Review on Lattice Muon g-2 HVP Calculation

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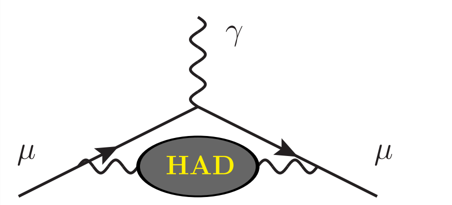
Hadron Vaccum Polarization (HVP) Contribution to Muon g - 2



$\hat{\Pi}(Q^2)$

$a_\mu^{exp.}$ vs. a_μ^{SM}

SM contribution	$a_\mu^{\text{contrib.}} \times 10^{10}$	Ref.
QED [5 loops]	11658471.8951 ± 0.0080	[Aoyama et al '12]
HVP-LO (pheno.)	692.6 ± 3.3	[Davier et al '16]
	694.9 ± 4.3	[Hagiwara et al '11]
	681.5 ± 4.2	[Benayoun et al '16]
	688.8 ± 3.4	[Jegerlehner '17]
HVP-NLO (pheno.)	-9.84 ± 0.07	[Hagiwara et al '11]
		[Kurz et al '11]
HVP-NNLO	1.24 ± 0.01	[Kurz et al '11]
HLbyL	10.5 ± 2.6	[Prades et al '09]
Weak (2 loops)	15.36 ± 0.10	[Gnendiger et al '13]
SM tot [0.42 ppm]	11659180.2 ± 4.9	[Davier et al '11]
[0.43 ppm]	11659182.8 ± 5.0	[Hagiwara et al '11]
[0.51 ppm]	11659184.0 ± 5.9	[Aoyama et al '12]
Exp [0.54 ppm]	11659208.9 ± 6.3	[Bennett et al '06]
Exp − SM	28.7 ± 8.0	[Davier et al '11]
	26.1 ± 7.8	[Hagiwara et al '11]
	24.9 ± 8.7	[Aoyama et al '12]

$$a_\mu^{\text{LO-HVP}}|_{NoNewPhys} \times 10^{10} \simeq 720 \pm 7,$$

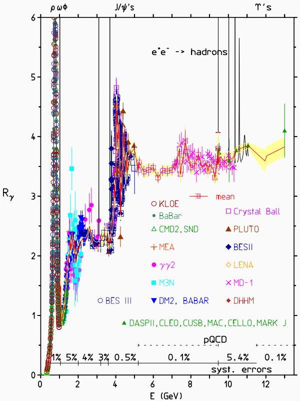
FNAL E989 (2017): 0.14-ppm, J-PARC E34: 0.1-ppm

Really $a_\mu^{exp.} \neq a_\mu^{\mathrm{SM}}$?

Motivation

HVP in Phenomenology

- The HVP in Pheno. is:
$$\hat{\Pi}(Q^2) = \int_0^\infty ds \frac{Q^2}{s(s+Q^2)} \frac{\mathrm{Im}\Pi(s)}{\pi}$$
$$= (Q^2/(12\pi^2)) \int_0^\infty ds \frac{R_{had}(s)}{s(s+Q^2)} \,,$$
- with R-ratio [right fig. Jegerlehner EPJ-Web2016] given by
$$R_{had}(s) \equiv \frac{\sigma(e^+e^- \to had.)}{4\pi\alpha^2(s)/(3s)} \,,$$
- where the systematics is challenging to control(next talk). Some tension among experiments in $\sigma(e^+e^- \to \pi^+\pi^-)$.



Requirement for Lattice QCD:

- Independent cross-check of Hadronic Vauccum Polarization Contribution to muon g-2 ($a_\mu^{\rm HVP}$),
- Permil-Level determination of $a_\mu^{\rm HVP}$ w.r.t. FNAL/J-PARC expr.

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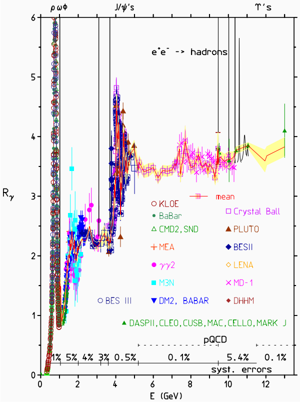

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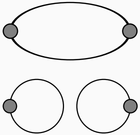
Objective in This Work

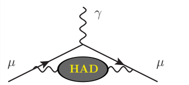
- **Hadron Vacuum Polarization (HVP):**

$$\Pi_{\mu\nu}(Q) = \int d^4x\ e^{iQx} \langle j_\mu(x) j_\nu(0)\rangle$$
$$= (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2)\Pi(Q^2)\ ,$$
$$j_\mu = \tfrac{2}{3}\bar{u}\gamma_\mu u - \tfrac{1}{3}\bar{d}\gamma_\mu d - \tfrac{1}{3}\bar{s}\gamma_\mu s + \tfrac{2}{3}\bar{c}\gamma_\mu c + \cdots\ .$$



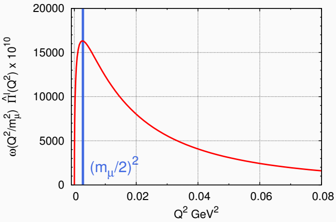
- **Leading-Order(LO) HVP Contr. to Muon g-2:**

$$a_\mu^{\text{LO-HVP}} = (\alpha/\pi)^2 \int_0^\infty dQ^2\ \ \omega(Q^2/m_\mu^2)\hat{\Pi}(Q^2)\ ,$$
$$\hat{\Pi}(Q^2) = \Pi(Q^2) - \Pi(0)\ .$$



- **HVP Time-Moments:**

$$\hat{\Pi}(Q^2) = \sum_{n=1} Q^{2n}\Pi_n\ ,$$
$$\Pi_n = \frac{1}{n!}\frac{d^n\hat{\Pi}(Q^2)}{(dQ^2)^n}\bigg|_{Q^2\to 0} = \sum_x \frac{(-\hat{x}_\nu^2)^{n+1}}{(2n+2)!}\langle j_\mu(x) j_\mu(0)\rangle .$$



Access to Deep IR: Pade and Time-Moment Rep.

Model Independent Approximants

Pade Approximant

- For $Q^2 < Q^2_{cut}$, lattice HVP data are fitted to

$$\hat{\Pi}(Q^2) = \frac{A_2 Q^2 + \cdots}{1 + B_2 Q^2 + \cdots} . \quad (1)$$

- The dispersion relation $\hat{\Pi}(Q^2) = \int_0^\infty ds \frac{Q^2}{s(s+Q^2)} \frac{\mathrm{Im}\Pi(s)}{\pi}$ is seen as so-called *Stieltjes Integral* [Aubin et.al., PRD2012], which guarantees a finite conversion radius.

Time-Momentum Representation (TMR)

- For $Q^2 < Q^2_{uv\text{-}cut}$, define [Bernecker and Meyera, EPJA2011],

$$\hat{\Pi}(Q^2) = \sum_t t^2 \left[1 - \left(\frac{\sin[Qt/2]}{Qt/2}\right)^2\right] \frac{1}{3} \sum_{i=1}^{3} \langle j_i(t) j_i(0) \rangle . \quad (2)$$

- The momentum Q is *Continuous*. The Sine-Cardinal $\sin[Qt/2]/(Qt/2)$ accounts for a pediodic feature of lattice correlators $\langle j_i(t) j_i(0) \rangle$.

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Example of TMR

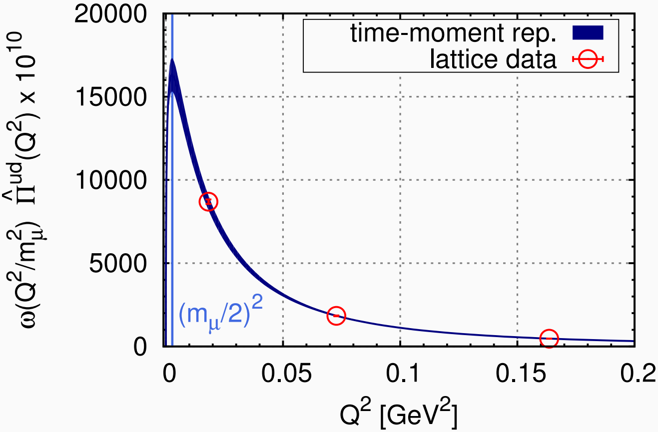

Figure: From BMW Ensemble ($a = 0.064$ fm) used in PRD2017 and PRL2018.

Table of Contents

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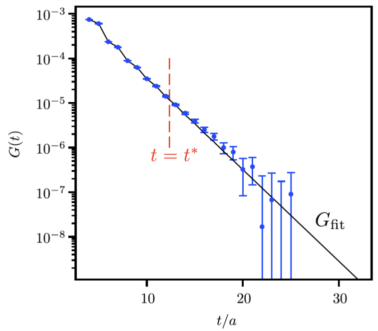
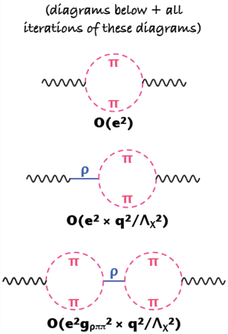
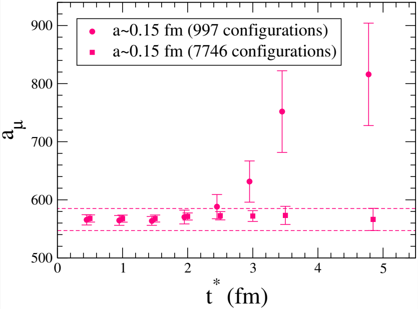
Multi-Exponential Fits [HPQCD PRD2017]

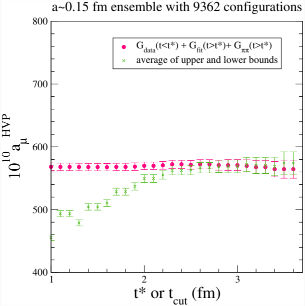




- **Left:** HPQCD PRD2017, vector-current correlator with ud-quarks and a fit line $t > t^*$: $G^{ud}(t, t^*) = G^{ud}_{data}(t < t^*)$ or $(G_{fit}(t > t^*) + G_{\pi\pi}(t > t^*))$, where $t^* \in [0.5, 1.5] fm$. Multi($N = 5$)-Exponential Ansatz are adopted and ρ-meson dominates.
- **Right:** From a slide of Van de Water at Mainz Workshop 2018. Diagrams in effective theory to correct missing effects in the fits. Taste-spliting and finite volume corrections are also taken account.

Multi-Exponential Fits [FNAL/HPQCD/MILC Preliminary]





- **Left:** The t^* dependence of

$$a_{\mu,ud}^{\text{LO-HVP}}(t^*) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 \;\; \omega(Q^2/m_\mu^2) \;\; \mathcal{FT}[G^{ud}(t,t^*),Q^2]_{with\ Pade}\,. \qquad (3)$$

  With high-statistics, $a_{\mu,ud}^{\text{LO-HVP}}$ get stable at larger t^*. For $t^* \lesssim 2$ fm, low-(used in PRD2017) and high-statistics are consistent.
- **Right:** The high-statistics in the left-panel is compared with *Bounding Method* (next page).

Bounding [BMW PRD2017 and PRL2018]

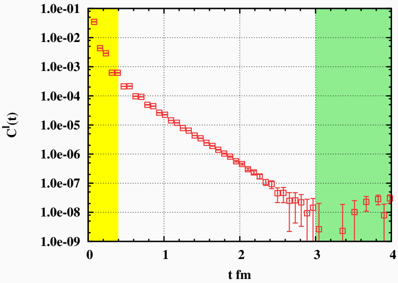


Figure shows

$$C^{ud}(t) = \frac{5}{9}\sum_{\vec{x}}\frac{1}{3}\sum_{i=1}^{3}\langle j_i^{ud}(\vec{x},t)j_i^{ud}(0)\rangle \ ,$$

by BMW Ensemble with $a = 0.078$ [fm] used in PRD2017/PRL2018.

- The connected-light correlator $C^{ud}(t)$ loses signal for $t > 3fm$. To control statistical error, consider $C^{ud}(t > t_c) \to C^{ud}_{\rm up/low}(t, t_c)$, where

  $$C^{ud}_{\rm up}(t, t_c) = C^{ud}(t_c)\ \varphi(t)/\varphi(t_c),$$
  $$C^{ud}_{\rm low}(t, t_c) = 0.0,$$

  with $\varphi(t) = \cosh[E_{2\pi}(T/2 - t)]$, and $E_{2\pi} = 2(M_\pi^2 + (2\pi/L)^2)^{1/2}$.
- Similarly, $C^{disc}(t) \to C^{disc}_{\rm up/low}(t, t_c)$,

  $$-C^{disc}_{\rm up}(t > t_c) = 0.1\,C^{ud}(t_c)\ \varphi(t)/\varphi(t_c),$$
  $$-C^{disc}_{\rm low}(t > t_c) = 0.0.$$
- By construction, $C^{ud,disc}_{low}(t, t_c) \le C^{ud,disc}(t) \le C^{ud,disc}_{up}(t, t_c)$.

Bounding [BMW PRL2018]

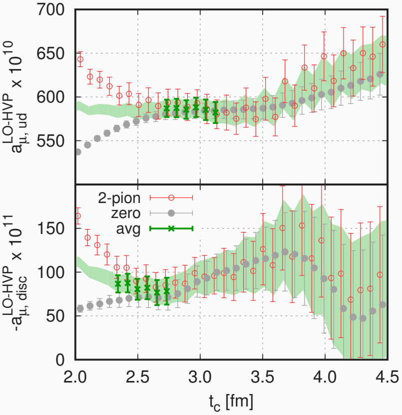


Figure: BMW, PRL2018.

- Corresponding to $C^{ud,disc}_{up/low}(t_c)$, we obtain upper/lower bounds for muon g-2: $a^{ud,disc}_{\mu,up/low}(t_c)$.
- Two bounds meet around $t_c = 3fm$. Consider the average of bounds: $\bar{a}^{ud,disc}_{\mu}(t_c) = 0.5(a^{ud,disc}_{\mu,up} + a^{ud,disc}_{\mu,low})(t_c)$, which is stable around $t_c = 3fm$.
- We pick up such averages $\bar{a}^{ud,disc}_{\mu}(t_c)$ with $4-6$ kinds of t_c around $3fm$. The average of average is adopted as $a^{\rm LO\text{-}HVP}_{\mu,\rm ud/disc}$ to be analysed, and a fluctuation over selected t_c gives systematic error.
- A similar method is proposed by C.Lehner in Lattice2016 and used in RBC/UKQCD-PRL2018. Improved bounding method with GEVP: [A. Meyer/C. Lehner, 27 Fri Hadron Structure].

Large Distance Control Using F_π, [Mainz CLS JHEP2017]

- **Isospin Decomp. of Vector-Current Correlator:**

$$G(t,L) = G^{I=1}(t,L) + G^{I=0}(t,L)\ , \quad G^{I=1}(t,L) = \sum_{n=1} |A_n|^2 e^{-\omega_n t}\ , \tag{4}$$

where $\omega_n = 2\sqrt{M_\pi^2 + k_n^2}$. Investigate the large distance behavior of $G^{I=1}(t)$.

- **Lüscher's Formula [NPB1991]:** The p-wave phase shift determines k_n,

$$\delta_{l=1}(k_n) + \phi(k_n L/(2\pi)) = n\pi\ , \tag{5}$$

where ϕ is a known kinematical function.

- **Meyer's Formula [PRL2011]:**

$$|F_\pi(\omega_n)|^2 = \frac{3\pi\omega_n^2}{2k_n^5}\left(k_n \frac{\partial\delta_1(k_n)}{\partial k_n} + q_n \frac{\partial\phi(q_n)}{\partial q_n}\right)|A_n|^2\ , \quad q_n = \frac{k_n L}{2\pi}\ , \tag{6}$$

which is analogous to Lellouch-Lüscher Formula [CMP2012].

- **Gounaris-Sakurai(GS) [PRL1968]** (c.f. Fransis et.al. [PRD2013]):

$$(k^3/\omega)\cot\delta_1^{GS}(k) = k^2 h(\omega) - k_\rho^2 h(M_\rho) + b[k_\rho, M_\rho, \Gamma_\rho](k^2 - k_\rho^2)\ ,$$

$$F_\pi^{GS}(\omega) = f_0[M_\pi, M_\rho, \Gamma_\rho]/((k^3/\omega)(\cot[\delta_1^{GS}(k)] - i))\ , \quad k_\rho^2 = (M_\rho^2/4) - M_\pi^2\ .$$

- **Construct $G^{I=1}(t)$:** For given lattice data $(M_{\pi,\rho})$, using GS formulae with Eqs. (5) and (6), $G^{I=1}_{lat}(t)$ is fitted to Eq. (4) to determine (A_n, k_n, Γ_ρ).

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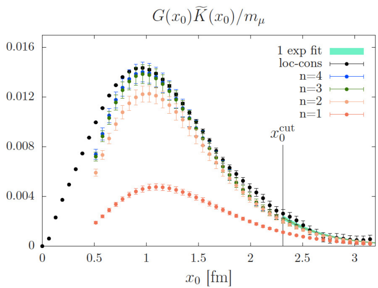


(A) $G_n^{I=1}(t,L) = \sum_{j=1}^{n} |A_j|^2 e^{-\sqrt{M_\pi^2+k_j^2}\;t}$,

(B) $G^{I=1}(t>t^*, L\to\infty) =$
$\frac{1}{48\pi^2}\int_{2M_\pi}^{\infty} d\omega\, \omega^2 (1-\frac{4M_\pi^2}{\omega^2})^{3/2} |F_\pi(\omega)|^2 e^{-\omega|t|}$.

- Figure: [Mainz Prelim], update of [Mainz Lat2017]. $(\tilde{K}(t)/m_\mu)G_n(t,L)$ vs $x_0 = t$ for $N_f = 2+1$, $M_\pi = 200$ MeV. G_n is given by Eq. (A). c.f. Talk by H. Wittig (27 Fri, Hadron Structure).
- The lowest mode ($n = 1$) becomes dominant at around 3 [fm]. A single exponential-fit provides a good approximation at long-distance.
- Using $F_\pi^{\rm GS}(\omega)$, the infinite-volume correlator $G^{I=1}(t, L\to\infty)$ is given by Eq. (B). Comparing $a_{\mu,ud}^{\rm LO\text{-}HVP}$ obtaind with $G^{I=1}(t>t^*, L\to\infty)$ or $G_{lat}^{I=1}(t>t^*, L)$, a finite volume effect can be estimated.

Large Distance Control Using F_π

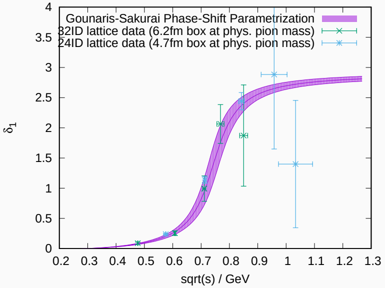


Figure: RBC/UKQCD Preliminary.

- $E_\rho = 0.766(21)$ [GeV] (c.f. PDG: 0.77549(34) [GeV]).
- $\Gamma_\rho = 0.139(18)$ [GeV] (c.f. PDG: 0.1462(7) [GeV]).

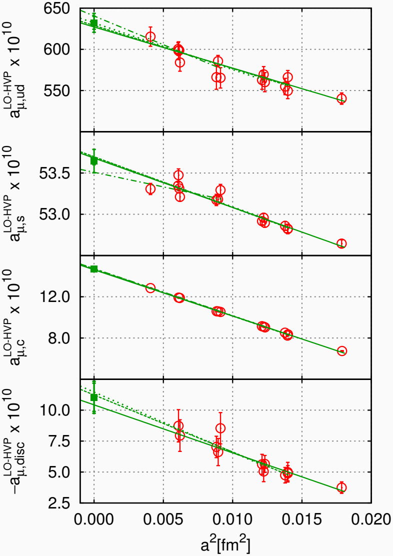
Finite Volume Effects

Consider $a^{\text{LO-HVP}}_{\mu,ud}(L_2) - a^{\text{LO-HVP}}_{\mu,ud}(L_1)$.

- $(L_1, L_2) = (4.66, 6.22)$[fm], physical M_π **[RBC/UKQCD Prelim., talk by C. Lehner (27 Fri, Hadron Structure)]**
  - XPT: 12.2×10^{-10} ,
  - LQCD: $21.6(6.3) \times 10^{-10}$,
  - GSL: $20(3) \times 10^{-10}$.
- $(L_1, L_2) = (5.4, 10.8)$[fm], $M_\pi = 135$[MeV] **[talk by E. Shintani (24 Tue, Hadron Spectroscopy), update of PACS 1805.04250]**
  - LQCD: $40(18) \times 10^{-10}$, 2.5 times larger than XPT estimates.
- $L_2 =$ large, $M_\pi L_1 \sim 4$
  - XPT/RBCUK-PRL18: $16(4) \times 10^{-10}$,
  - GSL/RBCUK-Prelim: $22(1) \times 10^{-10}$,
  - XPT/BMW-PRL18: $15(15) \times 10^{-10}$,
  - GSL/Mainz-Prelim: $20.4(4.2) \times 10^{-10}$,
  - GSL+dual/ETM-prelim: $31(6) \times 10^{-10}$.

Continuum Extrapolation

Controlled Continuum Extrap. [BMW PRL2018]

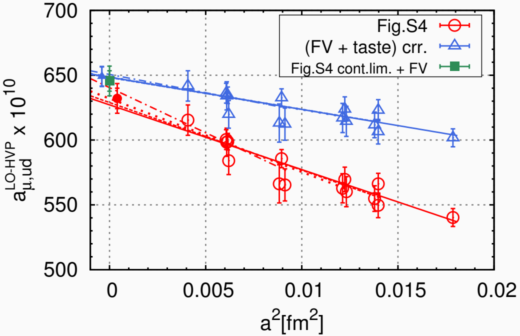


BMW Ensemble PRD2017 and PRL2018

- 6-β, 15 simulation with all physical masses.
- Nf=(2+1+1) staggered quarks.
- Large Volume: $(L, T) \sim (6, 9-12) fm$.
- AMA with 6000-9000 random-source meas. for disconnected. [c.f. Mainz-Lat2014, RBC/UKQCD-PRL2016, HPQCD-PRD2016]. [Poster by S. Yamamoto FNAL/HPQCD/MILC, 24 Tue].

- Get systematic uncertainty from various cuttings: no-cut, or cutting $a \geq 0.134,\ 0.111,$ or 0.095.
- Strong a^2 deps. for $a_{\mu,ud/disc}^{\text{LO-HVP}}$ due to taste violations, and for $a_{\mu,c}^{\text{LO-HVP}}$ due to large m_c.
- Get good χ^2/dof with extrapolation linear in a^2, and interpolation linear in M_K^2 (strange) or M_π^2 and $M_{\eta c}$ (charm).

Crosscheck of Continuum Extrapolation [BMW PRL2018]



1. Red open-circles are raw lattice data and continuum-extrapolated (red filled-circle). Then finite-volume correction using XPT is added to get the green-square point.
2. Similarly to HPQCD-PRD2017, raw data (red-circles) are first corrected with finite-volume and taste-partner effects to get blue open-triangles, which are continuum-extrapolated to get blue filled-triangle.

Continuum Extrapolation, Comparison

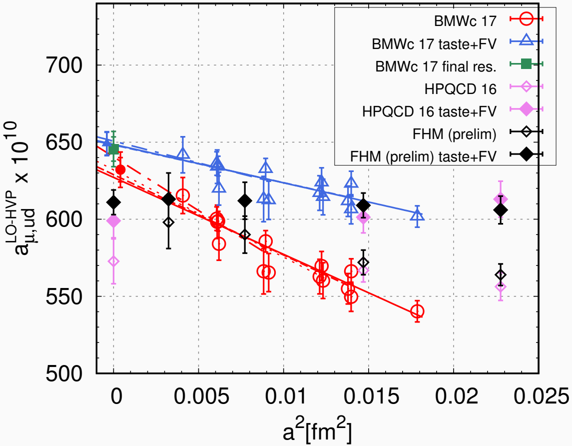


Figure: BMW-PRL2018 vs HPQCD-PRD2017 and FNAL/HPQCD/MILC-Prelim.

QED and Strong-Isospin Breaking Corrections

$$\mathcal{O}(\alpha) \sim \mathcal{O}\left(\frac{m_d - m_u}{\Lambda_{QCD}}\right) \sim 1\%\ \text{Correction} \quad .$$

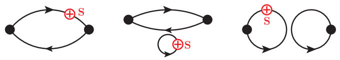
Strong Isospin Breaking (SIB)

Strong isospin breaking: $m_d - m_u = 2.41(6)(4)(9)$ [BMW PRL2016] in $\overline{MS}$-2[GeV].

- Direct Simulations with $m_u \neq m_d$ [FNAL/HPQCD/MILC-PRL2018].
- Perturbative Method [RM123-JHEP2012,RBC/UKQCD-JHEP17]:

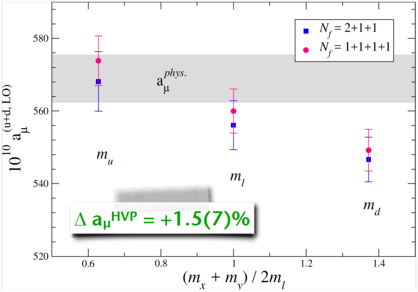
$$\langle O \rangle = \langle O \rangle_{m_{u/d}=\hat{m}} + (m_{u/d} - \hat{m}) \frac{\partial \langle O \rangle}{\partial m_{u/d}}\Big|_{m_u=m_d} + \mathcal{O}((m_{u/d} - \hat{m})^2) ,$$
$$= \langle O \rangle_{m_{u/d}=\hat{m}} - (m_{u/d} - \hat{m}) \langle OS \rangle_{m_{u/d}=\hat{m}} ,$$

where $\hat{m} = (m_u + m_d)/2$, and $S = \sum_x \bar{q}_{u/d} q_{u/d}(x)$.



Up: Strong Isospin Breaking Diagrams.

Right: FNAL/HPQCD/MILC-PRL2018 (Van de Water, Mainz g-2 workshop). Valence-quark dep. of $a_\mu^{\text{LO-HVP}}$ for (2+1+1) and (1+1+1+1) ensemble. Two ensemble results agree at $m_l = (m_u + m_d)/2$; sea-quark SIB are negligible. To quantify SIB, define, $\Delta a_\mu^{\text{LO-HVP}} = (4a_\mu^{\text{LO-HVP}}|_{m_u} + a_\mu^{\text{LO-HVP}}|_{m_d})/5 - a_\mu^{\text{LO-HVP}}|_{m_l}$. SIB corr. $= \Delta a_\mu^{\text{LO-HVP}} / a_\mu^{\text{LO-HVP}}|_{m_l} = 1.5(7)\%$



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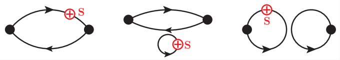

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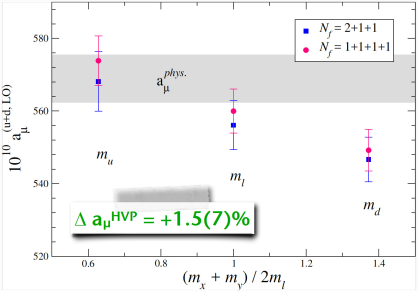

QED Correction

- Consider QCD + QED Eucridean partition function:

$$\langle O \rangle = \frac{1}{Z} \int \mathcal{D}[q, \bar{q}, U] \mathcal{D}[A] \; O \; e^{-S_F[q,\bar{q},U,A] - S_G[U]} e^{-S_\gamma[A]} \; . \tag{7}$$

- **Full QCD + QED:** First Come Out! [QCDSF-Prelim, talk by J. Zanotti (27 Fri, Hadron Structure)].
- **Stochastic Method:** Stochastic photon fields A_μ are generated with weight e^{-S_γ} independently of gluon fields U_μ (electro-quenched), and multiplied, $U_\mu(x) \to e^{-ieq_f A_\mu(x)} U_\mu(x)$ [Duncan et.al. PRL1996].
- **Perturbative Method:** QED can be treated in a perturbative way in $\alpha = e^2/(4\pi^2)$ [RM123-PRD2013]:

$$\langle O \rangle = \langle O \rangle_0 + \frac{e^2}{2} \frac{\partial^2 \langle O \rangle}{\partial e^2}\Big|_{e=0} + \mathcal{O}(\alpha^2) \; . \tag{8}$$

  The stochastic and perturbative methods gave consistent corrections [RBC/UKQCD-Lat2017].
- To control QED FV effects, QED_L prescription [Hayakawa PTP2008] is used; spatial zero-modes and the universal $1/L^{n=1,2}$ corrections to mass are removed [BMW Science2015], while a reflection positivity is preserved.

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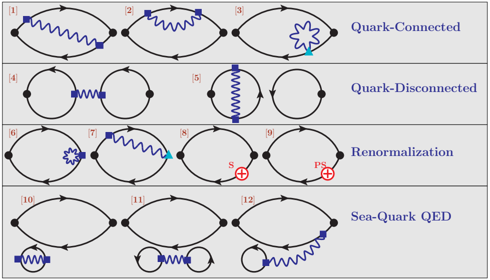
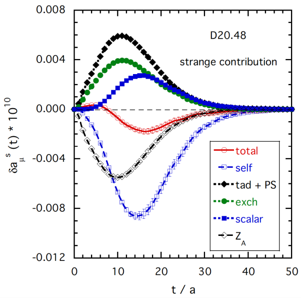
QED Correction Diagrams in Perturbative Approach





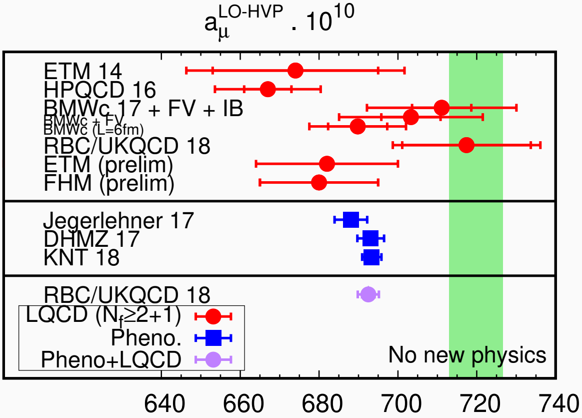
- **Left:** ■ = vector-current, ▲ = tadpole, ⊕ = (pseudo-)scalar insersions.
- **Right:** [ETMc JHEP2017, talk by D. Giusti, (27 Fri, Hadron structure)] with corrections [1],[2],[3],[8] (mass retuning) and [9] (keeping maximal twist) for strange component.
- RBCUKQCD (Domain-Wall) considered [1],[2],[3],[4]; the others $\sim 1/N_c$ or irrelevant. One must take are a double counting problem in [4] w.r.t. single-photon and additional glues [talks by RBC/UKQCD (27 Fri, Hadron Structure).]
- For diagram details, see [talk by A. Risch (24 Tue, Hadron Spectroscopy)].

SIB + QED Corrections, Short Summary

- **ETMc Preliminary**
  $\delta a_\mu^{\text{LO-HVP}} \times 10^{10} = 7(2)$ (quark connected and qQED).
- **BMW PRL2018**
  $\delta a_\mu^{\text{LO-HVP}} \times 10^{10} = 7.8(5.1)$ (pheno. $(\pi^0\gamma, \eta\gamma, \rho - \omega$ mix, $M_{\pi^\pm}))$.
- **RBC/UKQCD PRL2018**
  $\delta a_\mu^{\text{LO-HVP}} \times 10^{10} = 9.5(10.2)$ (quark connected + one disconnected and qQED. Also relevant to use tau decay input for HVP, [M. Bruno, 27 Fri Hadron Structure].)
- **FNAL/HPQCD/MILC PRL2018**
  $\delta a_\mu^{\text{LO-HVP}} \times 10^{10} = 9.5(4.5)$ (Strong Isospin Breaking only).
- **QCDSF Prelim:**
  $\delta a_\mu^{\text{LO-HVP}} / a_\mu^{\text{LO-HVP}} \lesssim 1\%$ (Dynamical QED, $M_\pi \sim 400[\text{MeV}]$)

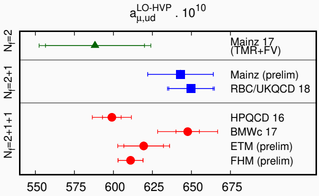
Table of Contents

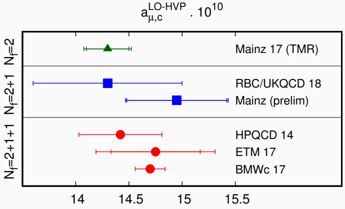
The obvious: $a_\mu^{\text{LO-HVP}}$

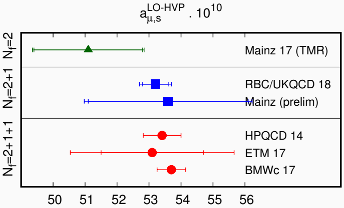


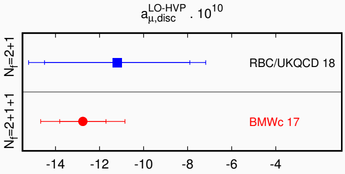
- Lattice errors $\sim$ 2% vs phenomenology errors $\sim$ 0.4%.
- Some lattice results suggest new physics others not but all compatible with phenomenology.

$a_\mu^{\text{LO-HVP}}$: flavor by flavor comparison





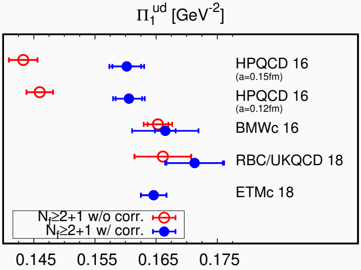


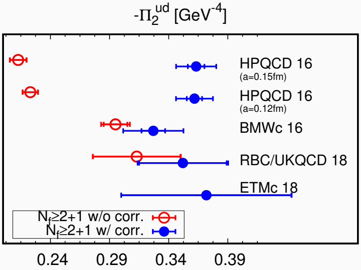


- $a_{\mu,\,s,c,\text{disc}}^{\text{LO-HVP}}$ already known with high enough precision for FNAL E989
- “Disagreement” is on $a_{\mu,\,ud}^{\text{LO-HVP}}$

Derivatives of $\Pi(Q^2)$ at $Q^2 = 0$: *ud* contribution

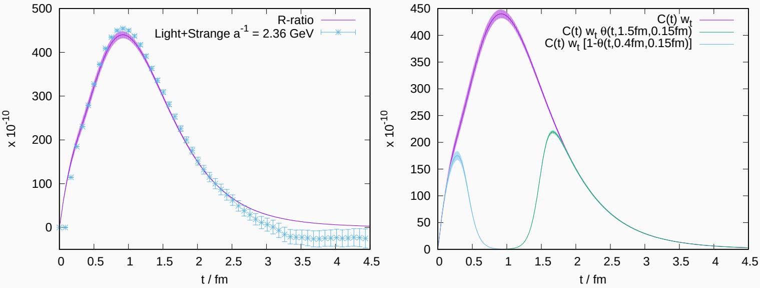
$$\Pi_n = \frac{1}{n!}\frac{d^n\hat{\Pi}(Q^2)}{(dQ^2)^n}\bigg|_{Q^2\to 0} = \sum_x \frac{(-\hat{x}_\nu^2)^{n+1}}{(2n+2)!}\langle j_\mu(x) j_\mu(0)\rangle.$$





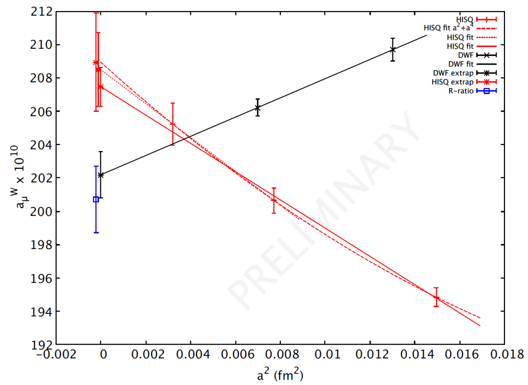
- In Pad picture, larger $\Pi_1(\Pi_2) \to$ larger (smaller) a_μ.
- HPQCD 16 has slightly smaller Π_1^{ud} and larger $-\Pi_2^{ud}$ than BMWc 16 and RBC/UKQCD 18 $\to$ combine to give smaller $a_{\mu,\,ud}^{\text{LO-HVP}}$
- Suggests that HPQCD 16 has smaller $C(t)$ for $t \sim 1\,\mathrm{fm}$ but larger for $t \gtrsim 2\,\mathrm{fm}$
- Difference comes from HPQCD 16's large corrections

Time window: lattice + phenomenology



- Figure: [RBC/UKQCD-PRL2018, talk by C. Lehner and Colleages (27 Fri, Hadron Structure)]. In $a_\mu^{\text{LO-HVP}} = (\alpha/\pi)^2 \sum_t W(t, Q^2/m_\mu^2) C(t)$, consider lattice/pheno correlators; $C_{lat}(t) = \sum_{\vec{x}} \frac{1}{3} \sum_{i=1}^{3} \langle j_i(\vec{x}, t) j_i^{ud}(0) \rangle$, $C_{pheno}(t) = \frac{1}{2} \int_0^\infty ds \sqrt{s} \frac{R(s)}{3} e^{-\sqrt{s}|t|}$. $C_{lat}(t)$ may be more precise in intermediate $t \sim 1$ [fm].
- Consider the decomposition $C(t) = (C^{\text{SD}} + C^{\text{W}} + C^{\text{LD}})(t)$, where $(C^{\text{SD}}, C^{\text{W}}, C^{\text{LD}})(t) = C(t)(1 - \Theta(t, t_0, \Delta), \Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta), \Theta(t, t_1, \Delta))$ with the smeared step function, $\Theta(t, t', \Delta) = (1 + \tanh[(t - t'/\Delta)])/2$.
- For $C^{\text{W}}(t)$, use lattice data C^{W}_{lat}. For the others, use phenomenological data $C^{\text{SD/LD}}_{pheno}$. $(t_0, t_1, \Delta) = (0.4, 1.0, 0.15)$[fm], $a_\mu^{\text{LO-HVP}} = 692.5(2.7) \cdot 10^{-10}$ [RBC/UKQCD-PRL2018].

Window Method: DWF vs HISQ vs Pheno.



- Fig.: T. Blum (27 Fri). Continuum extrapolation of $a_\mu^W = \sum_t C_{lat}^W(t)W(t, m_\mu)$, where $C_{lat}^W(t) = C_{lat}(t)((\Theta(t, t_0, \Delta) - \Theta(t, t_1, \Delta)))$ with $t_0 = 4.0$, $t_1 = 1.0$, $\Delta = 0.15$[fm].
- (2+1+1) HISQ(MILC ensemble) and DWF all physical points in 5.5 [fm] boxes. HISQ and DWF shows 2-3 σ tension; lattice spacing, statistics may be responsible. The DWF result is consistent with phenomenology.

Other Important Subjects

- Lattice ($Q^2 < Q^2_{cut}$) - Perturbation ($Q^2 \geq Q^2_{cut}$) Matching [BMW-PRL2018].
- Lattice results of Higher-Order HVP [FNAL/HPQCD/MILC, 1806.08190].
- Dual Propagator + Gounaris-Sakurai-Lüscher Propagator [ETMc-Prelim, Mainz g-2 Workshop].
- Omnès Formula for time-like pion form factor [Mainz Preliminary, talk by H. Wittig (27 Fri, Hadron Structure)].
- HVP for $\sin^2\theta_W$ [talk by Cè Marco, (27 Fri, Hadron Structure)].

Table of Contents

Summary and Conclusions

- Lattice computation of $a_\mu^{\text{LO-HVP}}$ has total error $\sim 2\% \gg\sim 0.4\%$ from phenomenology. Some results are consistent with no new physics and phenomenology, others with phenomenology and new physics
- Difference comes from *ud* contribution and most probably from treatment of long-distance physics, for which many progress have been done but need more understandings.
- Comparison of *ud* time moments suggests:
  - larger intermediate-distance contribution in [BMWc-PRL2018 and RBC/UKQCD-PRL2018]
  - larger long-distance contribution in [HPQCD-PRD2017], associated with model description
- With current lattice results, too early to make detailed comparisons with dispersive approach. However, combination of lattice and phenomenology [RBC/UKQCD PRL18, T. Blum Preliminary] may deliver a reliable 0.2% $a_\mu^{\text{LO-HVP}}$.
- Lattice combined with Experimental Data: Next Talk by Marina.

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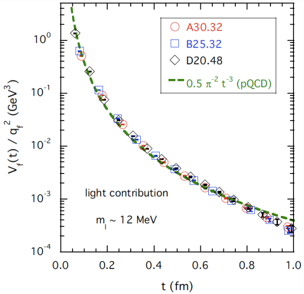
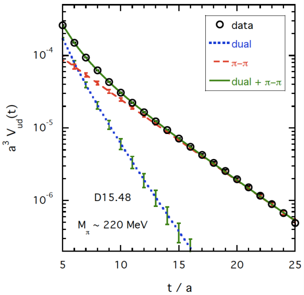
Table of Contents

Large Distance Control (GSL + SVZ) [ETMc Preliminary]





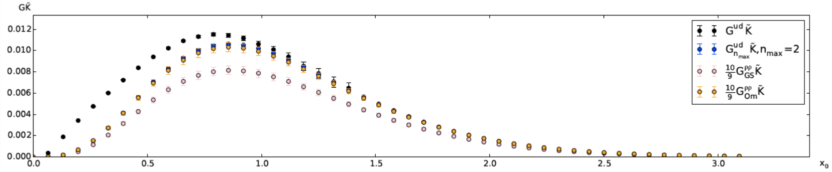
- Top panel [ETMc JHEP2017]: Vector-current correlator data are well described by 1-loop QCD up to $1\,fm > \hbar c/\Lambda_{QCD}$. This was interpreted as the onset of SVZ Quark-Hadron Duality [NPB1979].
- Motivated by the duality, consider the following expression for the vector-current correlator,

$$V_{dual}(t) = \frac{5R_{dual}}{72\pi^2}\int_{s_{dual}}^{\infty} ds\sqrt{s}e^{-\sqrt{s}t}R^{1l\text{-}QCD}(s)\ ,$$

where, $R^{1l\text{-}QCD}(s) = (1 - \frac{4m_{ud}^2}{s})^{1/2}(1 + \frac{2m_{ud}^2}{s})\ .$

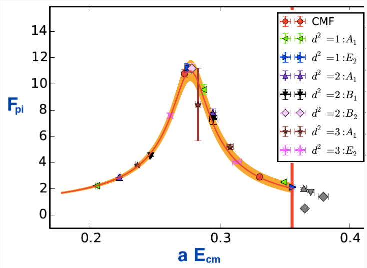
- This expression differs from 1-loop QCD by two fit params (R_{dual}, s_{dual}), and combined with 2-pion correlator $V_{\pi\pi}$ constructed via Gounaris-Sakurai F_π^{GS}.
- Bottom panel [ETMc Preliminary]: $(V_{dual} + V_{\pi\pi})$ describes well lattice data whole range. FV effects and other systematics can be studied with this.

Large Distance Control Omnès [Mainz Preliminary]

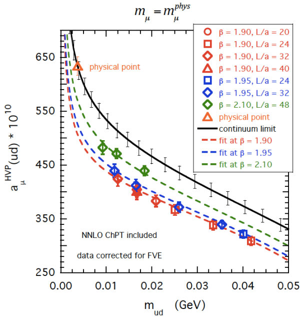


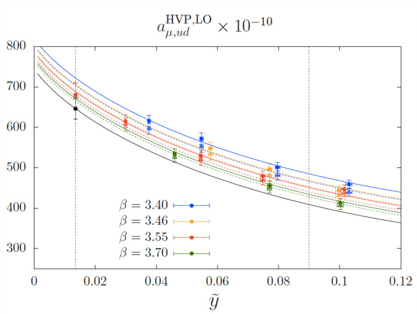
Omnès Formula (Nuovo Cimento (1958))

- Figs: Mainz Preliminary, Thanks to F.Erben (GSI-HIM).
- $F_\pi(\omega) =$
  $$\exp\left[\omega^2 P_{n-1}(\omega^2) + \frac{\omega^{2n}}{\pi}\int_{4M_\pi^2}^{\infty} ds \frac{\delta_1(s)}{s^n(s-\omega^2-i\epsilon)}\right].$$
- Lattice data are used for F_π and δ_1 and fit parameters are in the Polynomial P_{n-1}.
- Omnès gives a better description than GS in the middle range.



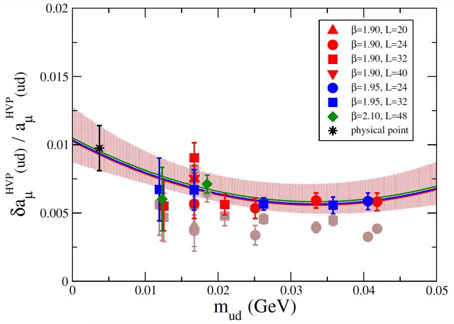
Continuum Extrapolation and Mass Dependence

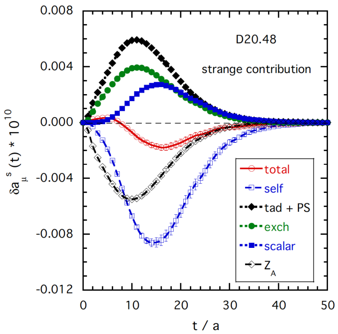




- **Left:** ETM Preliminary. From slide by S.Simula in Mainz g-2 workshop 2018. The continuum limit line (black-solid) becomes sensitive to m_{ud} at physical point.
- **Right:** Mainz Preliminary. From slide by H.Meyer in Mainz g-2 workshop 2018. $\tilde{y} = (M_\pi/(4\pi f_\pi))^2$.

ISB + QED Corrections, [ETMc JHEP2017 and Preliminary]

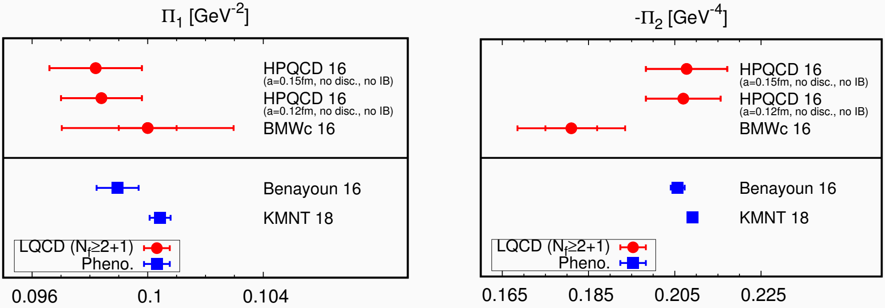




- **Left:** ETMc Preliminary, (SIB + QED) corrections for light components. The chiral/continuum-extrapolation is investigated with FV effects taken account.
- **Right:** ETMc JHEP2017, (SIB + QED) corrections for strange component integrand for each diagrams shown previous pages. The charm is also investigated. In both, partial cancellations among the various diagrams.

Comparison of derivatives of $\Pi(Q^2)$ at $Q^2 = 0$

$$\Pi_n = \frac{1}{n!}\frac{d^n\hat{\Pi}(Q^2)}{(dQ^2)^n}\Big|_{Q^2\to 0} = \sum_x \frac{(-\hat{x}_\nu^2)^{n+1}}{(2n+2)!}\langle j_\mu(x) j_\mu(0)\rangle.$$



BMWc 16 has Π_1 comparable to phenomenology but smaller $-\Pi_2$

$\rightarrow$ suggests that BMWc (and RBC/UKQCD) has $C(t)$ slightly larger for $t \sim 1\,\mathrm{fm}$ and smaller for $t \gtrsim 2\,\mathrm{fm}$