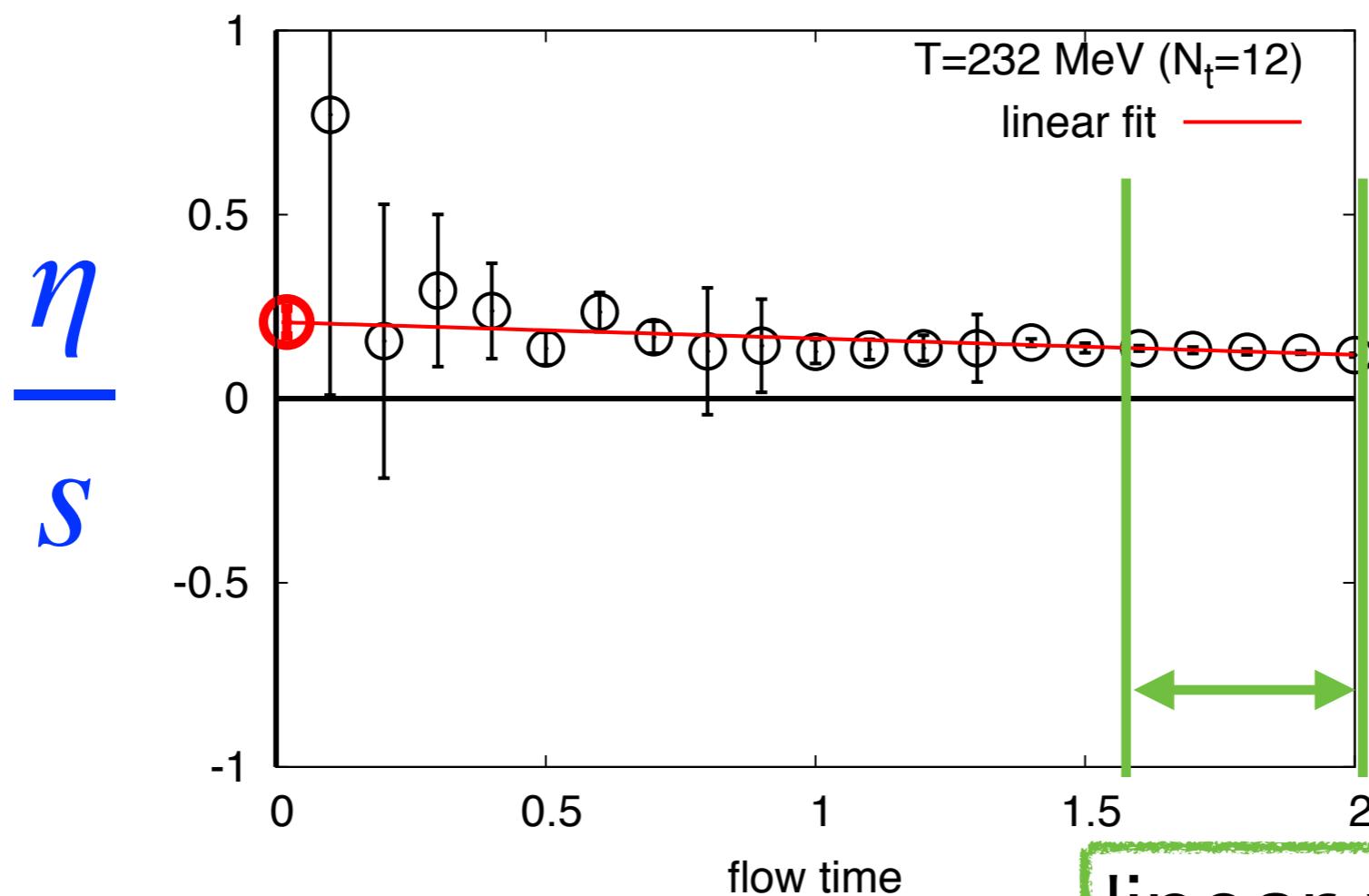


$$\frac{\rho(\omega)}{\omega} = \frac{2\eta}{1 + b^2\omega^2} + \theta(\omega - \omega_0) \frac{A\omega^3}{\tanh \frac{\omega}{4T}}$$

Shear viscosity

Hard thermal loop ansatz

T=232 MeV (N_t=12)



preliminary!

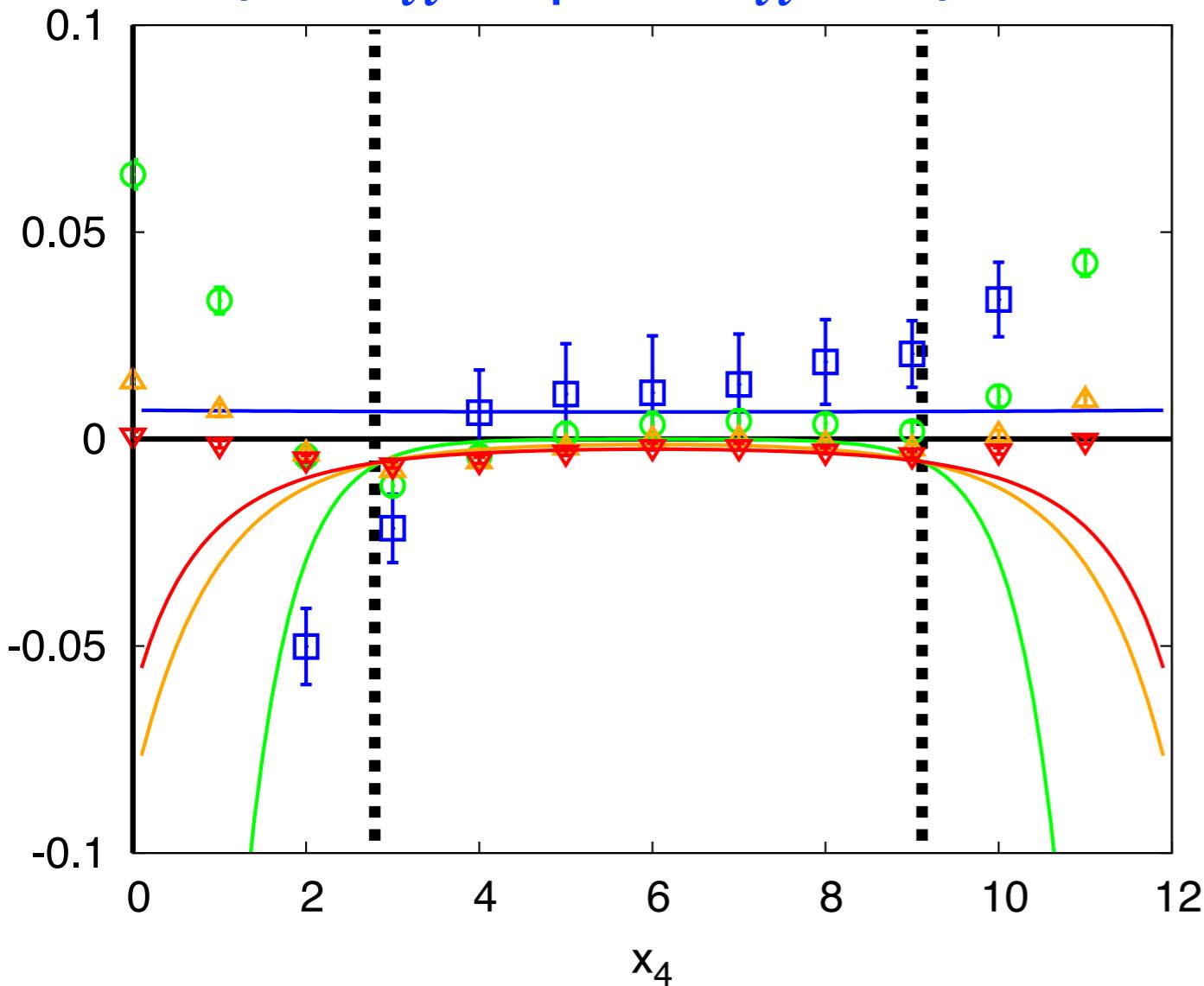
$$\frac{\eta}{s} = 0.208(38)$$

Bulk viscosity

Breit-Wigner ansatz

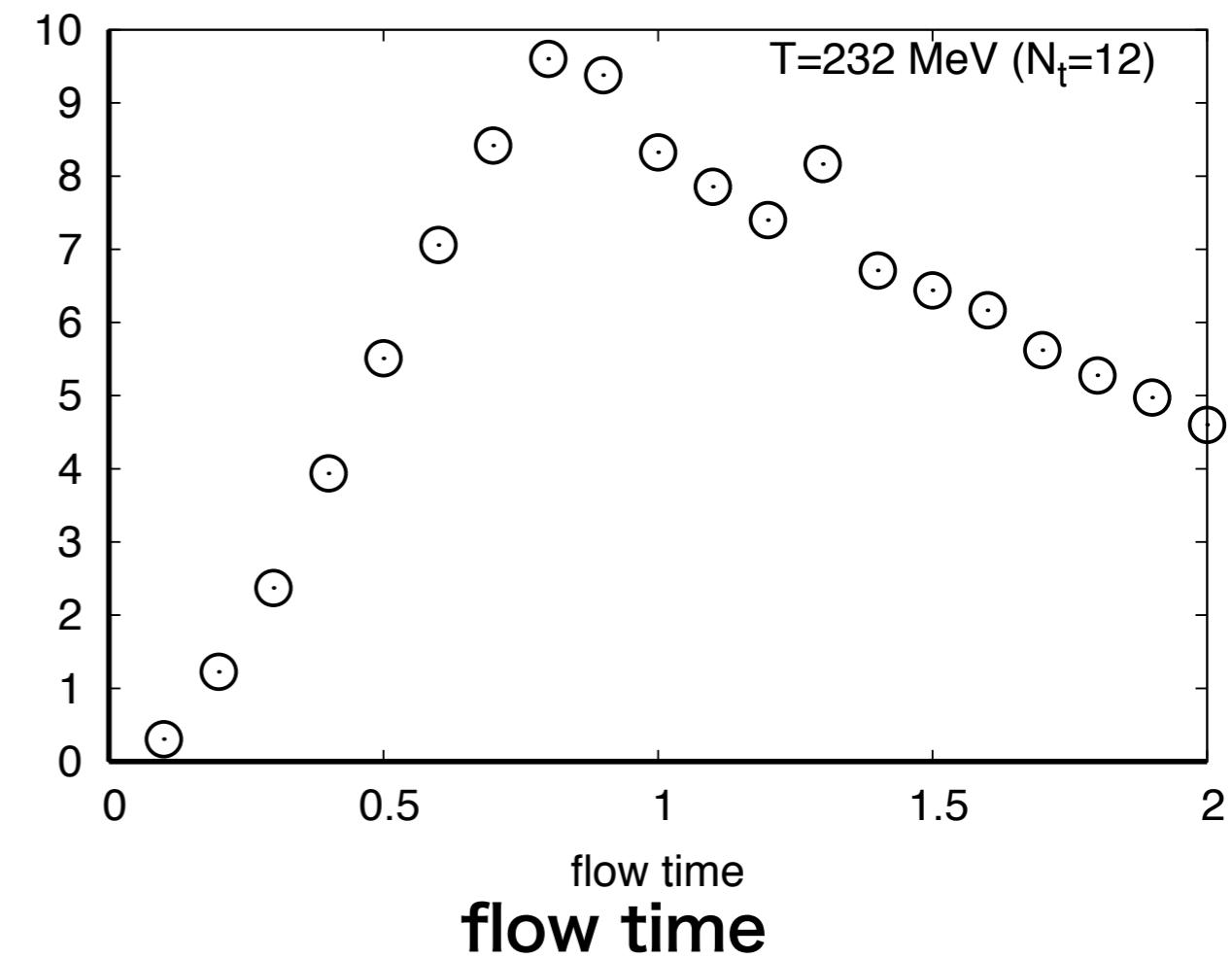
T=232 MeV (N_t=12)

$$\langle \Delta T_{ii}(x_4) \Delta T_{ii}(0) \rangle$$



flow time=0.5
 flow time=1.0
 flow time=1.5
 flow time=2.0

$$\chi^2/\text{dof}$$

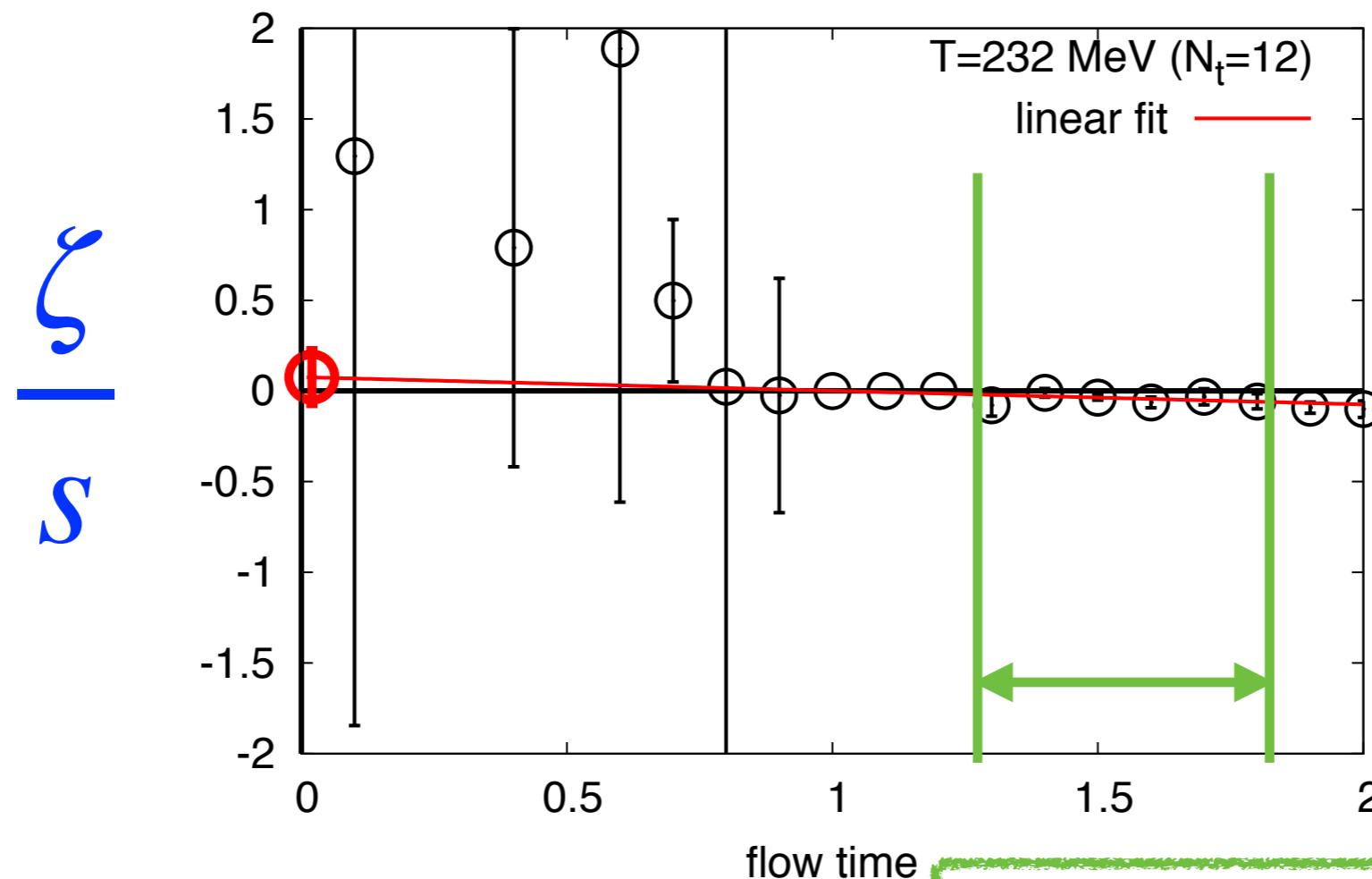


$$\frac{\rho(\omega)}{\omega} = \frac{F}{1 + b^2(\omega - \omega_0)^2} + \frac{F}{1 + b^2(\omega + \omega_0)^2}$$

Bulk viscosity

Breit-Wigner ansatz

T=232 MeV (Nt=12)



$$\zeta = \frac{F}{1 + b^2 \omega_0^2}$$

linear window

very preliminary!

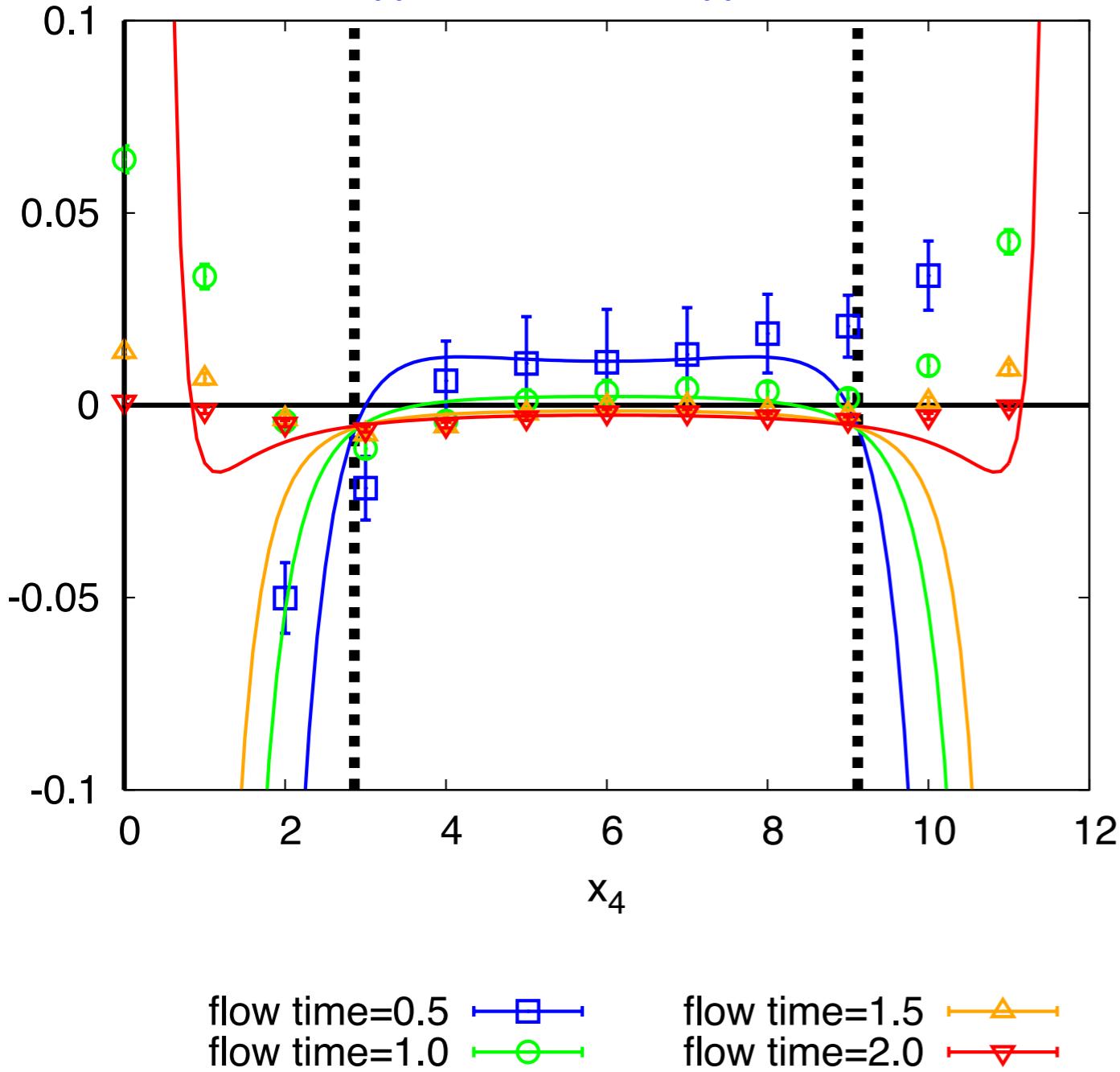
$$\frac{\zeta}{s} = 0.08(15)$$

Bulk viscosity

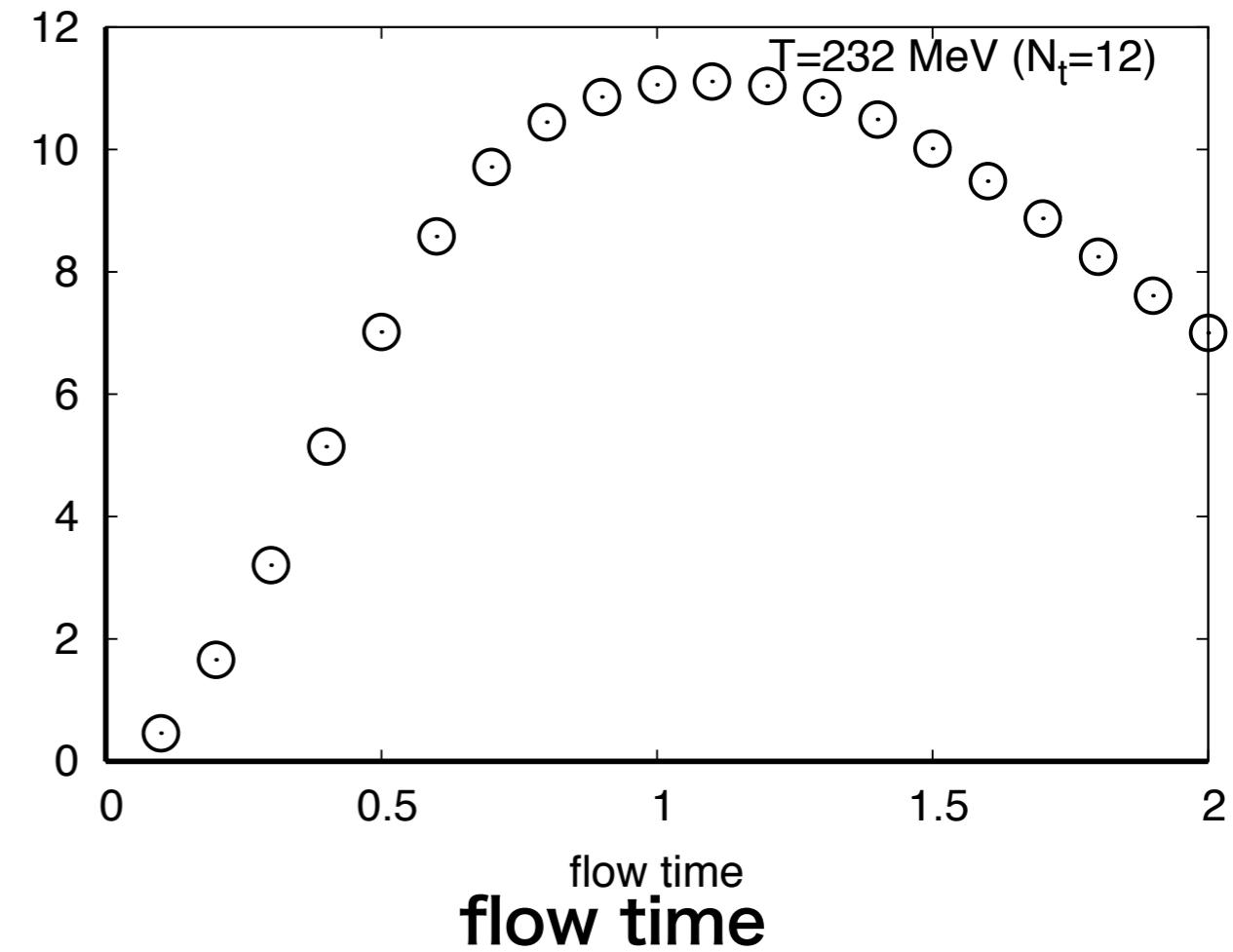
Hard thermal loop ansatz

T=232 MeV (N_t=12)

$$\langle \Delta T_{ii}(x_4) \Delta T_{ii}(0) \rangle$$



$$\chi^2/\text{dof}$$

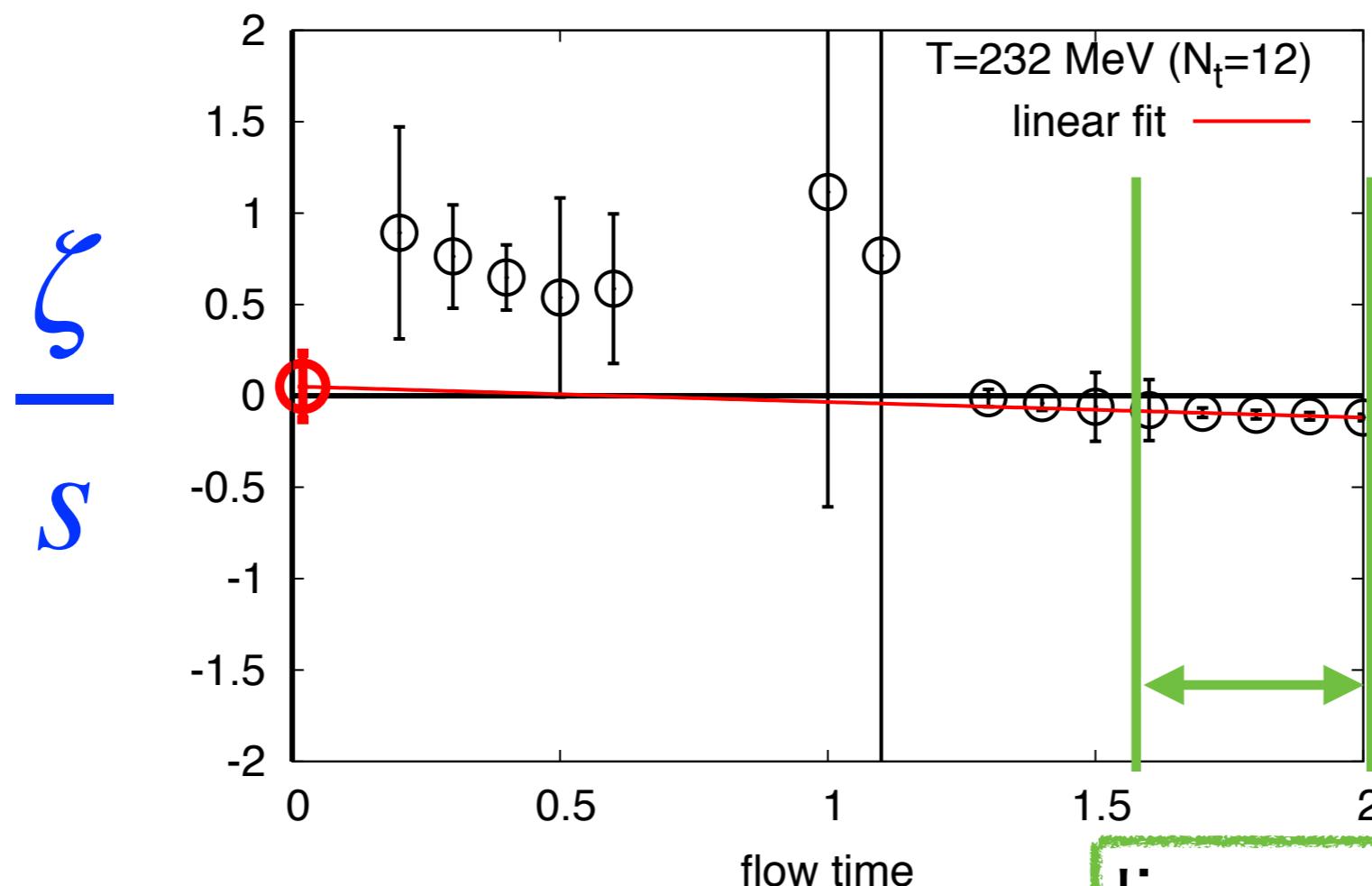


$$\frac{\rho(\omega)}{\omega} = \frac{2\eta}{1 + b^2\omega^2} + \theta(\omega - \omega_0) \frac{A\omega^3}{\tanh \frac{\omega}{4T}}$$

Bulk viscosity

Hard thermal loop ansatz

T=232 MeV (N_t=12)

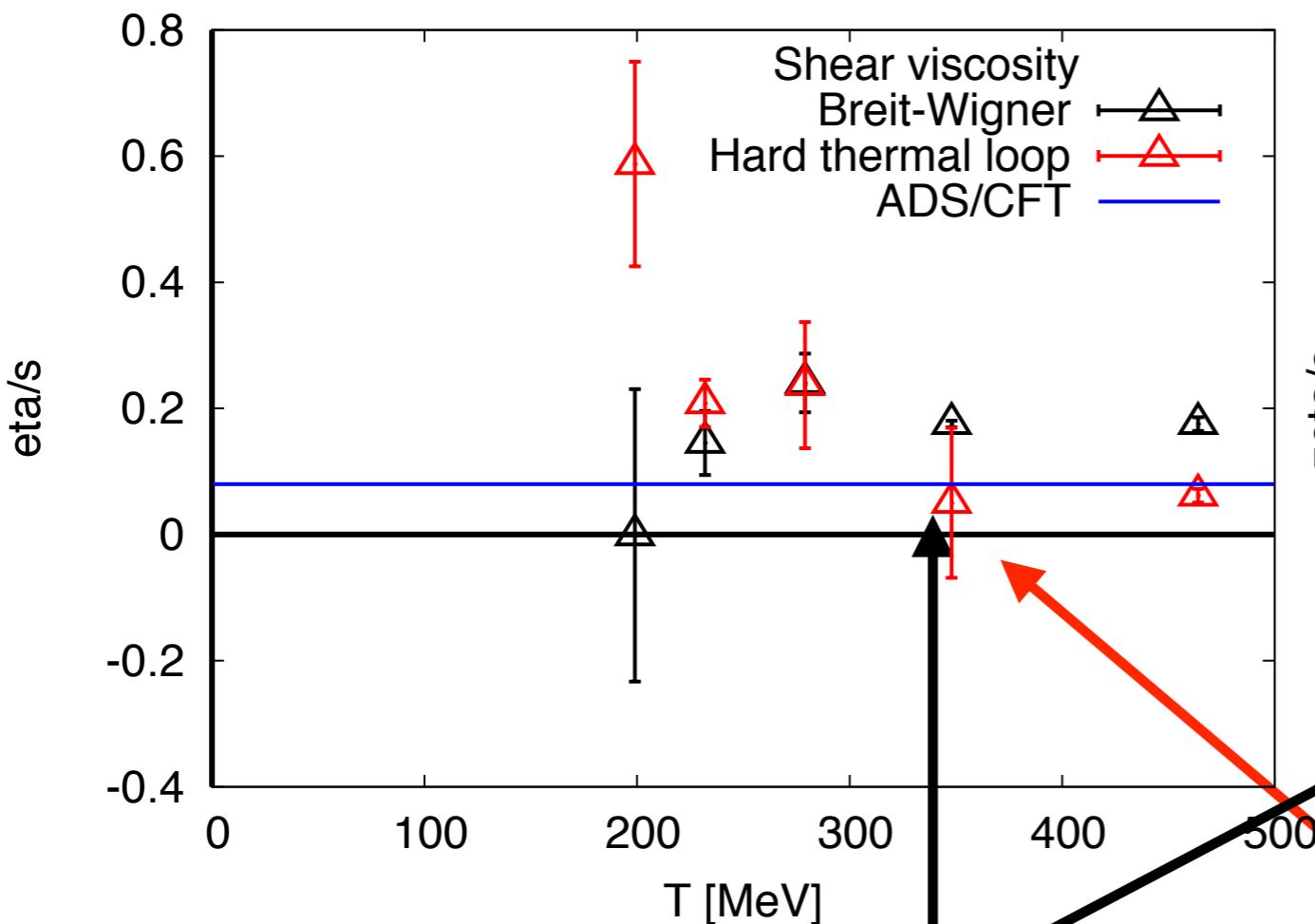


very preliminary!

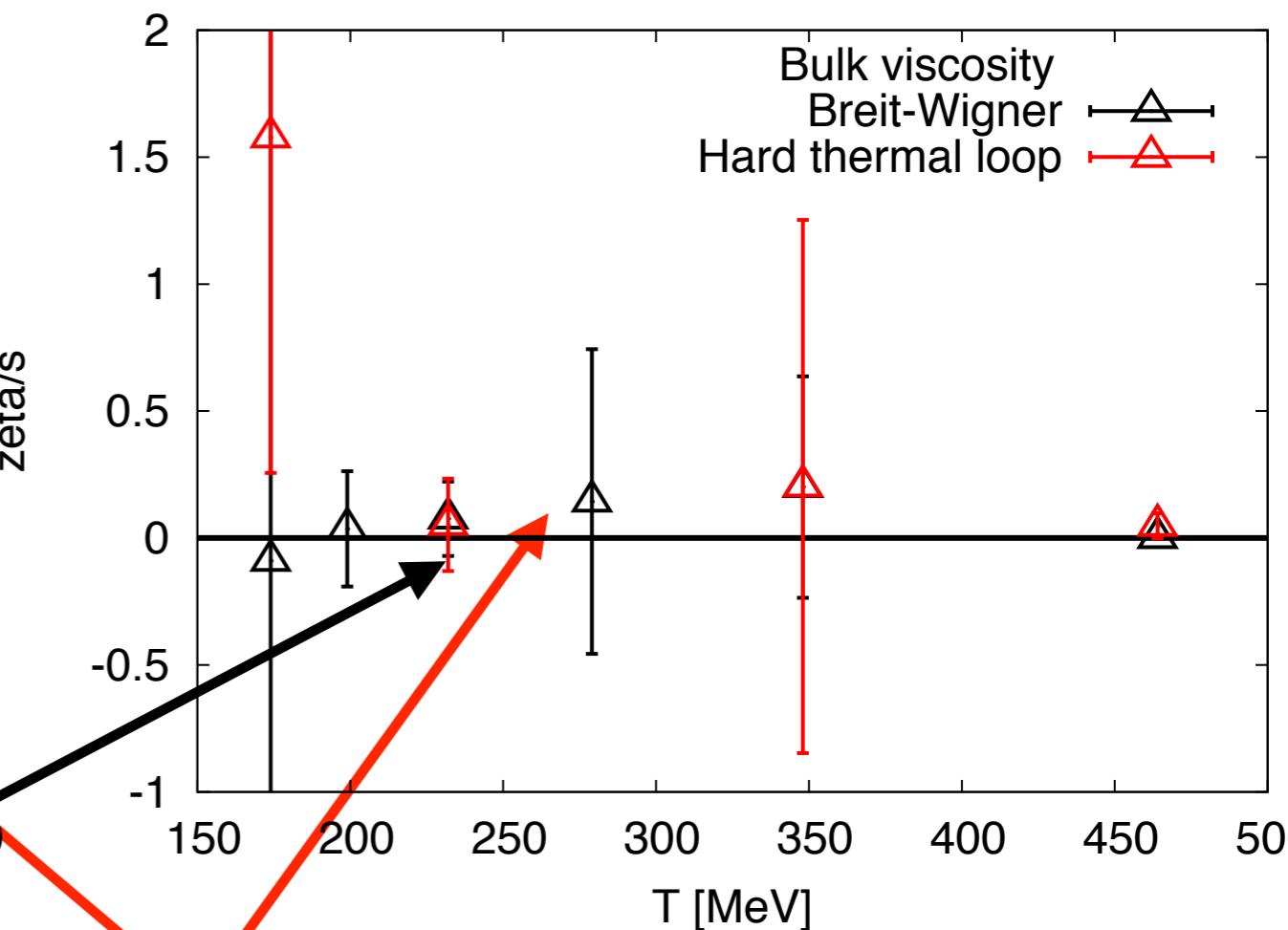
$$\frac{\zeta}{s} = 0.05(18)$$

Viscosity as a function of temperature

Shear viscosity



Bulk viscosity

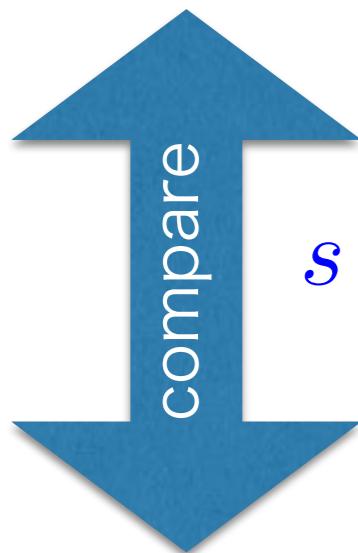


Breit-Wigner fit ansatz

Hard thermal loop fit ansatz

Check of correlation function using entropy density

Thermodynamical relation



Maxwell's relation

$$s = \left(\frac{\partial S}{\partial V} \right)_T = \left(\frac{\partial p}{\partial T} \right)_V = \frac{\epsilon + p}{T} = \frac{\langle T_{00} + T_{ii} \rangle}{T}$$

integrable condition of entropy

Linear response relation

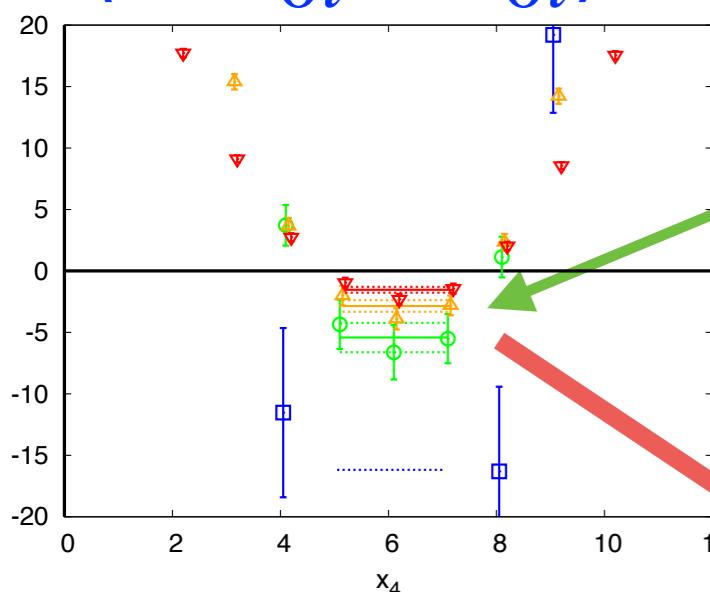
$$s = \left(\frac{\partial p}{\partial T} \right)_V = \frac{\partial \langle T_{ii} \rangle}{\partial T} \quad \langle T_{ii} \rangle = \frac{1}{Z} \text{Tr} \left(T_{ii} e^{-H/T} \right)$$

$$s = \frac{1}{T^2} \langle \delta H \delta T_{ii} \rangle = \frac{1}{T^2} \int_{V_3} d^3x (\langle \delta T_{00}(t; x_0, \vec{x}) \delta T_{ii}(t; 0) \rangle)$$

$$\epsilon + p = \left. \frac{\partial \langle T_{01} \rangle}{\partial v_1} \right|_{\vec{v}=0} = \frac{1}{T} \int_{V_3} d^3x (\langle \delta T_{0i}(t; x_0, \vec{x}) \delta T_{0i}(t; 0) \rangle)$$

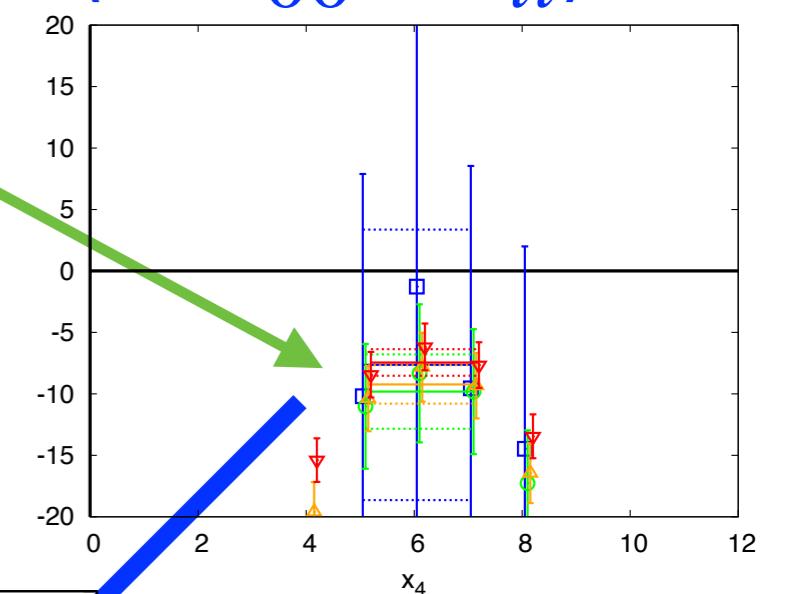
Nf=2+1 QCD

$$\langle \Delta T_{0i} \Delta T_{0i} \rangle / T^5$$



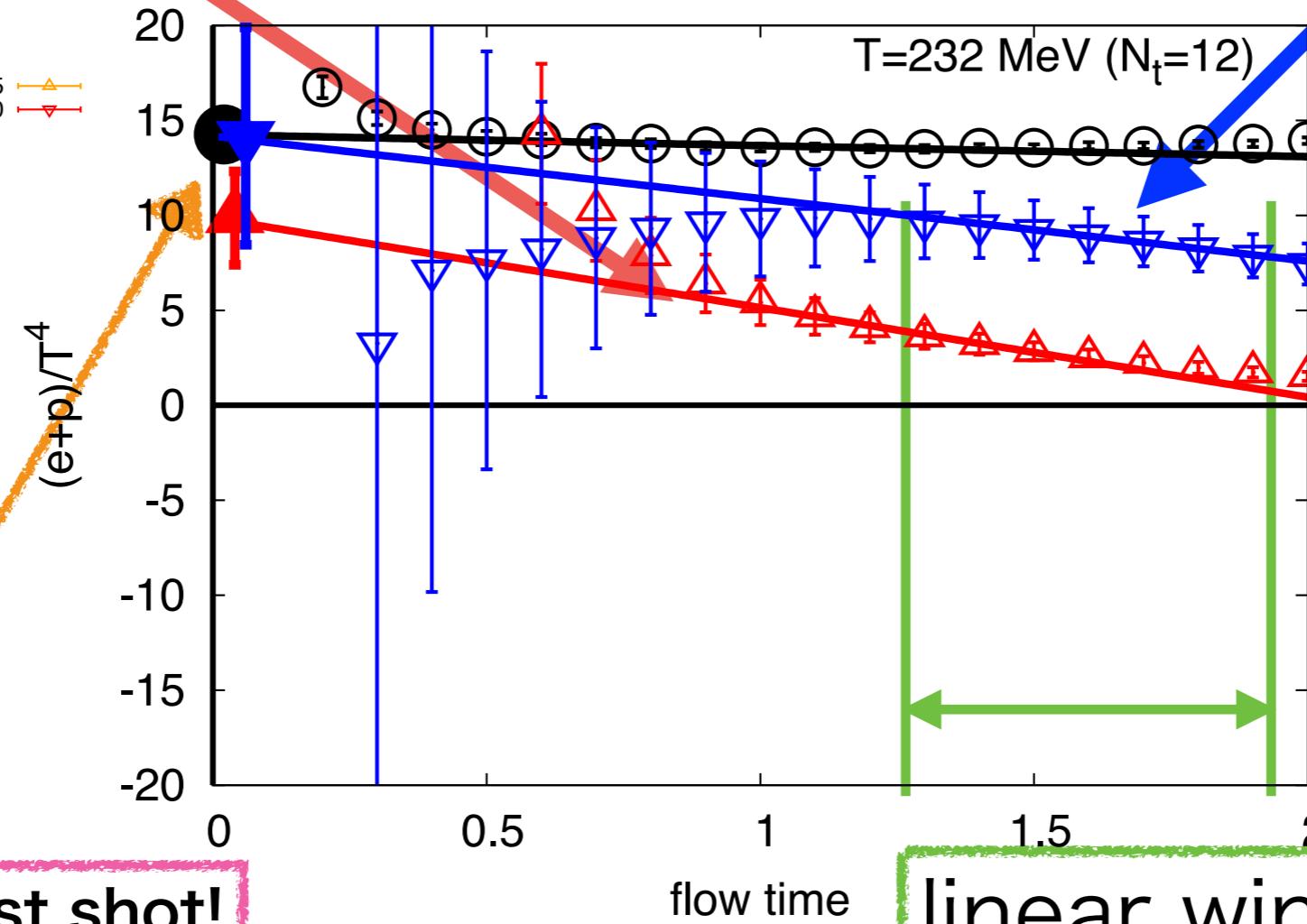
$T=232$ MeV ($N_t=12$)

$$\langle \Delta T_{00} \Delta T_{ii} \rangle / T^5$$



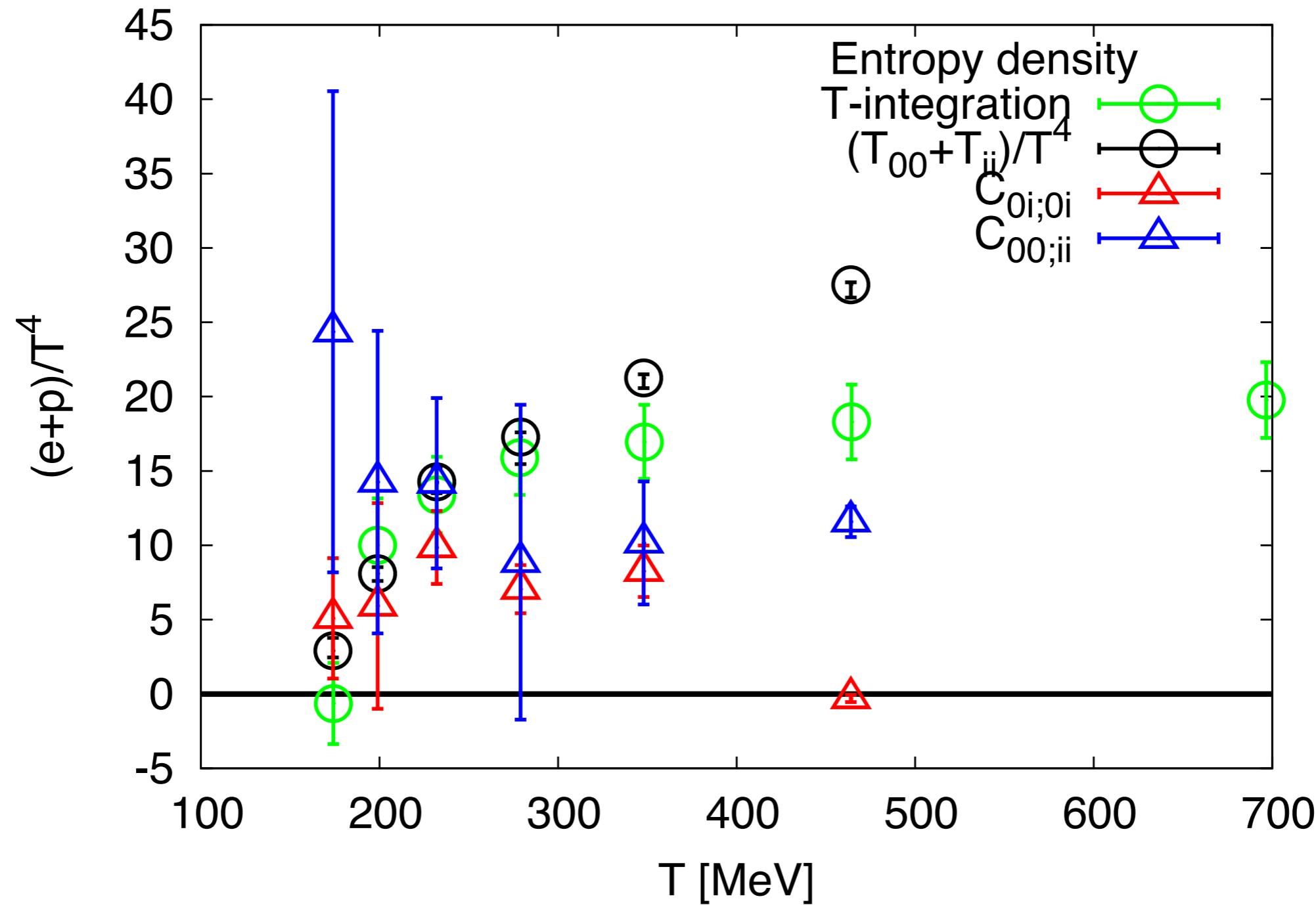
flow time=0.5 (blue squares)
flow time=1.0 (green circles)
flow time=1.5 (orange triangles)
flow time=2.0 (red inverted triangles)

agrees in
 $t \rightarrow 0$ limit



This is the best shot!

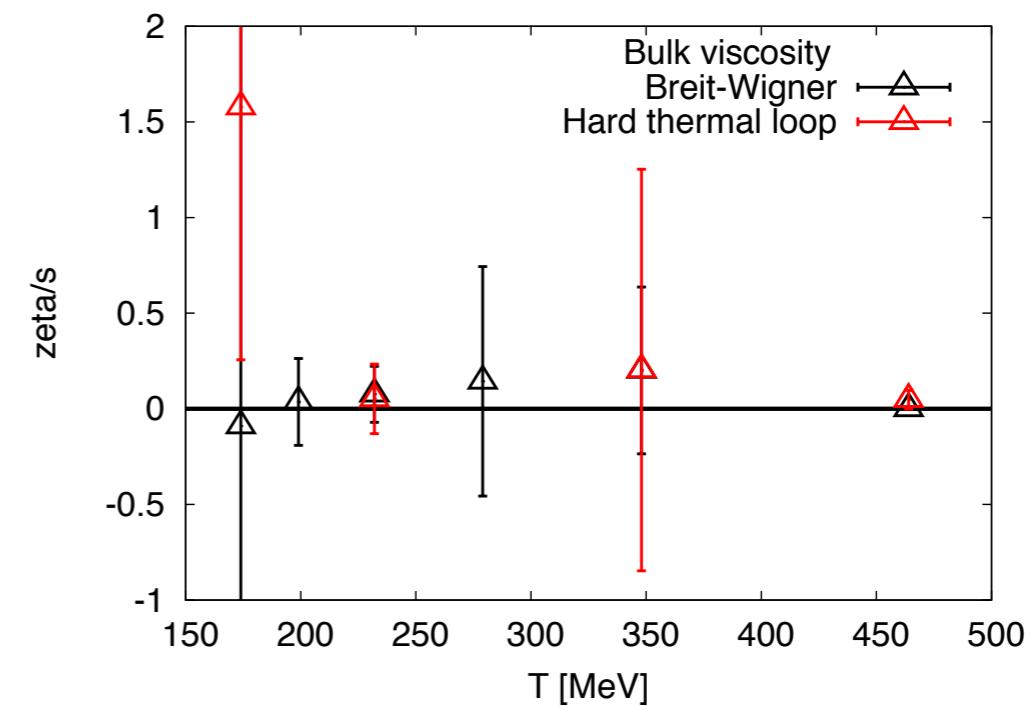
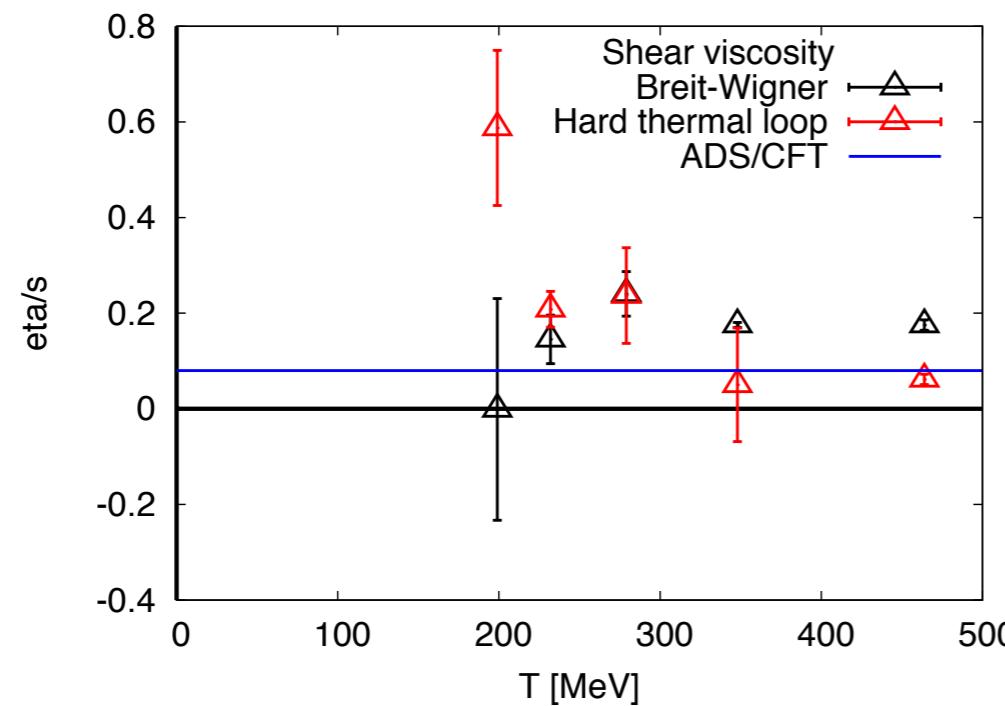
Entropy density as a function of temperature



Summary

- We calculate EM tensor correlation functions
- EM tensor is given using gradient flow
- Spectral function is given with model fit
- Viscosity is derived from spectral function

Strategy seems to work well!



Future plan

Alternative derivation of spectral function

- Maximal entropy method
- Backus-Gilbert method
- Machine learning (Heng-Tong Ding. (Fri.))