Relation between scattering amplitude and Bethe-Salpeter wave function in quantum field theory

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Reference: TY and Kuramashi, PRD96:114511,11(2017)

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Purpose

(re)introduce a simple relation between scattering amplitude

and Bethe-Salpeter (BS) wave function inside interaction range R

rather than HALQCD method

Outline

- Wave function in finite volume method
- BS wave function outside R
- BS wave function inside R
- Fundamental relation in quantum mechanics
- Expansion of reduced BS wave function
- Summary

Lüscher's finite volume method

[Lüscher, NPB354:531(1991)]

spinless two-particle elastic S-wave scattering in center of mass frame

Quantum mechanics

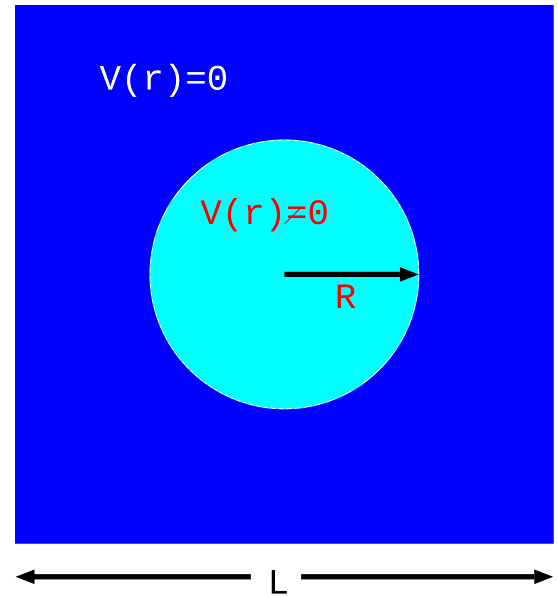
Important assumption

1. Two-particle interaction is localized.

   $\to$ Interaction range R exists.

   $$V(r) \begin{cases} \neq 0 & (r \leq R) \\ = 0 \ (\sim e^{-cr}) & (r > R) \end{cases}$$

2. $V(r)$ is not affected by boundary. $\to R < L/2$



Two-particle wave function $\phi(\vec{r};k)$ in $r > R$ satisfies Helmholtz equation.

$$\left(\Delta + k^2\right)\phi(\vec{r};k) = 0 \text{ in } r > R, \quad E_k = 2\sqrt{m^2 + k^2}$$

Lüscher's finite volume method

[Lüscher, NPB354:531(1991)]

Helmholtz equation on L^3

1. Solution of $(\Delta + k^2)\phi(\vec{r};k) = 0$ in $r > R$

$$\phi(\vec{r};k) = G(\vec{r};k) = C \cdot \sum_{\vec{n}\in Z^3} \frac{e^{i\vec{r}\cdot\vec{n}(2\pi/L)}}{\vec{n}^2 - q^2}, \quad q^2 = (Lk/2\pi)^2 \neq \text{integer}$$

2. Expansion by spherical Bessel $j_l(kr)$ and Neumann $n_l(kr)$ functions

$$\begin{aligned}\phi(\vec{r};k) &= \beta_0(k) n_0(kr) + \alpha_0(k) j_0(kr) + (l \geq 4) \\ &= e^{i\delta(k)} \sin(kr + \delta(k))/kx + (l \geq 4)\end{aligned}$$

3. S-wave scattering phase shift $\delta(k)$ in infinite volume

$$\frac{\beta_0(k)}{\alpha_0(k)} = \boxed{\tan\delta(k) = \frac{\pi^{3/2} q}{Z_{00}(1;q^2)}} \quad Z_{00}(s;q^2) = \frac{1}{\sqrt{4\pi}} \sum_{\vec{n}\in Z^3} \frac{1}{(\vec{n}^2 - q^2)^s}$$

Relation between $\boxed{\delta(k) \text{ and } k}$ $\left(E_k = 2\sqrt{m^2 + k^2}\right)$

$\phi(\vec{r};k)$ disappears in final formula.

BS wave function through LSZ reduction formula

Quantum field theory

[Lin *et al.*, NPB619:467(2001)]

BS wave function of two pions in infinite volume (Only S-wave)

$$\phi(x;k) = \langle 0|\pi_1(\vec{x}/2)\pi_2(-\vec{x}/2)|\pi_1(\vec{k})\pi_2(-\vec{k});\text{in}\rangle$$

Inelastic scattering contribution and unnecessary overall factors are neglected.

NOT exactly same as one in BS equation

$\phi(x;k)$ from 4-point correlation function $C(\vec{x},t)$ on lattice

$$\begin{aligned} C(\vec{x},t-t_s) &= \langle 0|\pi_1(\vec{x}/2,t)\pi_2(-\vec{x}/2,t)\Omega_{\pi\pi}(t_s)|0\rangle \\ &= \sum_k C_k \phi(x;k) e^{-E_k(t-t_s)} \end{aligned}$$

where $\Omega_{\pi\pi}(t_s)$ = two-pion operator, $C_k = \langle 0|\Omega_{\pi\pi}(0)|E_k\rangle$

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Inelastic scattering contribution and unnecessary overall factors are neglected.

NOT exactly same as one in BS equation

Half off-shell amplitude $H(p;k)$

$$H(p;k) = \frac{E_p+E_k}{8E_pE_k}M(p;k)$$

$M(p;k)$ defined by LSZ reduction formula

$$e^{-i\mathbf{q}\cdot\mathbf{x}}\frac{-i\sqrt{Z}M(p;k)}{-\mathbf{q}^2+m^2-i\varepsilon} = \int d^4z d^4y_1 d^4y_2 K(\mathbf{p},\mathbf{z})K(-\mathbf{k}_1,\mathbf{y}_1)K(-\mathbf{k}_2,\mathbf{y}_2)G_4(\mathbf{z},\mathbf{x},\mathbf{y}_1,\mathbf{y}_2)$$

$$K(\mathbf{p},\mathbf{z}) = \frac{i}{\sqrt{Z}}e^{i\mathbf{p}\cdot\mathbf{z}}(-\mathbf{p}^2+m^2), \quad G_4(\mathbf{z},\mathbf{x},\mathbf{y}_1,\mathbf{y}_2) = \langle 0|T[\pi_1(\mathbf{z})\pi_2(\mathbf{x})\pi_1(\mathbf{y}_1)\pi_2(\mathbf{y}_2)|0\rangle$$

$$\mathbf{p} = (E_p,\vec{p}), \quad \mathbf{k}_1 = (E_k,\vec{k}), \quad \mathbf{k}_2 = (E_k,-\vec{k}), \quad \underline{\mathbf{q} = (2E_k-E_p,-\vec{p})}$$

off-shell momentum

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Inelastic scattering contribution and unnecessary overall factors are neglected.

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Half off-shell amplitude $H(p;k)$

$$H(p;k) = \frac{E_p+E_k}{8E_pE_k}M(p;k)$$

$M(p;k)$ at on-shell $p=k$

$$M(k;k) = \frac{16\pi E_k}{k}e^{i\delta(k)}\sin\delta(k) \Rightarrow H(k;k) = \frac{4\pi}{k}e^{i\delta(k)}\sin\delta(k)$$

BS wave function outside R

[CP-PACS, PRD71:094504(2005)]

Quantum field theory

Reduced BS wave function

$$h(x;k) = (\Delta + k^2)\phi(x;k)$$

Assumption: $h(x;k) = 0$ outside interaction range ($x > R$)

c.f. $(\Delta + k^2)\phi(x;k) = mV(x)\phi(x;k)$ in quantum mechanics

Using the assumption and $\phi(x;k) = e^{i\vec{k}\cdot\vec{x}} + \int \frac{d^3p}{(2\pi)^3} \frac{H(p;k)}{p^2 - k^2 - i\epsilon} e^{i\vec{p}\cdot\vec{x}}$ in $x > R$

$$\phi(x;k) = e^{i\delta(k)} \frac{\sin(kx + \delta(k))}{kx}$$

agrees with wave function in quantum mechanics

Following derivation in quantum mechanics,

finite volume formula can be derived from BS wave function.

BS wave function outside R

[CP-PACS, PRD71:094504(2005)]

Quantum field theory

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c.f. $(\Delta + k^2)\phi(x;k) = mV(x)\phi(x;k)$ in quantum mechanics

$I = 2$ S-wave two-pion BS wave function

- assumption is valid in lattice QCD
- k^2 from $\phi(r;k)$ in $r > R$

  using $G(\vec{r};k)$: Solution of Helmholtz equation on L^3

$\phi(x;k)$ $r > R$ can be used for calculation of $\delta(k)$.

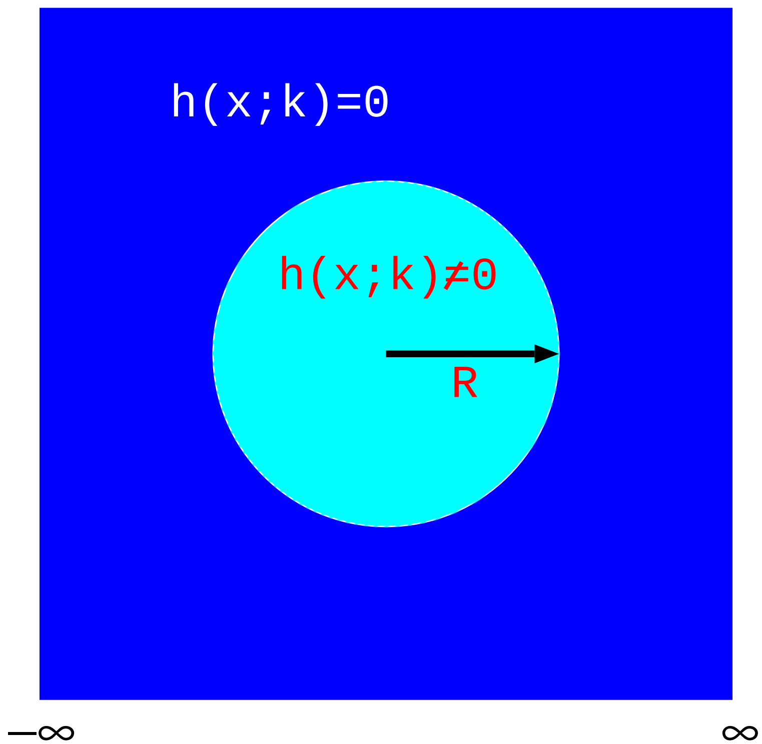
Our results

BS wave function inside R

[TY and Kuramashi, PRD96:114511,11(2017)]

Reduced BS wave function

$$h(x;k) = (\Delta + k^2)\phi(x;k)$$



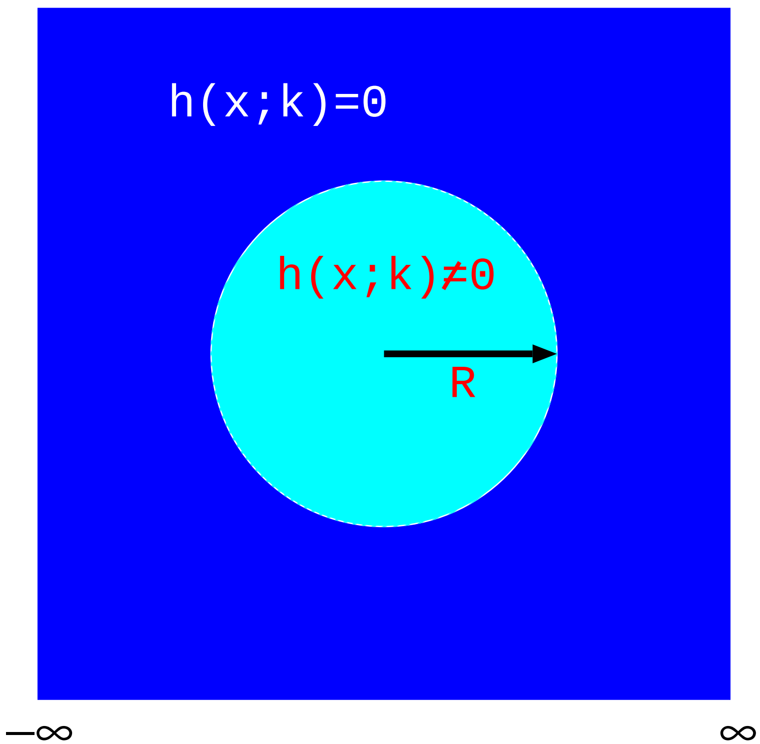
BS wave function inside R

[TY and Kuramashi, PRD96:114511,11(2017)]

Reduced BS wave function

$$h(x;k) = (\Delta + k^2)\phi(x;k) = -\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} H(p;k)$$

$$\because \phi(x;k) = e^{i\vec{k}\cdot\vec{x}} + \int \frac{d^3p}{(2\pi)^3} \frac{H(p;k)}{p^2 - k^2 - i\epsilon} e^{i\vec{p}\cdot\vec{x}}$$



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$\Downarrow$ Fourier transformation

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$$\Downarrow$$ Fourier transformation

Fundamental relation in this talk

$$H(p;k) = -\int d^3x\, e^{-i\vec{p}\cdot\vec{x}} h(x;k)$$

Relation between $H(p;k)$ and $h(x;k)$ *i.e.* $\phi(x;k)$ inside R

BS wave function inside R

[TY and Kuramashi, PRD96:114511,11(2017)]

Reduced BS wave function

$$h(x;k) = (\Delta + k^2)\phi(x;k) = -\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} H(p;k)$$

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Relation between $H(p;k)$ and $h(x;k)$ *i.e.* $\phi(x;k)$ inside R

At on-shell $p = k$

c.f. [CP-PACS, PRD71:094504(2005)]

$$H(k;k) = \frac{4\pi}{k} e^{i\delta(k)} \sin\delta(k) = -\int d^3x\; e^{-i\vec{k}\cdot\vec{x}} h(x;k)$$

$h(x;k)$ is essential to calculate $H(p;k)$.

BS wave function inside R

[TY and Kuramashi, PRD96:114511,11(2017)]

Reduced BS wave function

$$h(x;k) = (\Delta + k^2)\phi(x;k) = -\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} H(p;k)$$

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Relation between $H(p;k)$ and $h(x;k)$ *i.e.* $\phi(x;k)$ inside R

can be used for calculation on finite volume

Exploratory study with fundamental relation

$\rightarrow$ Next talk by Namekawa

Fundamental relation in quantum mechanics

[TY and Kuramashi, PRD96:114511,11(2017)]

Interpretation of HALQCD method in this frame work

$V(x;k)$ is defined by $h(x;k)$ as

$$V(x;k) = \begin{cases} \frac{1}{m}\frac{h(x;k)}{\phi(x;k)} & (x \leq R) \\ 0 & (x > R) \end{cases}$$

corresponding to LO HALQCD method

$V(x;k)$ is regarded as potential in Shrödinger equation.

$$(\Delta + p^2)\overline{\phi}(x;p) = mV(x;k)\overline{\phi}(x;p)$$

$\overline{\phi}(x;p)$ is a solution of the equation with given p.

Scattering phase shift $\overline{\delta}(p)$ from Shrödinger equation

[textbook of quantum mechanics]

$$\frac{e^{i\overline{\delta}(p)}\sin\overline{\delta}(p)}{p} = -\frac{m}{4\pi}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}V(x;k)\overline{\phi}(x;p)$$

Fundamental relation in quantum mechanics

[TY and Kuramashi, PRD96:114511,11(2017)]

Scattering phase shift $\overline{\delta}(p)$ from Shrödinger equation

$$\frac{e^{i\overline{\delta}(p)}\sin\overline{\delta}(p)}{p} = -\frac{1}{4\pi}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\frac{h(x;k)}{\phi(x;k)}\overline{\phi}(x;p)$$

At $p = k$, $\overline{\phi}(x;k) = \phi(x;k) \quad \because (\Delta + k^2)\phi(x;k) = h(x;k)$

$$\frac{e^{i\overline{\delta}(k)}\sin\overline{\delta}(k)}{k} = -\frac{1}{4\pi}\int d^3x\, e^{-i\vec{k}\cdot\vec{x}}h(x;k) = \frac{H(k;k)}{4\pi} = \frac{e^{i\delta(k)}\sin\delta(k)}{k}$$

$$\overline{\delta}(k) = \delta(k)$$

At $p \neq k$, $\overline{\phi}(x;p) \neq \phi(x;k)$ in general

$$\frac{e^{i\overline{\delta}(p)}\sin\overline{\delta}(p)}{p} = -\frac{1}{4\pi}\int d^3x\, e^{-i\vec{p}\cdot\vec{x}}\frac{h(x;k)}{\phi(x;k)}\overline{\phi}(x;p) \neq \frac{e^{i\delta(p)}\sin\delta(p)}{p}$$

Same $\delta(k)$ is obtained at only $p = k$, where $V(x;k)$ is defined.

Above discussion corresponding to LO HALQCD method

Expansion of reduced BS wave function

[TY and Kuramashi, PRD96:114511,11(2017)]

Derivative expansion in HALQCD method

$$\begin{aligned} h(x;k) &= (\Delta + k^2)\phi(x;k) \\ &= \sum_{n=0}^{\infty} V_n(x)\Delta^n\phi(x;k), \quad V_n(x) \text{ independent of } k \end{aligned}$$

[HALQCD, PTEP2018:043B04(2018)]

Convergence of expansion is unclear.

$\rightarrow$ Large number of terms would be necessary in general.

A few terms are not enough for convergence test.

c.f.) convergent test with two terms [HALQCD, arXiv:1805.02365]

Expansion of reduced BS wave function

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$$h(x;k) = \sum_{n=0}^{\infty} V_n(x)\Delta^n\phi(x;k), \quad V_n(x) \text{ independent of } k$$

[HALQCD, PTEP2018:043B04(2018)]

Truncated in practical determination of $V_n(x)$

$\rightarrow V_n(x)$ depends on input k.

Approximation $h(x;k) = V_0(x) + V_1(x)\Delta\phi(x;k)$ with inputs $h(x;k_1), h(x;k_2)$

$$V_0(x) = \frac{k_1^2\phi(x;k_1)h(x;k_2) - k_2^2\phi(x;k_2)h(x;k_2)}{\phi(x;k_1)h(x;k_2) - \phi(x;k_2)h(x;k_1) + \phi(x;k_1)\phi(x;k_2)(k_1^2 - k_2^2)}$$

$$V_1(x) = \frac{\phi(x;k_1)h(x;k_2) - \phi(x;k_2)h(x;k_1)}{\phi(x;k_1)h(x;k_2) - \phi(x;k_2)h(x;k_1) + \phi(x;k_1)\phi(x;k_2)(k_1^2 - k_2^2)}$$

$V_0(x), V_1(x)$ change with k_1, k_2.

Expansion of reduced BS wave function

[TY and Kuramashi, in preparation]

time-dependent HALQCD method [HALQCD, PLB712:437(2012)]

4-point function with different operator $n = 1, \cdots, N_O$

$$C_i(x,t) = \sum_{\alpha=1}^{N_\alpha} A_{i\alpha}(t)\phi_\alpha(x), \quad A_{i\alpha}(t) = B_{i\alpha}e^{-E_\alpha t}, \phi_\alpha(x) = \phi(x;k_\alpha)$$

Truncated approximation $h_\alpha(x) = h(x;k_\alpha) = \sum_{n=0}^{N_V-1} V_n(x)\Delta^n\phi_\alpha(x)$

Common $V_n(x)$ in all α

Simultaneous equations

$$M(x,t)V(x) = (\Delta + f(\partial_t))C(x,t)$$

$$(\Delta + f(\partial_t))C_i(x,t) = \sum_{\alpha=1}^{N_\alpha} A_{i\alpha}(t)h_\alpha(x), \quad f(\partial_t)A_{i\alpha}(t) = k_\alpha^2 A_{i\alpha}(t)$$

$$M_{in}(x,t) \equiv \Delta^n C_i(x,t) = \sum_{\alpha=1}^{N_\alpha} A_{i\alpha}(t)\Delta^n\phi_\alpha(x)$$

Expansion of reduced BS wave function

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4-point function with different operator $n = 1, \cdots, N_O$

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Common $V_n(x)$ in all α

Simultaneous equations

$$(\Delta + f(\partial_t))C_i(x,t) = \sum_{\alpha=1}^{N_\alpha} A_{i\alpha}(t)h_\alpha(x)$$

$$M(x,t)V(x) = A(t)h(x)$$

$$M_{in}(x,t) \equiv \Delta^n C_i(x,t)$$

$$[N_O \times N_V][N_V] = [N_O \times N_\alpha][N_\alpha]$$

sizes for matrices and vectors

Expansion of reduced BS wave function

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Common $V_n(x)$ in all α

Simultaneous equations

$$M(x,t)V(x) = A(t)h(x) \qquad M_{in}(x,t) \equiv \Delta^n C_i(x,t)$$

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sizes for matrices and vectors

Necessary condition to determine $V(x)$

$N_O = N_V$ for $(M(x,t))^{-1}$

Expansion of reduced BS wave function

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Truncated approximation $h_\alpha(x) = h(x;k_\alpha) = \sum_{n=0}^{N_V-1} V_n(x)\Delta^n\phi_\alpha(x)$

Common $V_n(x)$ in all α

Simultaneous equations

$$M(x,t)V(x) = A(t)h(x) \qquad M_{in}(x,t) \equiv \Delta^n C_i(x,t)$$

$$[N_O \times N_V][N_V] = [N_O \times N_\alpha][N_\alpha]$$

sizes for matrices and vectors

Necessary condition to determine $V(x)$

$N_O = N_V$ for $(M(x,t))^{-1}$ and $N_O = N_\alpha$ for $(A(t))^{-1}$

otherwise operator dependence remains in $V(x)$.

Expansion of reduced BS wave function

[TY and Kuramashi, in preparation]

time-dependent HALQCD method [HALQCD, PLB712:437(2012)]

4-point function with different operator $n = 1, \cdots, N_O$

$$C_i(x,t) = \sum_{\alpha=1}^{N_\alpha} A_{i\alpha}(t)\phi_\alpha(x), \quad A_{i\alpha}(t) = B_{i\alpha}e^{-E_\alpha t}, \phi_\alpha(x) = \phi(x;k_\alpha)$$

Truncated approximation $h_\alpha(x) = h(x;k_\alpha) = \sum_{n=0}^{N_V-1} V_n(x)\Delta^n\phi_\alpha(x)$

Common $V_n(x)$ in all α

Simultaneous equations

$$M(x,t)V(x) = A(t)h(x) \qquad M_{in}(x,t) \equiv \Delta^n C_i(x,t)$$

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sizes for matrices and vectors

Necessary condition to determine $V(x)$

$N_O = N_V$ for $(M(x,t))^{-1}$ and $N_O = N_\alpha$ for $(A(t))^{-1}$

otherwise operator dependence remains in $V(x)$.

the same condition for generalized eigenvalue problem

need data in large t region to satisfy $N_O = N_\alpha$

Summary

Simple relation between BS wave function inside R

and half off-shell scattering amplitude $H(p;k)$

$$H(p;k) = -\int d^3x\ e^{-i\vec{p}\cdot\vec{x}} h(x;k), \quad H(k;k) = \frac{4\pi}{k} e^{i\delta(k)} \sin\delta(k)$$

Reduced BS wave function $h(x;k) = (\Delta + k^2)\phi(x;k)$

exploratory study: next talk by Namekawa

might be possible to derive similar relations in more than two particles

$\overline{\delta}(p)$ from Shrödinger equation with $V(x;k) = h(x;k)/\phi(x;k)$

At $p = k$, $\overline{\delta}(k) = \delta(k)$, but at $p \neq k$, $\overline{\delta}(p) \neq \delta(p)$.

Derivative expansion of $h(x;k)$

- convergence is unclear.
- $V_n(x)$ depends on k if truncated in finite Δ^n terms
- time-dependent HALQCD method: same necessary condition to GEVP
  $\rightarrow$ large t data necessary as in calculation of energy