Pion Transition Form Factor from Lattice QCD in Position Space

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Matrix Element

Neutral Pion Transition Form Factor [Ji and Jung, 2001, Feng et al., 2012, Gardin et al., 2016]

$$M_{\mu\nu}(q_1, q_2) = \int d^4 u e^{-iq_1\cdot u - iq_2 \cdot v} \langle 0| T\{iJ_\mu(u) iJ_\nu(v)\} |\pi^0(p)\rangle$$
$$= \frac{i}{4\pi^2 F_\pi} \epsilon_{\mu\nu\rho\sigma} q_{1,\rho} q_{2,\sigma} F(q_1^2, q_2^2)$$

Coordinate Space Formulation

$$\langle 0| T\{iJ_\mu(u) iJ_\nu(v)\} |\pi^0(p)\rangle = \frac{i}{4\pi^2 F_\pi} \epsilon_{\mu\nu\rho\sigma} (-i\partial_\rho^u)(-i\partial_\sigma^v) \left[F'(p\cdot(u-v), (u-v)^2) e^{ipv}\right]$$

Give the Result with the Pion Operator:

$$= \langle 0| \frac{i}{4\pi^2 F_\pi} \epsilon_{\mu\nu\rho\sigma} \int_0^1 dx \left[-\partial_\rho^u F_c(x, (u-v)^2)\right] \partial_\sigma \pi^0(xu + (1-x)v) |\pi^0(\vec{p})\rangle$$

where

$$F'(p\cdot(u-v), (u-v)^2) = \int_{-\infty}^{\infty} dx F_c(x, (u-v)^2) e^{ixp\cdot(u-v)}$$

$$F_c(x, (u-v)^2) = 0 \text{ if } x < 0 \text{ or } x > 1$$

Proof

When $u_t = v_t$ (also $p \cdot (u-v)$ is a real number), define $F_c(x,(u-v)^2)$ to be the Fourier transformation of $F'(p \cdot (u-v),(u-v)^2)$:

$$F'(p \cdot (u-v),(u-v)^2) = \int_{-\infty}^{\infty} dx F_c(x,(u-v)^2) e^{ixp\cdot(u-v)}$$

Then, we have:

$$\langle 0| T\{iJ_\mu(u) iJ_\nu(v)\} \left|\pi^0(p)\right\rangle = \frac{i}{4\pi^2 F_\pi}\epsilon_{\mu\nu\rho\sigma}(-i\partial^u_\rho)(-i\partial^v_\sigma)\int_{-\infty}^{\infty} dx F_c(x,(u-v)^2) e^{ip\cdot(xu+(1-x)v)}$$

Consider the following three point function (assuming u_t, $v_t > w_t$) by inserting the pion projection operator $\hat{P}_{\pi^0} = \int \frac{d^3p}{(2\pi)^3}\left|\pi^0(\vec{p})\right\rangle \frac{1}{2E_{\pi^0,\vec{p}}}\left\langle \pi^0(\vec{p})\right|$:

$$\begin{aligned}&\langle 0| T\{iJ_\mu(u) iJ_\nu(v)\}\hat{P}_{\pi^0}\pi^0(w)|0\rangle \\ =&\frac{i}{4\pi^2 F_\pi}\epsilon_{\mu\nu\rho\sigma}\int_{-\infty}^{\infty} dx\left[-\partial^u_\rho F_c(x,(u-v)^2)\right]\partial_\sigma G(xu+(1-x)v-w)\end{aligned}$$

$F_c(x,(u-v)^2)$ in $x \in [0,1]$

Select $\omega = x'u + (1-x')v + \varepsilon$, where ε is a very small distance, we get:

$$\langle 0| T\{iJ_\mu(u)iJ_\nu(v)\}\hat{P}_{\pi^0}\pi^0(\omega)|0\rangle$$
$$=\frac{i}{4\pi^2 F_\pi}\epsilon_{\mu\nu\rho\sigma}\int_{-\infty}^{\infty} dx \left[-\partial_\rho^u F_c(x,(u-v)^2)\right]\partial_\sigma G((x-x')(u-v)-\varepsilon)$$

In the area of $x' > 1$ or $x' < 0$, $\langle 0| T\{iJ_\mu(u)iJ_\nu(v)\}\hat{P}_{\pi^0}\pi^0(\omega)|0\rangle$ should be finite when $\varepsilon \to 0$, but the above formula suggest a singularity behavior of $\partial_\sigma G((x-x')(u-v)-\varepsilon)$. This implies that:

$$F_c(x,(u-v)^2) = 0 \text{ if } x < 0 \text{ or } x > 1$$

Then the integral becomes from 0 to 1:

$$\langle 0| T\{iJ_\mu(u)iJ_\nu(v)\}\left|\pi^0(\vec{p})\right\rangle$$
$$=\frac{i}{4\pi^2 F_\pi}\epsilon_{\mu\nu\rho\sigma}\int_0^1 dx \left[-\partial_\rho^u F_c(x,(u-v)^2)\right] ip_\sigma e^{ip\cdot(xu+(1-x)v)}$$

or, in pion operator expression:

$$=\langle 0|\frac{i}{4\pi^2 F_\pi}\epsilon_{\mu\nu\rho\sigma}\int_0^1 dx \left[-\partial_\rho^u F_c(x,(u-v)^2)\right]\partial_\sigma\pi^0(xu+(1-x)v)\left|\pi^0(\vec{p})\right\rangle$$

Convert F_c to Momentum Space Form Factor and the Property of F_c

Let $v = 0$ and $p = q_1 + q_2$, we get mapping from F_c to momentum space form factor:

$$F(q_1^2, q_2^2) = \int d^4u\ e^{-iq_1 \cdot u} \int_0^1 dx\ F_c(x, u^2) e^{ixp \cdot u}$$

In the Chiral limit and $\vec{p} = 0$, $q_1 = 0$, $F(q_1^2 \to 0, q_2^2 \to 0) = 1$, and:

$$\int d^4u \int_0^1 dx\ F_c(x, u^2) = 1$$

OPE analysis

When u and v are very close, we have such approximation [Gerardin et al., 2016]:

$$T\psi(u)\bar{\psi}(v) \approx \int \frac{d^4p}{(2\pi)^4}\frac{e^{ip\cdot(u-v)}}{ip_\rho\gamma_\rho + m} = \frac{(u-v)_\rho\gamma_\rho}{2\pi^2((u-v)^2)^2}$$

$$\begin{aligned}
&T\left[i\bar{\psi}(u)\gamma_\mu\psi(u)\right]\left[i\bar{\psi}(v)\gamma_\mu\psi(v)\right] \\
\approx& -\frac{(u-v)_\rho}{2\pi^2((u-v)^2)^2}\bar{\psi}(u)\gamma_\mu\gamma_\rho\gamma_\nu\psi(v) - \frac{(v-u)_\rho}{2\pi^2((v-u)^2)^2}\bar{\psi}(v)\gamma_\nu\gamma_\rho\gamma_\mu\psi(u) \\
=& -\frac{\epsilon_{\mu\nu\rho\sigma}(u-v)_\rho}{2\pi^2((u-v)^2)^2}\left[\bar{\psi}(u)\gamma_\sigma\gamma_5\psi(v) + \bar{\psi}(v)\gamma_\sigma\gamma_5\psi(u)\right]
\end{aligned}$$

Consider the definition of pion form factor:

$$\langle 0|\,\bar{u}(u)\gamma_\sigma\gamma_5 u(u)\,|\pi^0(\vec{p})\rangle = F_\pi p_\sigma e^{ip\cdot u} = \langle 0| - iF_\pi\partial_\sigma\pi^0(u)\,|\pi^0(\vec{p})\rangle$$

Analogously, the two current operator can be shown as follow:

$$T\{iJ_\mu(u)iJ_\nu(v)\} \xrightarrow{\mu,\ \nu \text{ are close}} \frac{i}{4\pi^2 F_\pi}\epsilon_{\mu\nu\rho\sigma}2(u-v)_\rho\left[\frac{2F_\pi^2}{3}\frac{1}{((u-v)^2)^2}\right]\partial_\sigma\pi^0\left(\frac{u+v}{2}\right)$$

Parametrization

Do Fourier series expansion of $F_c(x,(u-v)^2)$ (or other parametrization), using the fact that $0 \leq x \leq 1$:

$$-\partial_\rho^u F_c(x,(u-v)^2) = 2(u-v)_\rho \left[\frac{2F_\pi^2}{3}\frac{1}{((u-v)^2)^2}\right] \times \sum_{n=0}^{\infty} f_n(|u-v|)\frac{(2n+1)\pi}{2} sin((2n+1)\pi x)$$

$$\begin{aligned} &\langle 0| \, TiJ_\mu(u)iJ_\nu(v) \left|\pi^0(\vec{p})\right\rangle \\ =&\frac{i}{4\pi^2 F_\pi}\epsilon_{\mu\nu\rho\sigma}2(u-v)_\rho ip_\sigma \left[\frac{2F_\pi^2}{3}\frac{1}{((u-v)^2)^2}\right] \\ &\times \sum_{n=0}^{\infty} f_n(|u-v|)\frac{(2n+1)\pi}{2}\int_0^1 sin((2n+1)\pi x)e^{ip\cdot(xu+(1-x)v)}dx \end{aligned}$$

Study r Dependence of the Coordinate Space Formulation

Define $f(|r|)$:

$$\int_0^1 dx \left[-\partial^u_\rho F_c(x, r^2)\right] = 2r_\rho \left[\frac{2F_\pi^2}{3}\frac{1}{(r^2)^2}\right] f(|r|)$$

that is:

$$f(|r|) = \sum_{n=0}^{\infty} f_n(|r|)$$

Let $u = r/2$, $v = -r/2$ and $r_t = 0$, $\vec{p} = 0$, we have:

$$\langle 0| \, TiJ_\mu(0, \vec{r}/2) iJ_\nu(0, -\vec{r}/2) \left|\pi^0(\vec{p} = 0)\right\rangle = \frac{i}{4\pi^2 F_\pi}\epsilon_{\mu\nu\rho\sigma} 2r_\rho i p_\sigma \left[\frac{2F_\pi^2}{3}\frac{1}{(r^2)^2}\right] f(|r|)$$

Remember we also have the constrains from the pion decay width, which imply $f(|r|)$ should satisfy:

$$\frac{\pi^2}{2}\int_0^\infty \frac{2F_\pi^2}{3} f(|r|) 2r dr = 1$$

Point Source Propagator and Contraction

Propagator:

- 1024 point source propagator in each configuration
- 256 random area group chosen from the lattice
- 4 random points per group

Three-Point Correlation Function:

$$C^{\text{conn}}_{\mu\nu}(|\vec{r}|, \vec{p}, t_\pi) = \sum_{|\vec{r'}|=|\vec{r}|, \vec{x}} \langle 0| J_\mu(0, \vec{r'}/2) J_\nu(0, -\vec{r'}/2) P^+(\vec{x}, t_\pi) |0\rangle \, e^{i\vec{p}\cdot\vec{x}}$$

Ensembles

- 24c64 Ensemble

Observable	Fit	% err.
$am'_{\text{res}}(m_l)$	0.0022824(70)	0.31
am_π	0.13975(10)	0.07
am_K	0.504154(89)	0.02
am_Ω	1.6726(25)	0.15
am'_Ω	2.040(63)	3.09
af_π	0.13055(11)	0.09
af_K	0.15815(13)	0.09
Z_A	0.73457(11)	0.02
Z_V^π	0.72672(35)	0.05
Z_V^K	0.7390(11)	0.15
$aE^2_{\pi\pi}$	0.28175(21)	0.08
$a\delta E^2_\pi$	0.002246(52)	2.31
ap	0.01775(21)	1.16
$\delta^2_0(p)$	-0.339(12)°	3.40
$m_\pi a^2_0$	-0.0464(10)	2.23

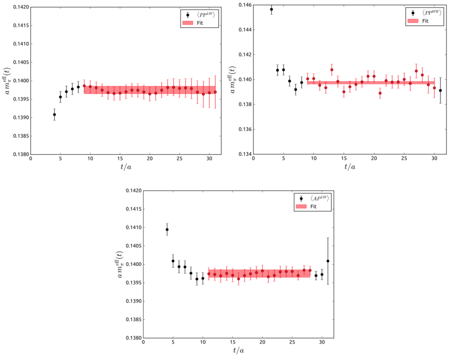
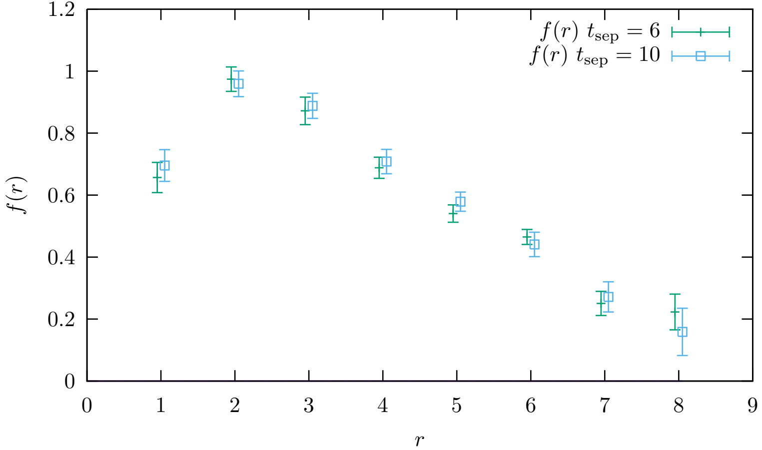


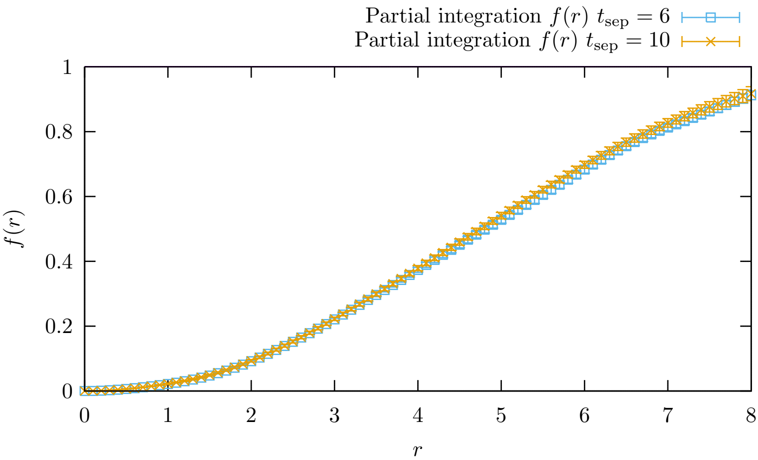
Figure: pion mass

Plots

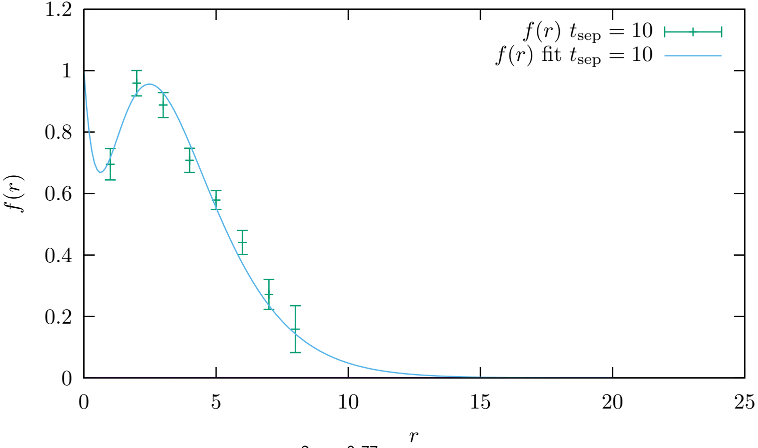
24c64 Lattice, 1024 Point Source Propagator



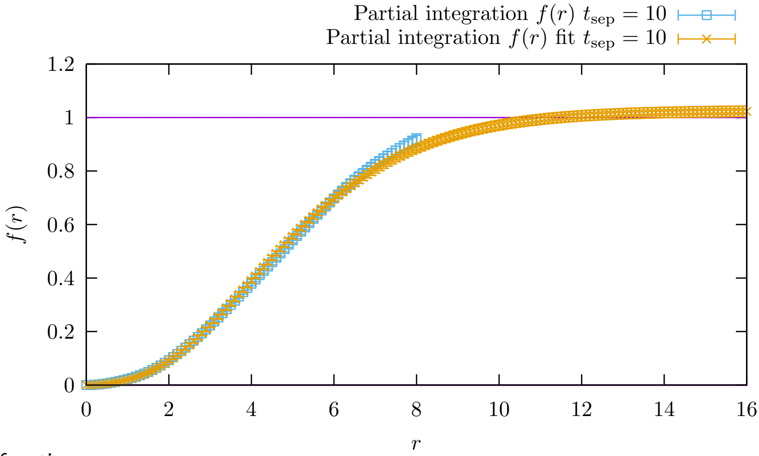
$$\int_0^1 dx \left[-\partial_\rho^u F_c(x, r^2)\right] = 2r_\rho \left[\frac{2F_\pi^2}{3}\frac{1}{(r^2)^2}\right] f(|r|)$$



$$\int_0^1 dx \left[-\partial_\rho^u F_c(x, r^2)\right] = 2r_\rho \left[\frac{2F_\pi^2}{3}\frac{1}{(r^2)^2}\right] f(|r|)$$



Fitting Formular: $f(r) = (c_0 + c_1 r + c_2 r^2)e^{-0.77r}$
based on 24c lattice, 16 configurations.



Integral function:

$$\frac{\pi^2}{2}\int_0^\infty \frac{2F_\pi^2}{3} f_{\text{fit}}(r)2rdr$$