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# The hadronic vacuum polarisation contribution to $(g-2)_\mu$ from 2+1 flavours of $O(a)$ improved Wilson quarks

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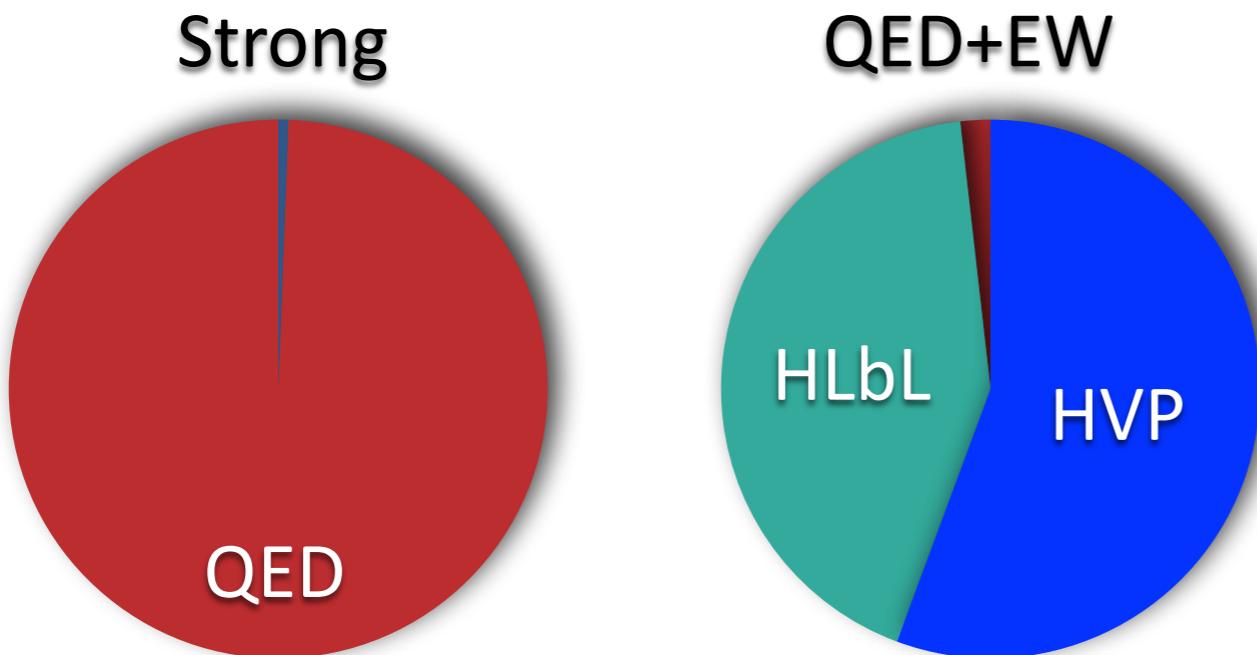
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22–28 July 2018

# The muon anomalous magnetic moment

$$a_\mu \equiv \frac{1}{2}(g - 2)_\mu = \begin{cases} 116\,592\,080(54)(33) \cdot 10^{-11} & \text{E821 @ BNL} \\ 116\,591\,825(34)(26)(1) \cdot 10^{-11} & \text{SM prediction} \end{cases}$$

- \* SM estimate dominated by QED; error dominated by QCD

$$a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{strong}}$$

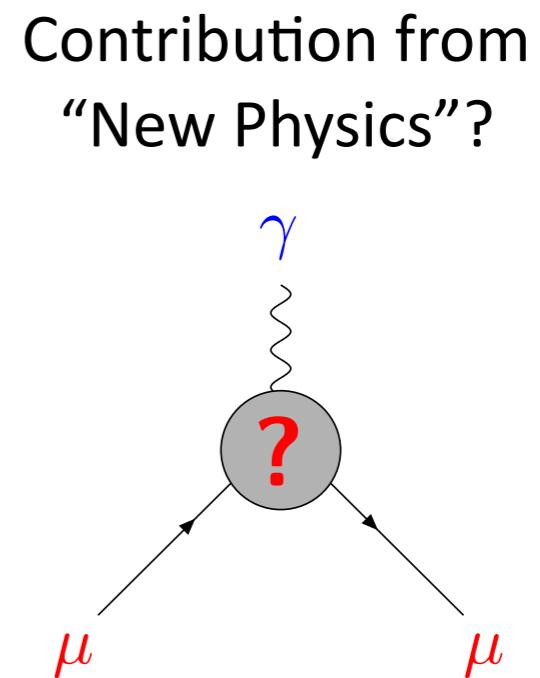
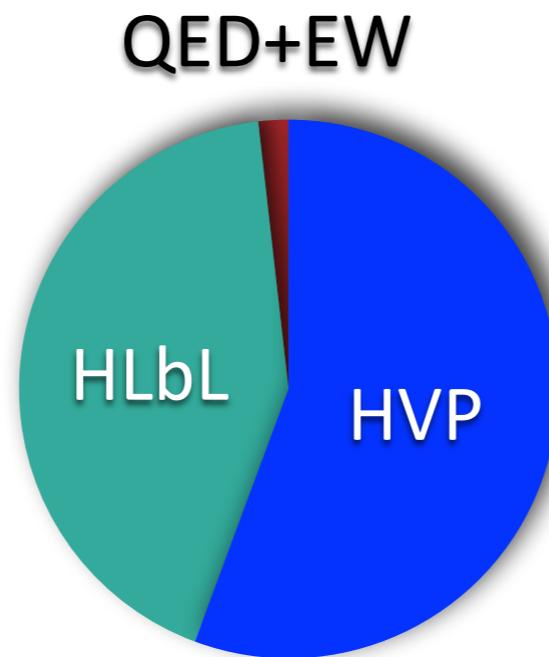
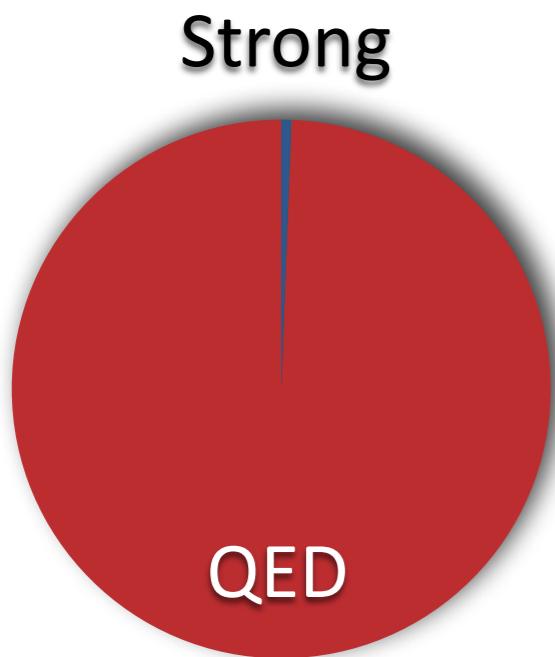


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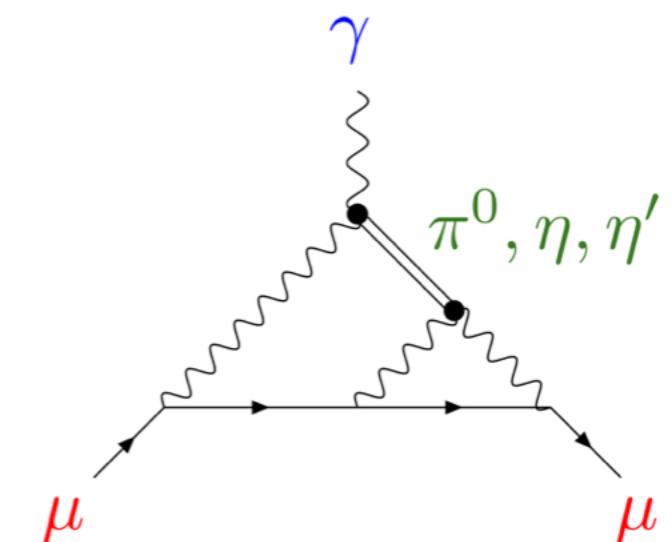
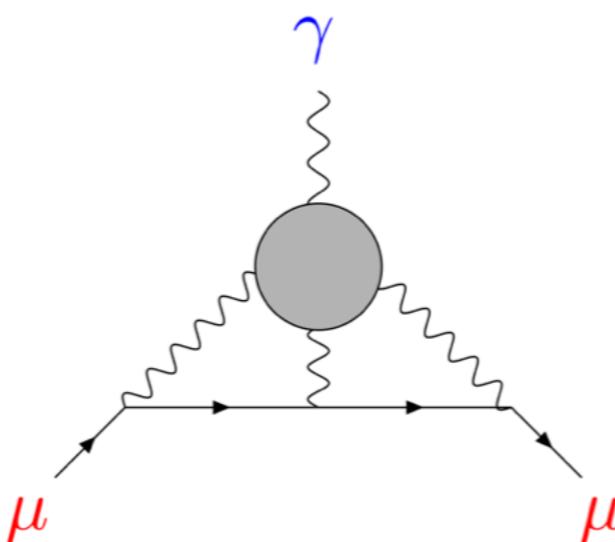
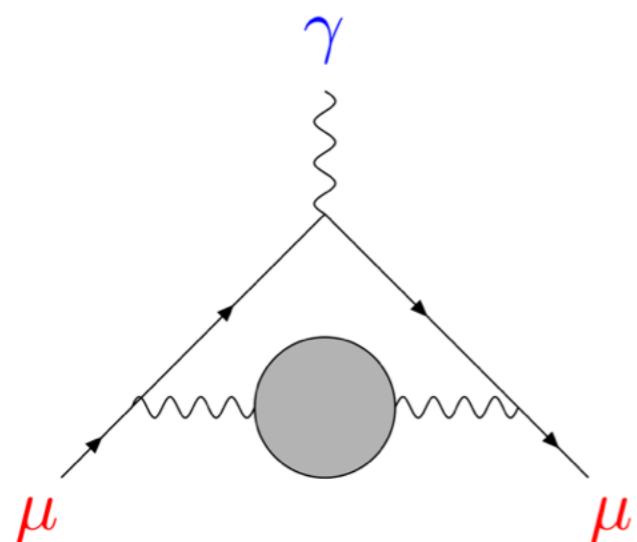
$$a_\mu = a_\mu^{\text{QED}} + a_\mu^{\text{weak}} + a_\mu^{\text{strong}} + a_\mu^{\text{NP?}}$$



# The Mainz $(g - 2)_\mu$ project

## Collaborators:

N. Asmussen, M. Cè, A. Gérardin, O. Gryniuk, G. von Hippel, B. Hörz,  
H.B. Meyer, A. Nyffeler, K. Ottnad, V. Pascalutsa, A. Risch, T. San José Perez, HW  
J. Green, B. Jäger, G. Herdoíza



- Direct determinations of LO  $a_\mu^{\text{hvp}}$
- Running of  $\alpha$  and  $\sin^2\theta_W$
- Exact QED kernel
- Forward scattering amplitude
- Transition form factor for  $\pi^0 \rightarrow \gamma^*\gamma^*$

# Lattice QCD approach to HVP

- \* Vacuum polarisation tensor:

[Meyer & HW, arXiv:1807.09370]

$$\Pi_{\mu\nu}(Q) = i \int d^4x e^{iQ \cdot (x-y)} \langle J_\mu(x) J_\nu(y) \rangle \equiv (Q_\mu Q_\nu - \delta_{\mu\nu} Q^2) \Pi(Q^2)$$

- \* Electromagnetic current:

$$J_\mu = \frac{2}{3} \bar{u} \gamma_\mu u - \frac{1}{3} \bar{d} \gamma_\mu d - \frac{1}{3} \bar{s} \gamma_\mu s + \dots$$

- \* Convolution integral over Euclidean momenta:

[Lautrup & de Rafael; Blum]

$$a_\mu^{\text{hvp}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dQ^2 f(Q^2) \hat{\Pi}(Q^2), \quad \hat{\Pi}(Q^2) = 4\pi^2 (\Pi(Q^2) - \Pi(0))$$

- \* Weight function  $f(Q^2)$  strongly peaked near muon mass

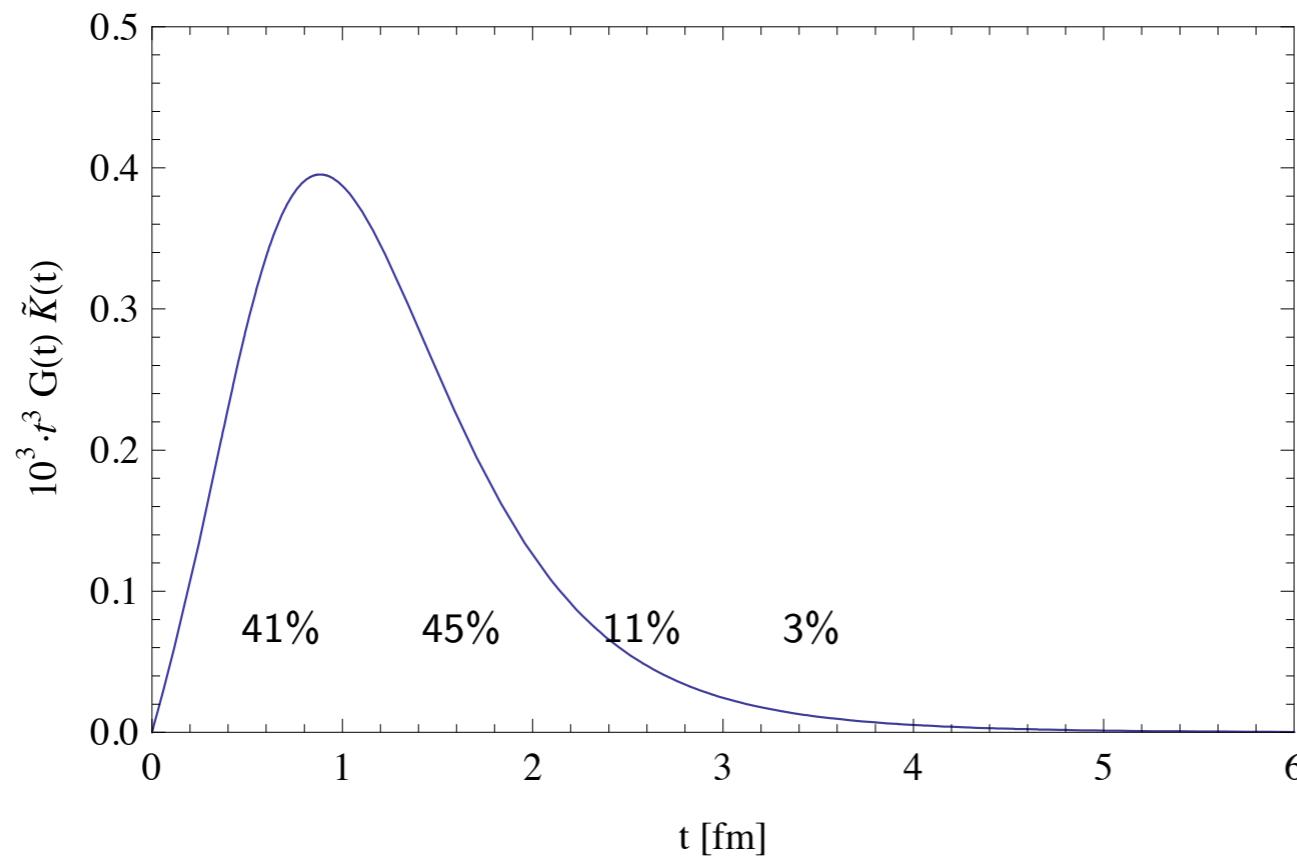
# Lattice QCD approach to HVP

- \* Time-momentum representation (TMR):

[Bernecker & Meyer]

$$a_\mu^{\text{hvp}} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0 \tilde{K}(x_0) G(x_0), \quad G(x_0) = -a^3 \sum_{\vec{x}} \langle J_k(x) J_k(0) \rangle$$

$$\tilde{K}(x_0) = 4\pi^2 \int_0^\infty dQ^2 f(Q^2) \left[ x_0^2 - \frac{4}{Q^2} \sin^2 \left( \frac{1}{2} Q x_0 \right) \right]$$



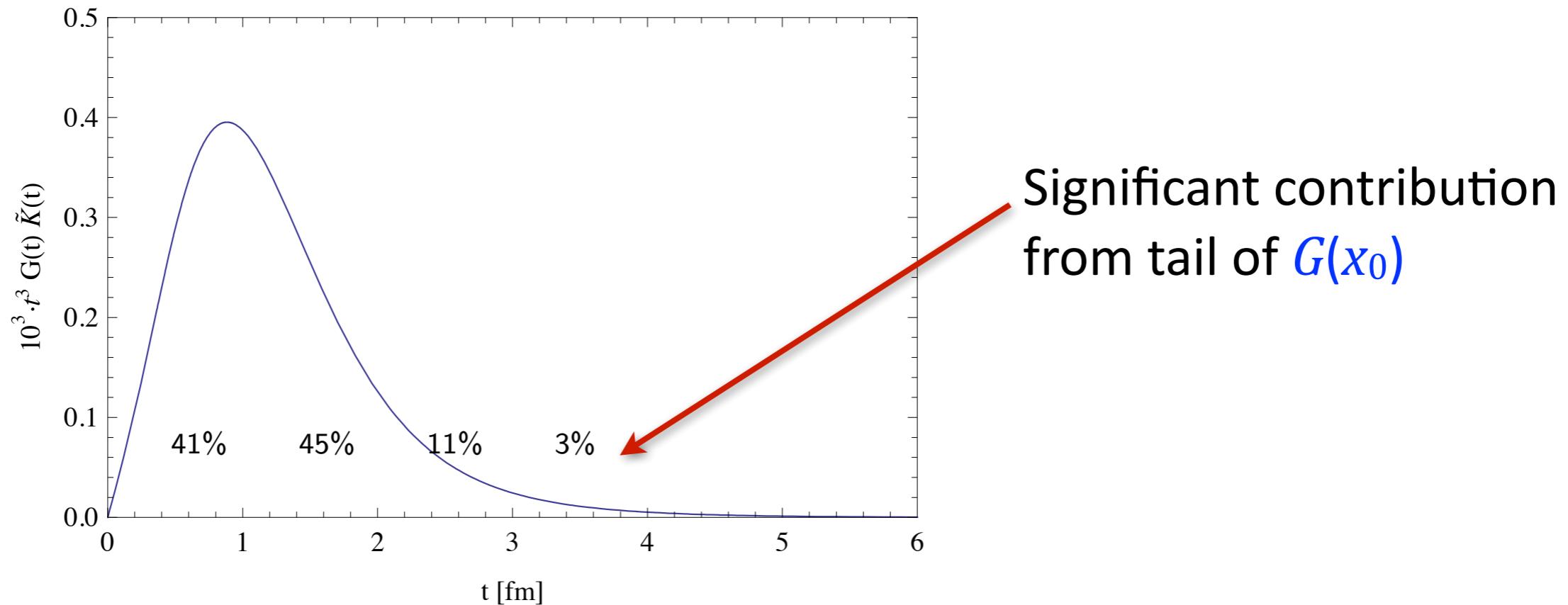
# Lattice QCD approach to HVP

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# Past and present calculations

Published result in two-flavour QCD:

[*Della Morte et al., JHEP 10 (2017) 020*]

$$a_\mu^{\text{hyp}} = (654 \pm 32_{\text{stat}} \pm 17_{\text{syst}} \pm 10_{\text{scale}} \pm 7_{\text{FV}} {}^{+0}_{-10} \text{disc}) \cdot 10^{-10}$$

Extension to QCD with  $N_f = 2+1$ :

- \* Use  $O(a)$  improved vector currents  $\rightarrow$  continuum extrapolation in  $a^2$
- \* Scale setting:  $\sqrt{8t_0} = 0.415(4)(2) \text{ fm}$  [Bruno et al., PRD 95 (2017) 074504]
- \* Dedicated calculation of the  $I = \ell = 1$  scattering phase shift and timeline pion form factor
- \* Disconnected diagrams:  
Hierarchical probing; covariant coordinate-space method

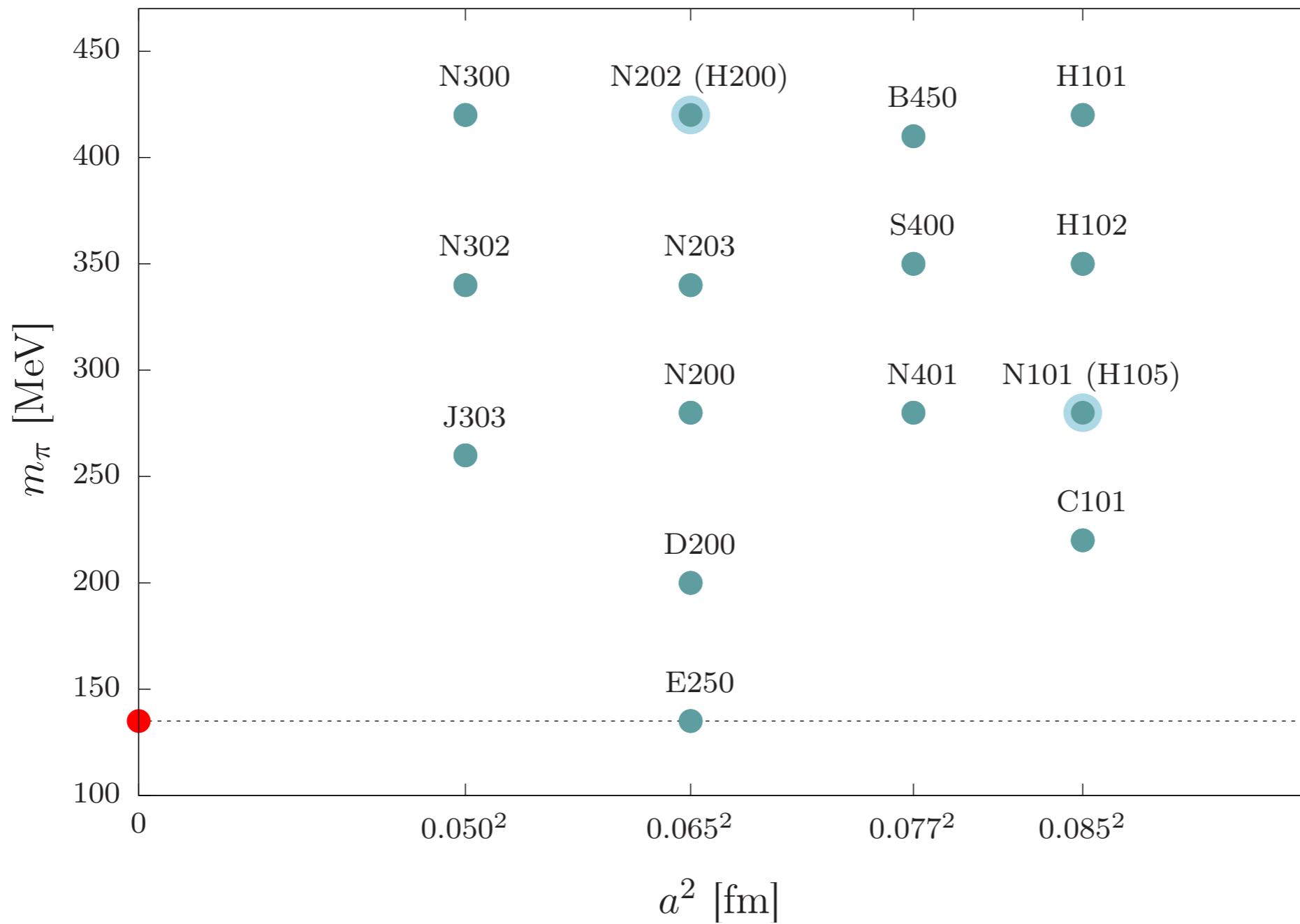
[H.B. Meyer, EPJC 77 (2017) 616]

# CLS ensembles

- \*  $N_f = 2+1$  flavours of  $O(a)$  improved Wilson fermions
- \* Tree-level Symanzik gauge action
- \* Four values of the lattice spacing:  
 $a = 0.085, 0.077, 0.065, 0.050 \text{ fm}$
- \* Open boundary conditions — topology freezing
- \* Pion masses and volumes:  $m_\pi^{\min} \approx 135 \text{ MeV}, \quad m_\pi L > 4$
- \* Quark mass trajectory:  $\text{Tr } M = \text{const.}$

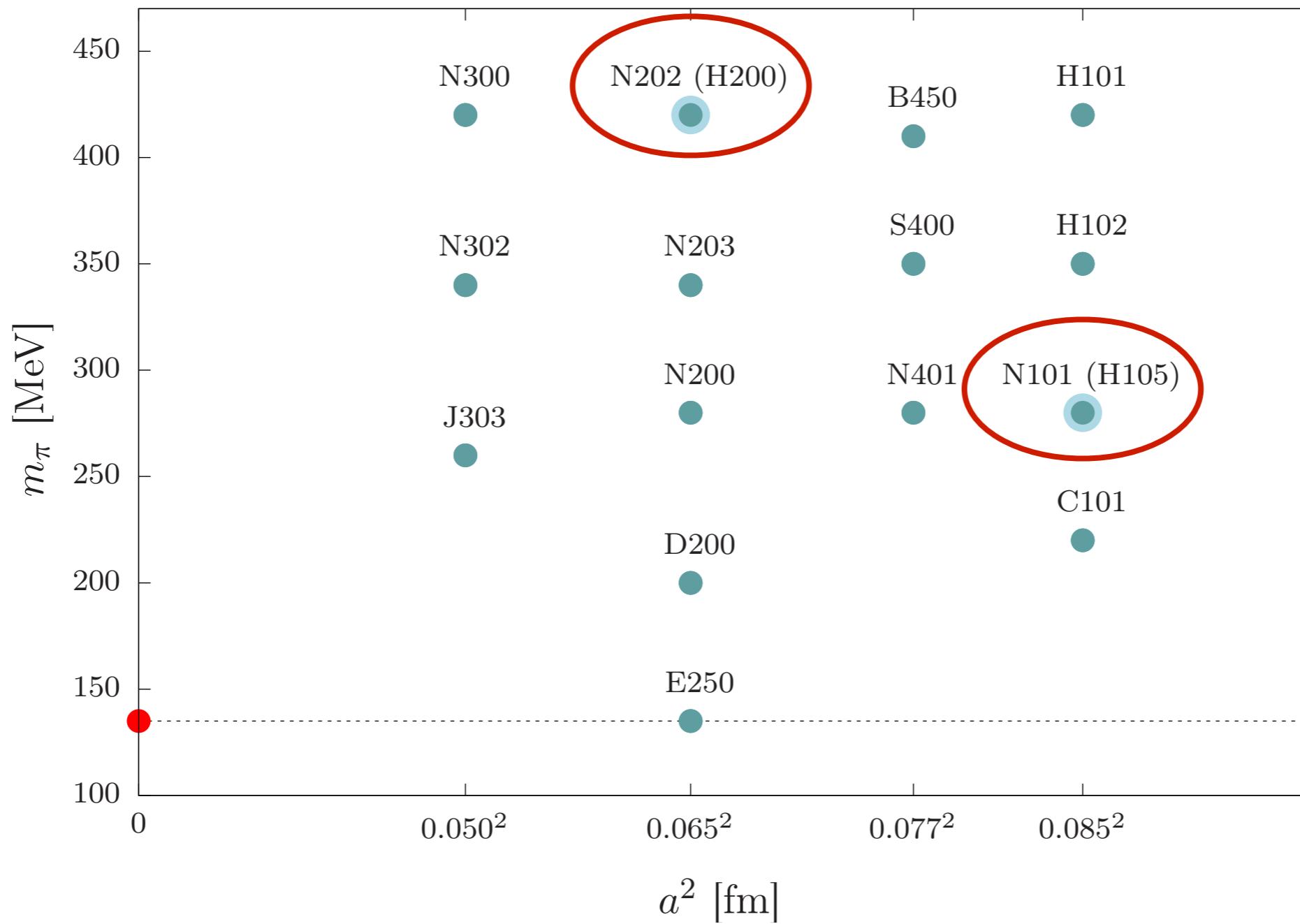
# Current data sets

- \* “Landscape” of CLS ensembles for HVP determination



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# Vector currents and correlators

- \* Quark-connected TMR correlators:

$$G^f(x_0) = -\frac{a^3}{3} \sum_k \sum_x q_f^2 Z_V^f \left\langle V_{k,f}^{\text{con}}(x_0, x) V_{k,f}^{\text{loc}}(0) \right\rangle, \quad f = u, d, s, c$$

$$G^f(x_0) = -\frac{a^3}{3} \sum_k \sum_x q_f^2 (Z_V^f)^2 \left\langle V_{k,f}^{\text{loc}}(x_0, x) V_{k,f}^{\text{loc}}(0) \right\rangle,$$

⇒ Can be applied with open boundary conditions

- \*  $O(a)$  improved local and conserved currents:

$$(V_k^A)^{\text{imp}}(x) = (V_k^A)(x) + a c_V^A \tilde{\partial}_0 T_{k0}(x), \quad A = \text{loc, con}$$

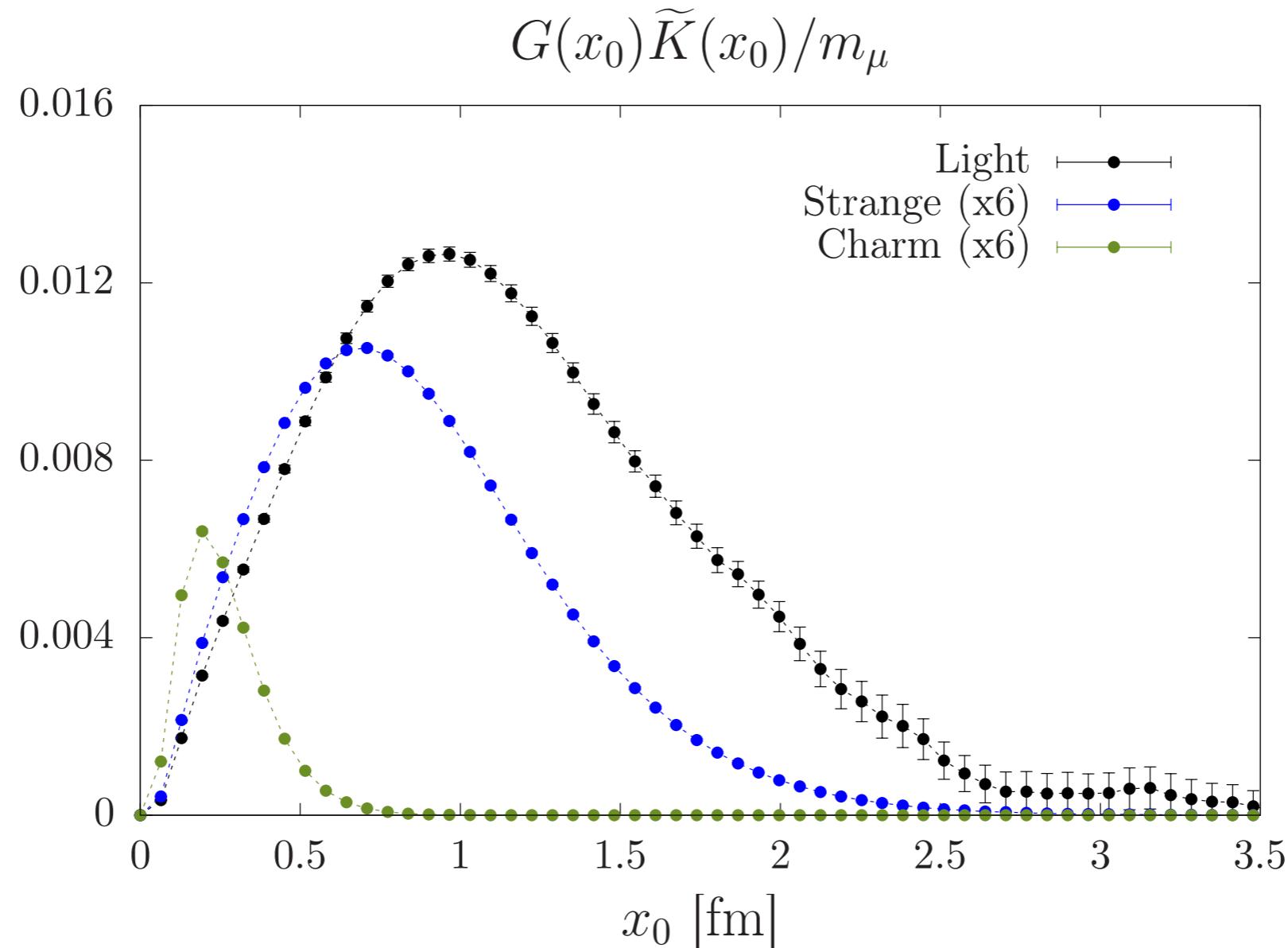
$c_V^A$  : determined non-perturbatively

- \*  $Z_V$  determined from ratio of three- and two-point functions;

[Della Morte et al., arXiv:1705.01775]

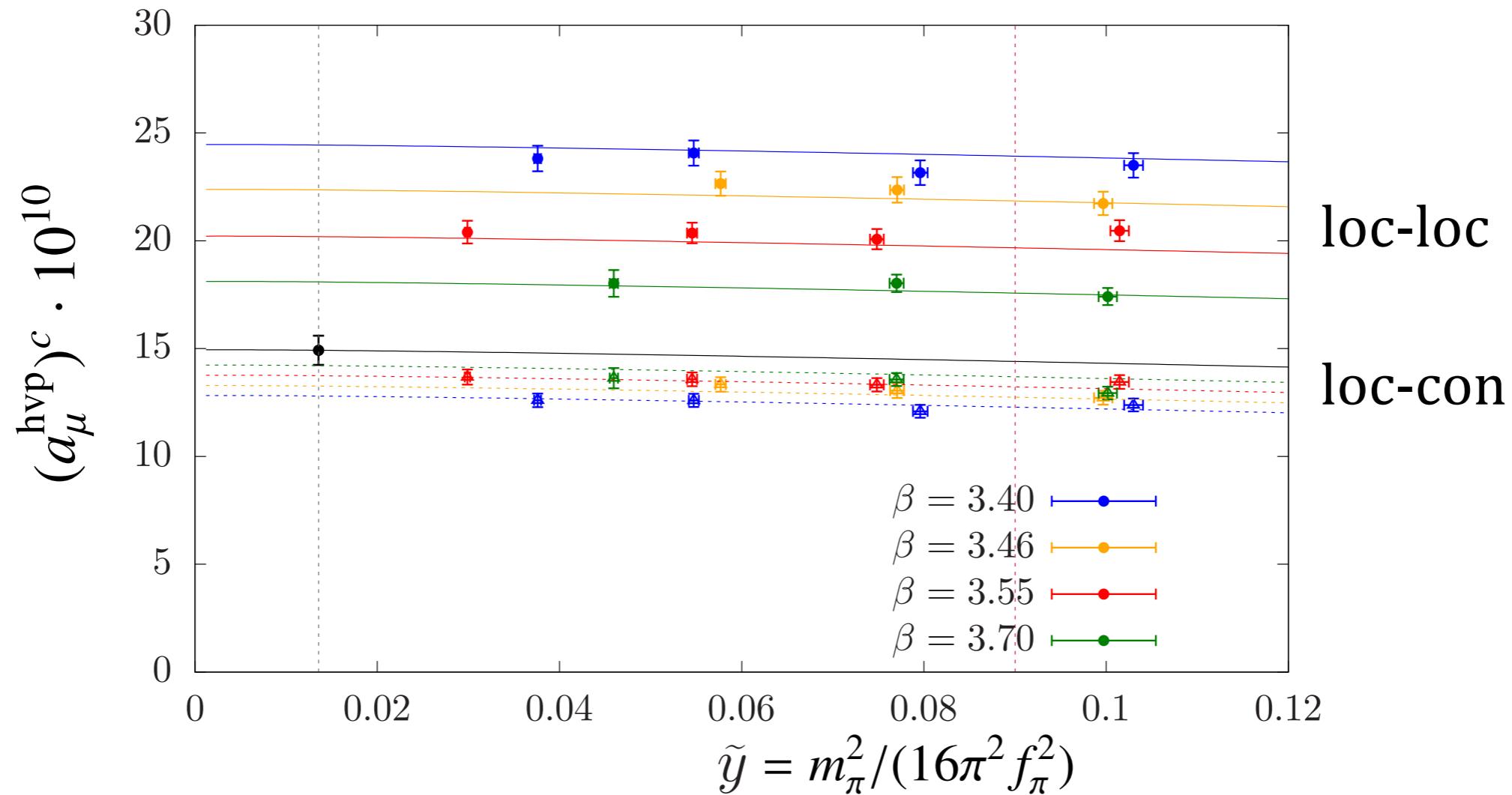
# Preliminary results: ud, s, c

- \*  $ud$ ,  $s$  and  $c$  contributions on “D200”, i.e.  $m_\pi = 200$  MeV,  $a = 0.065$  fm



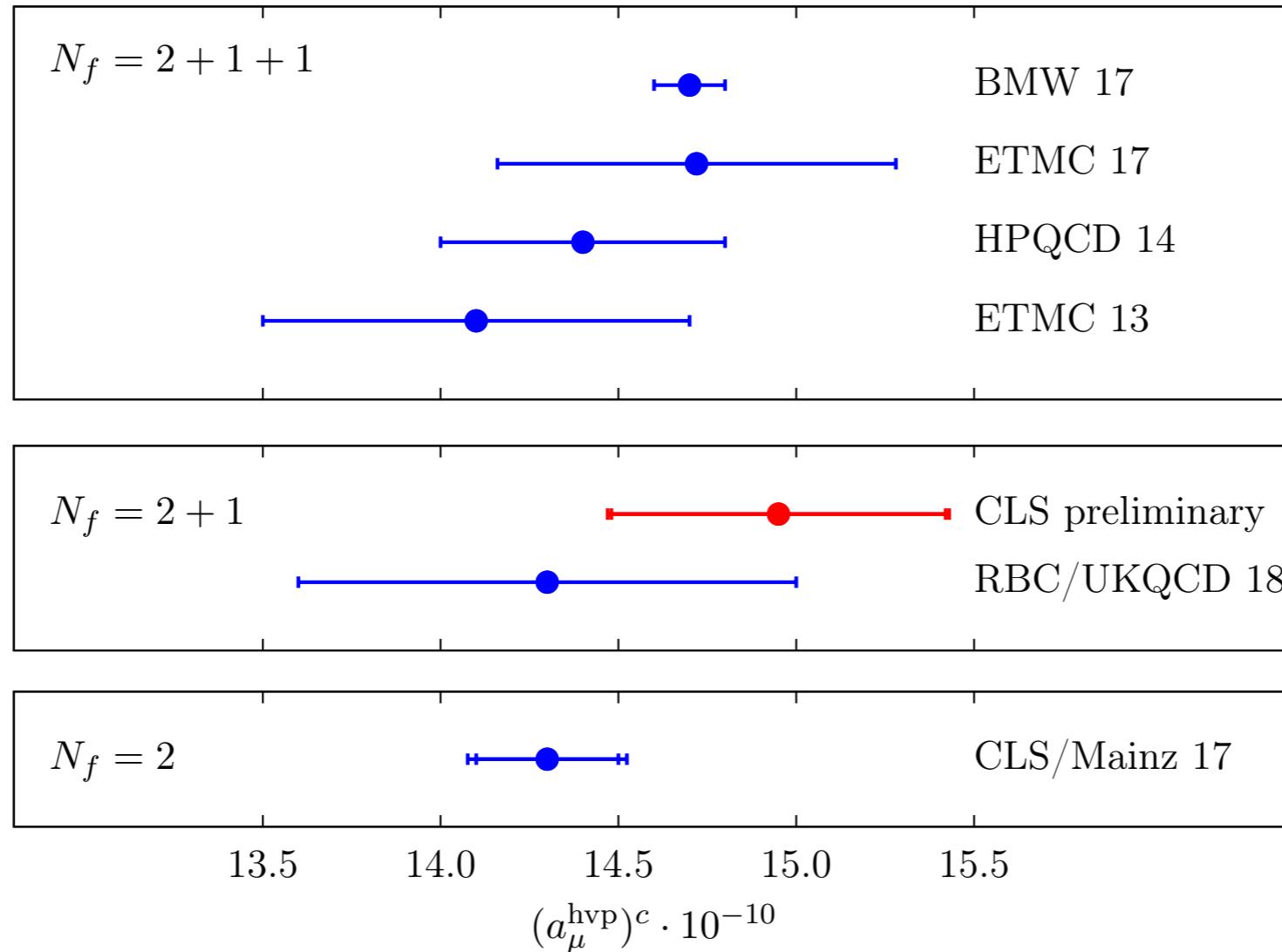
# Preliminary results: charm

- \* Chiral and continuum extrapolation



- \* Simultaneous fits using two discretisations work well
- \* Physical pion mass to be added — determine  $K_{\text{charm}}$

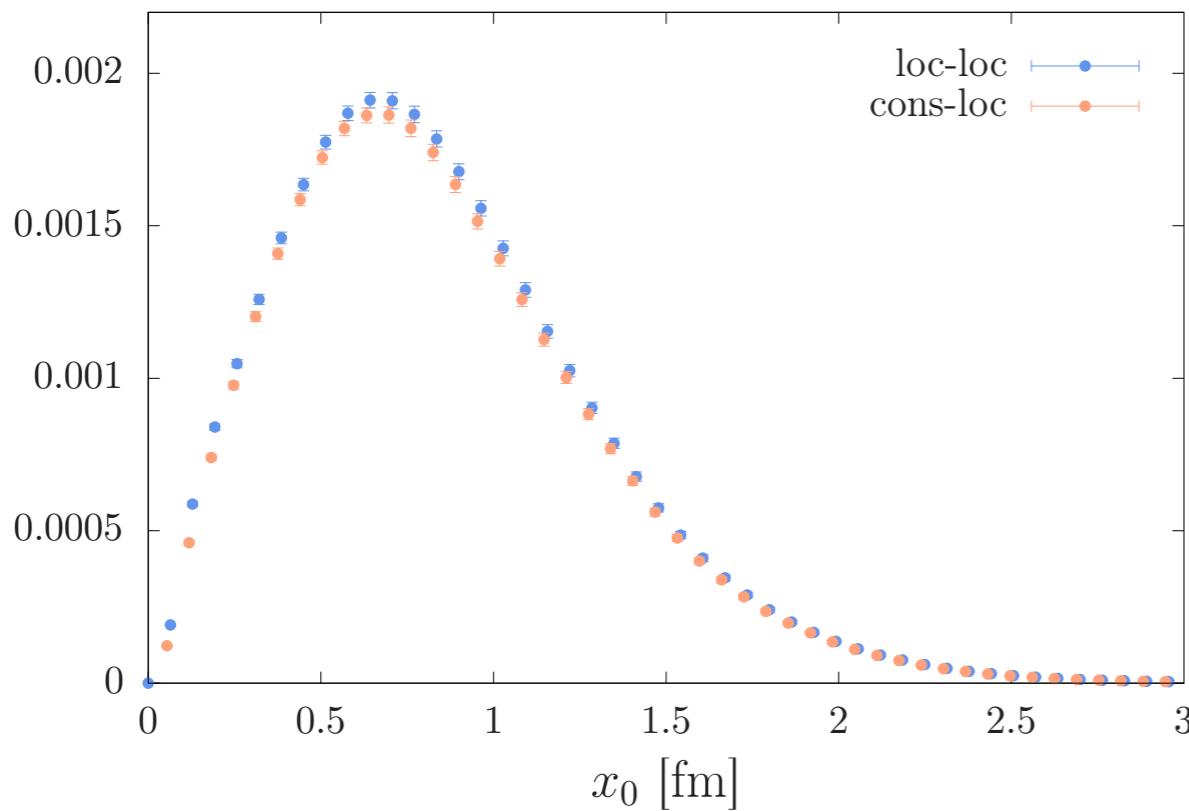
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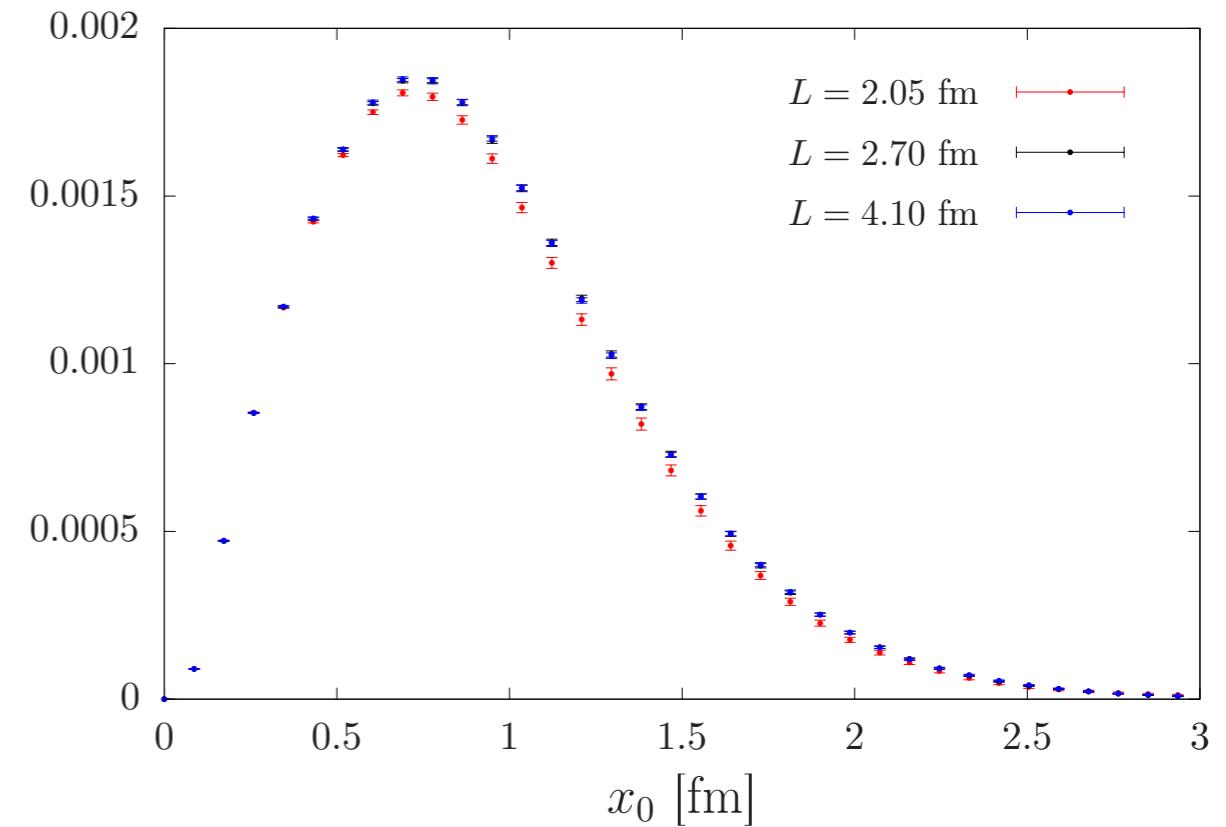
- \* Scale setting error dominates  $(a_\mu^{\text{hyp}})^c = 14.95(47)_{\text{stat}}(11)_\chi \cdot 10^{-10}$
- \* Systematic error from pion mass cut at  $m_\pi = 360 \text{ MeV}$

# Preliminary results: strange

Physical pion mass



FV effects @  $m_\pi = 280$  MeV



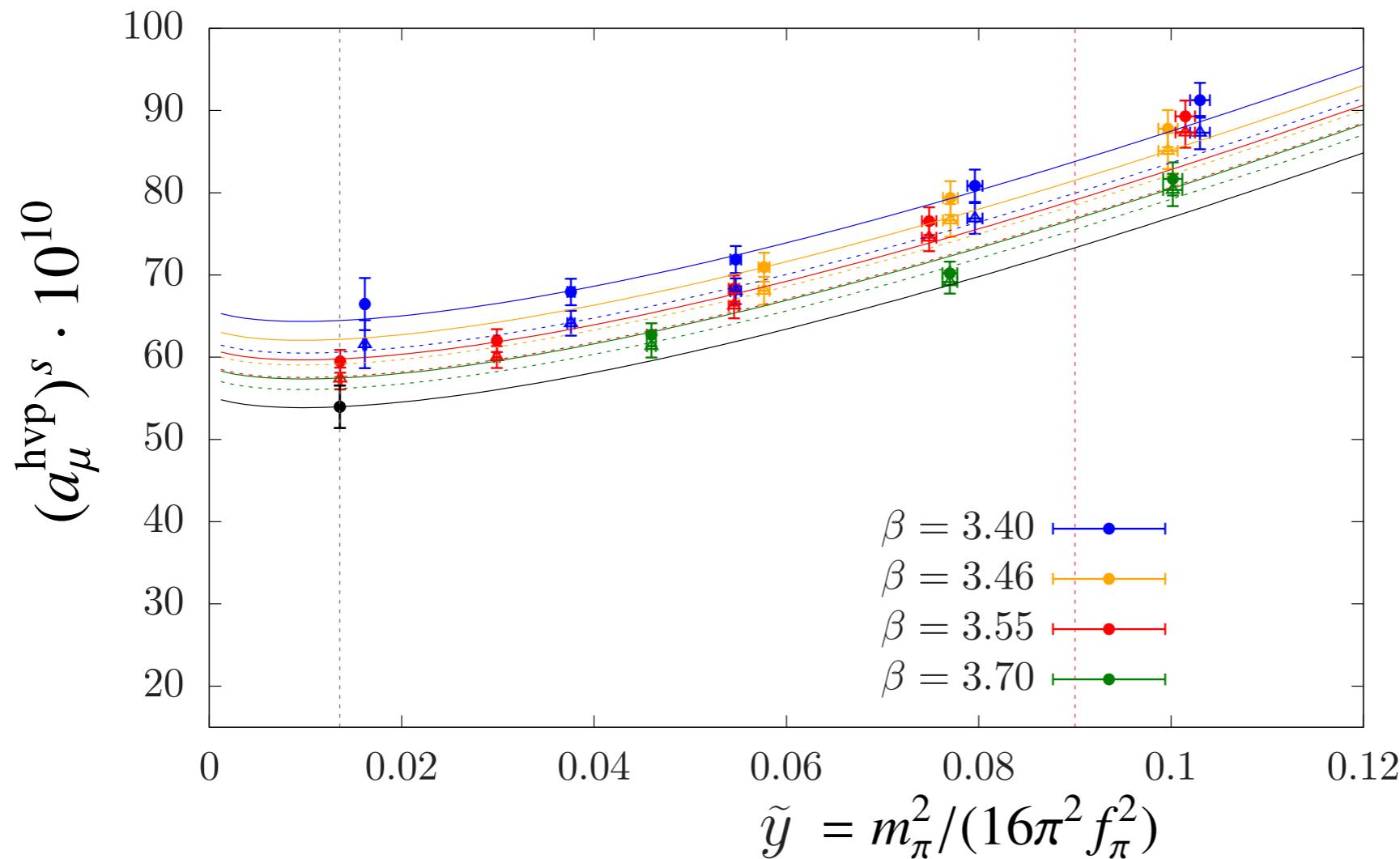
- \* Small remaining discretisation effects
- \* Finite-volume effects negligible for  $L \geq 2.7$  fm

$L$ [fm]	loc-con	loc-loc
2.05	65.8(0.6)	69.5(0.6)
2.70	68.0(0.4)	71.8(0.4)
4.10	68.0(0.3)	71.9(0.3)

# Preliminary results: strange

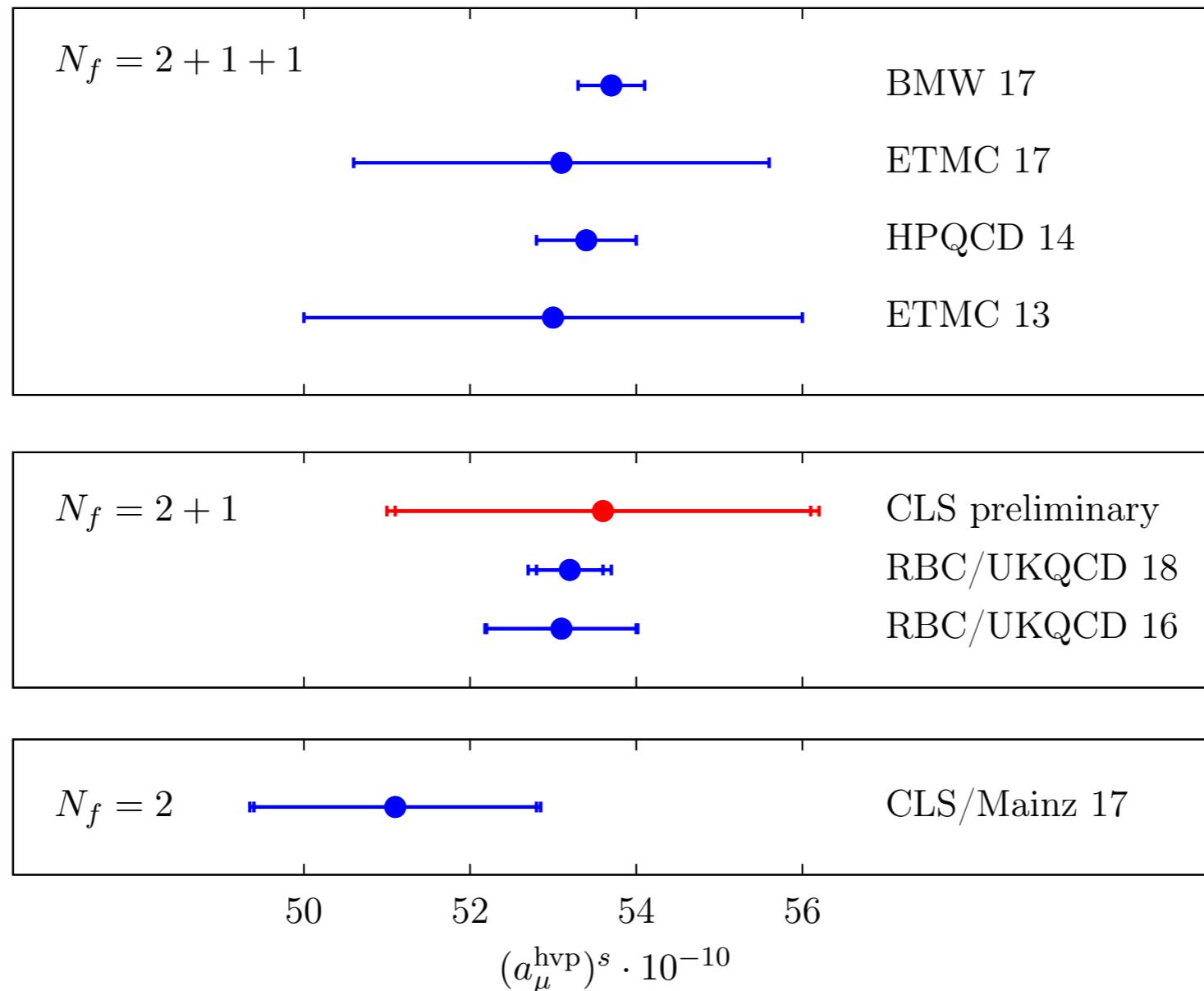
- \* Simultaneous chiral and continuum extrapolation

$$(a_\mu^{\text{hyp}})^s = \gamma_1 + \gamma_2 m_\pi^2 + \gamma_3 m_\pi^2 \ln m_\pi^2 + \gamma_4 a^2$$



- \* Apply pion mass cut to estimate systematic error

# Preliminary results: strange



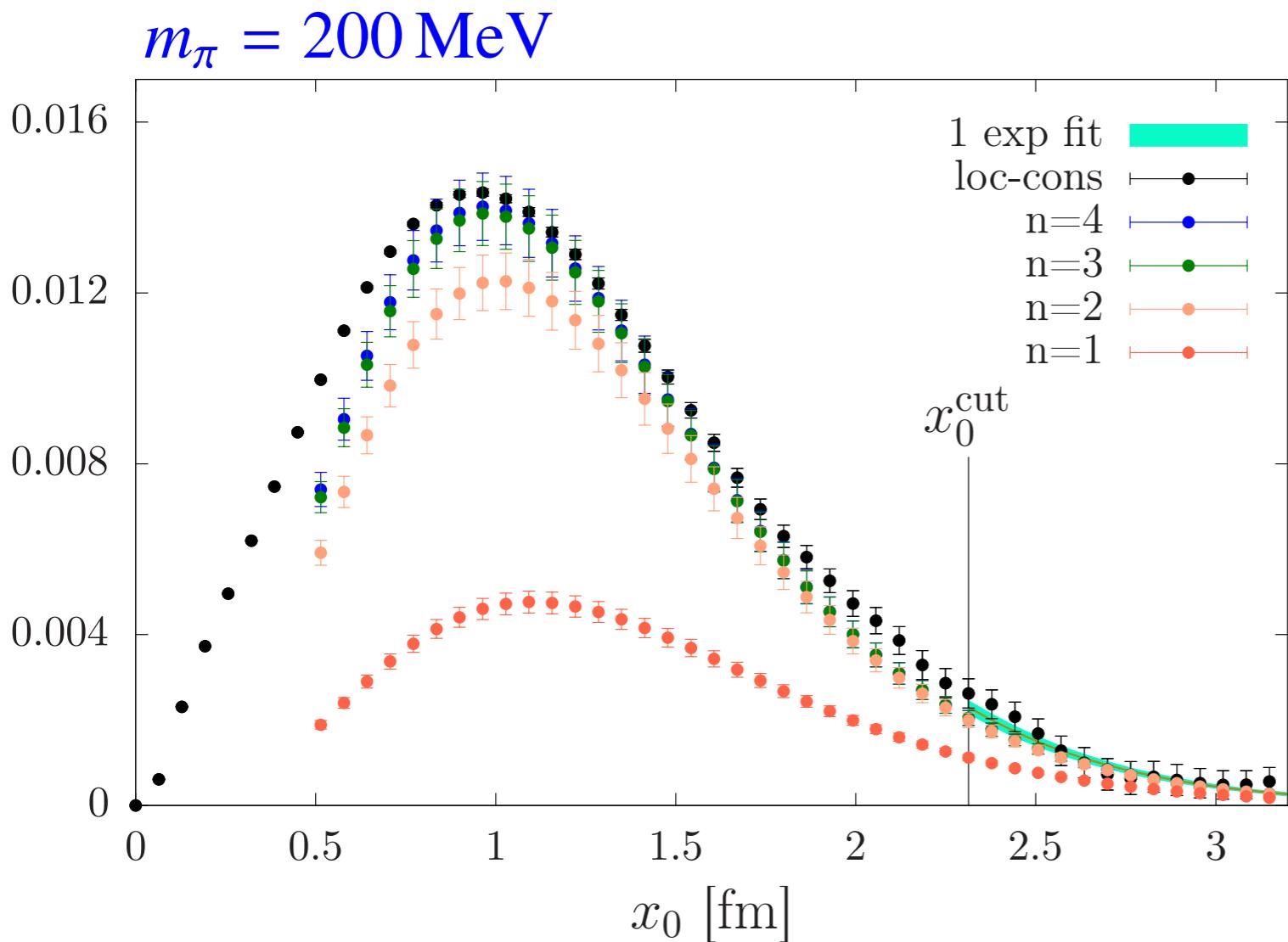
\* Scale setting error dominates

$$(a_\mu^{\text{hvp}})^s = 53.6(2.5)_{\text{stat}}(0.8)_\chi \cdot 10^{-10}$$

# Preliminary results: light quarks

- \* Long-distance regime of iso-vector TMR correlator

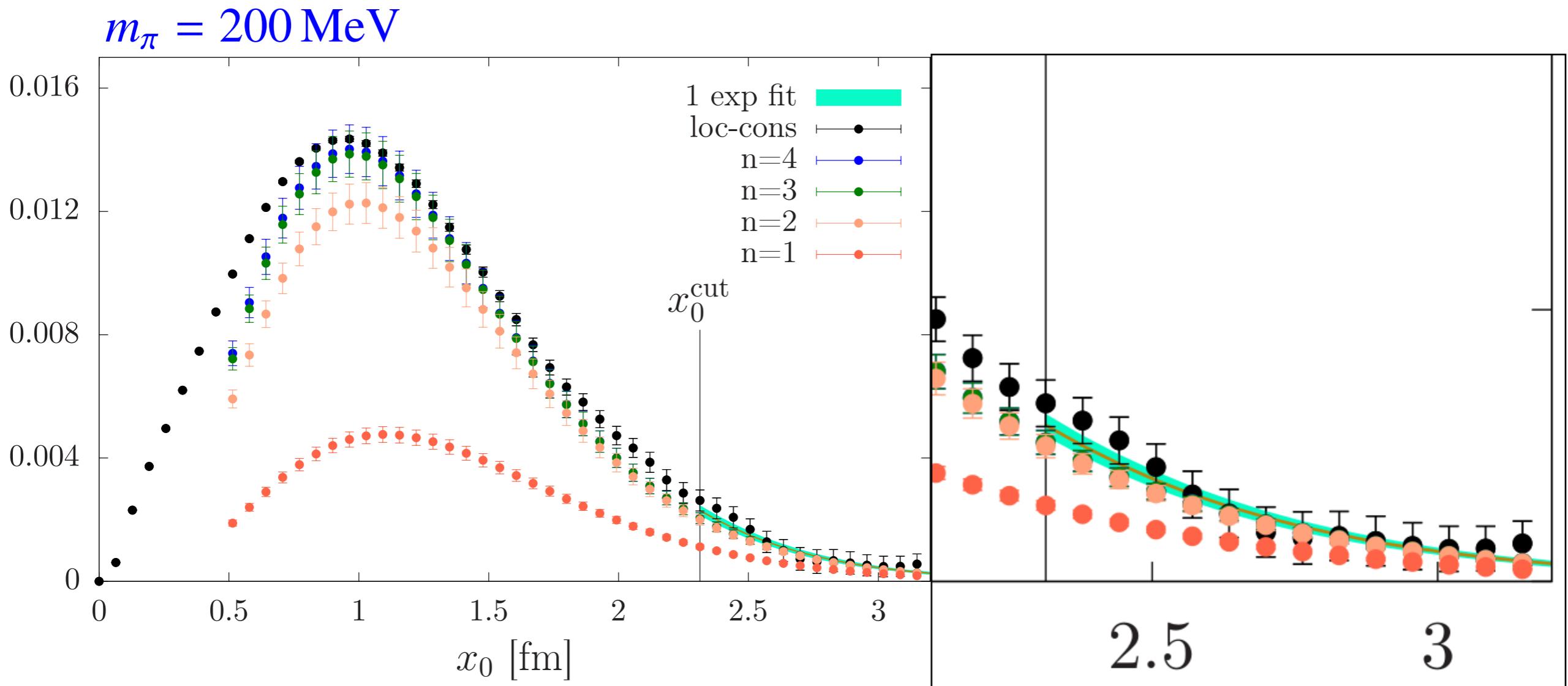
$$G^{\rho\rho}(x_0, L) \stackrel{x_0 \rightarrow \infty}{=} \sum_n |A_n|^2 e^{-\omega_n x_0}, \quad \omega_n = 2 \sqrt{m_\pi^2 + k^2}$$



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# Preliminary results: light quarks

- \* Finite-volume correction

$$a_\mu^{\text{hvp}}(\infty) - a_\mu^{\text{hvp}}(L) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0 \tilde{K}(x_0) [G(x_0, \infty) - G(x_0, L)]$$

- \* Isospin decomposition:  $G(x_0) = G^{\rho\rho}(x_0) + G^{I=0}(x_0)$

- \* Iso-vector correlator in infinite volume

$$G^{\rho\rho}(x_0, \infty) = \int_0^\infty d\omega \omega^2 \rho(\omega^2) e^{-\omega|x_0|}, \quad \rho(\omega^2) = \frac{1}{48\pi^2} \left(1 - \frac{4m_\pi^2}{\omega^2}\right)^{3/2} |F_\pi(\omega)|^2$$

- \* Finite volume:

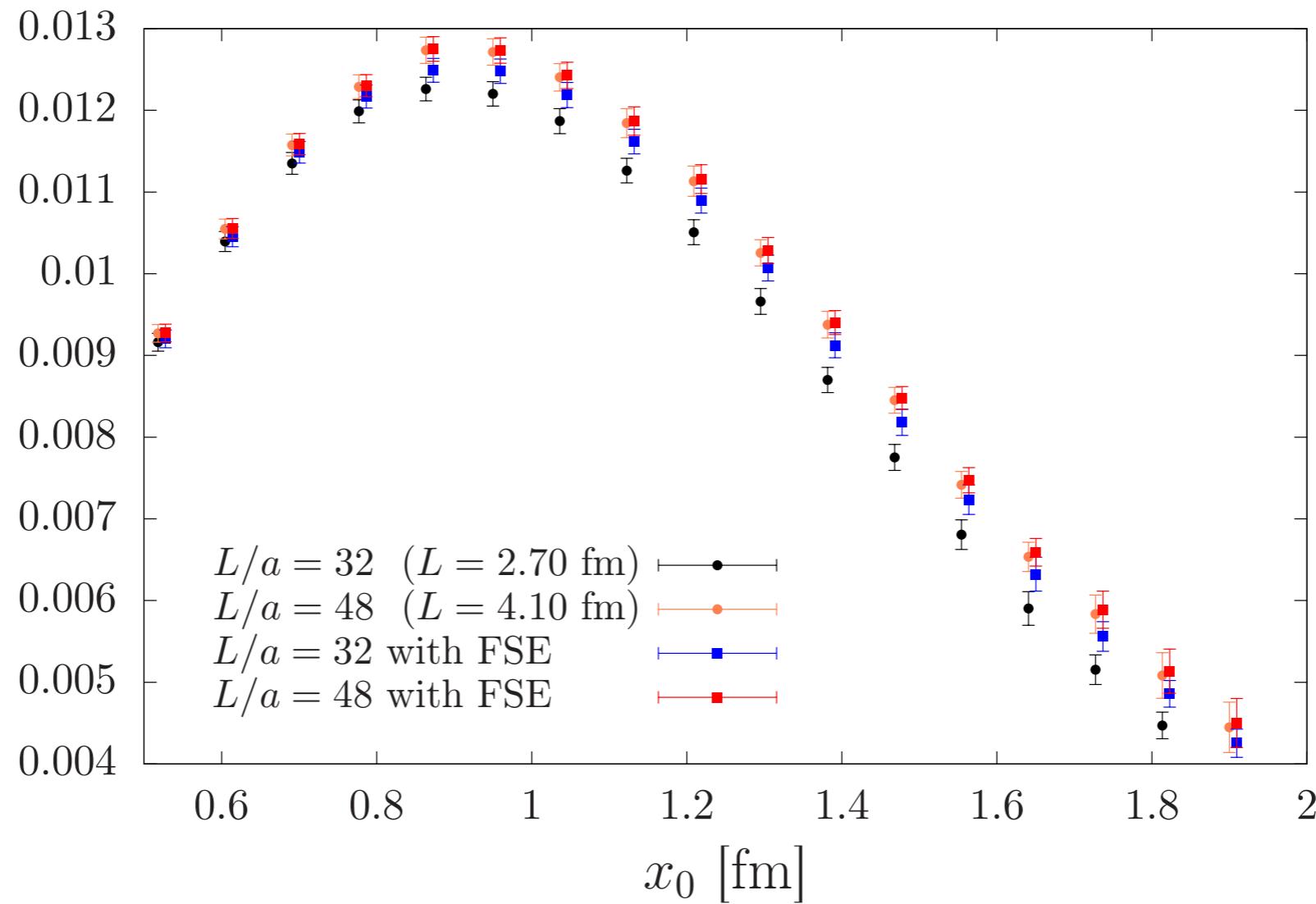
$$G^{\rho\rho}(x_0, L) \stackrel{x_0 \rightarrow \infty}{=} \sum_n |A_n|^2 e^{-\omega_n x_0}, \quad |A_n|^2 = \frac{2k^5}{3\pi\omega^2} \frac{|F_\pi(\omega)|^2}{\{k\phi'(k) + k\delta'_1(k)\}}$$

- \* Use Gounaris-Sakurai parameterisation of  $F_\pi(\omega)$  in terms of  $(m_\rho, \Gamma_\rho)$

[Francis et al., PRD 88 (2013) 054502; Della Morte et al., JHEP 10 (2017) 020]

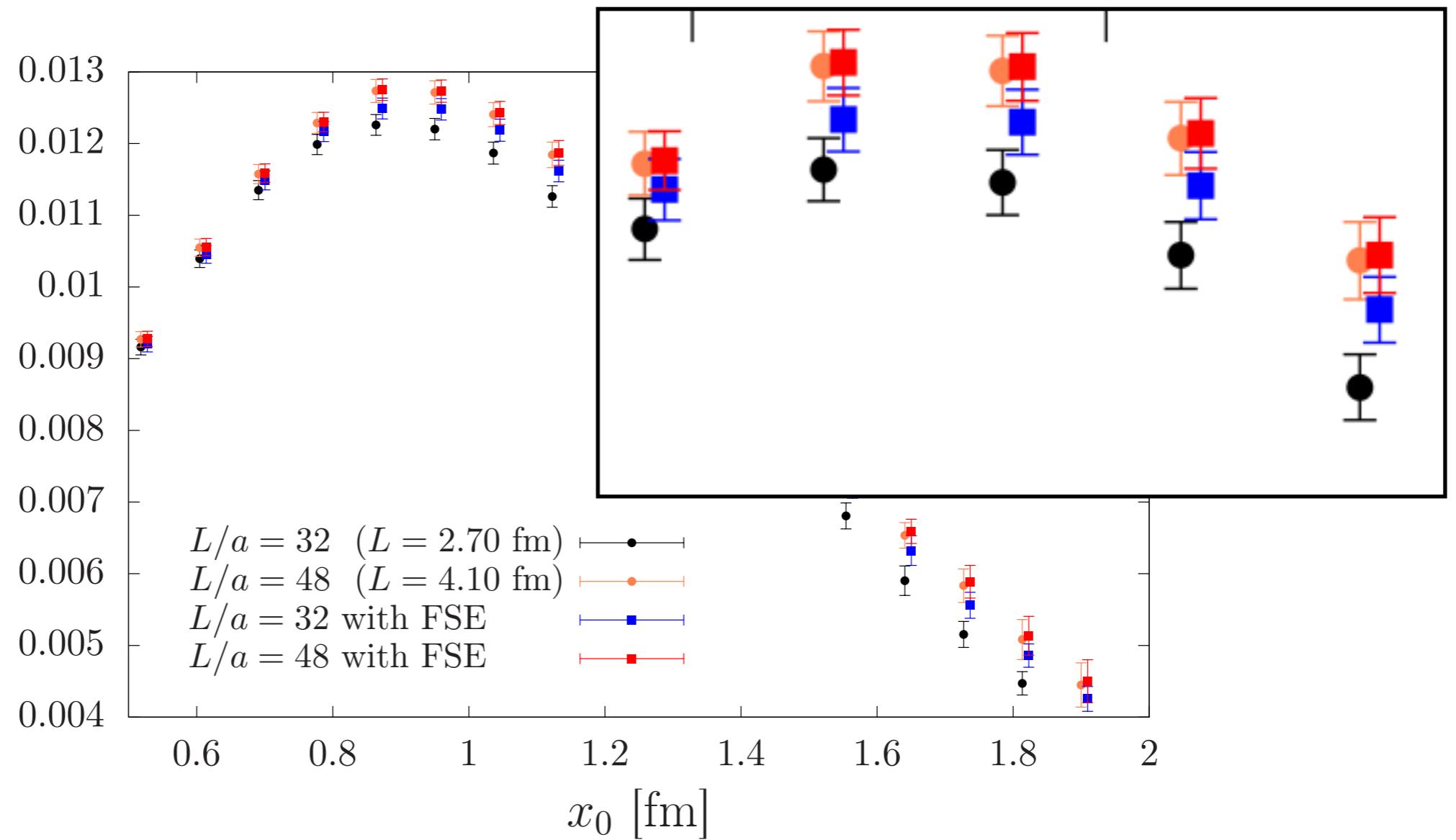
# Preliminary results: light quarks

- \* Cross check at  $m_\pi = 280$  MeV (H105, N101:  $m_\pi L = 3.8, 5.8$ )



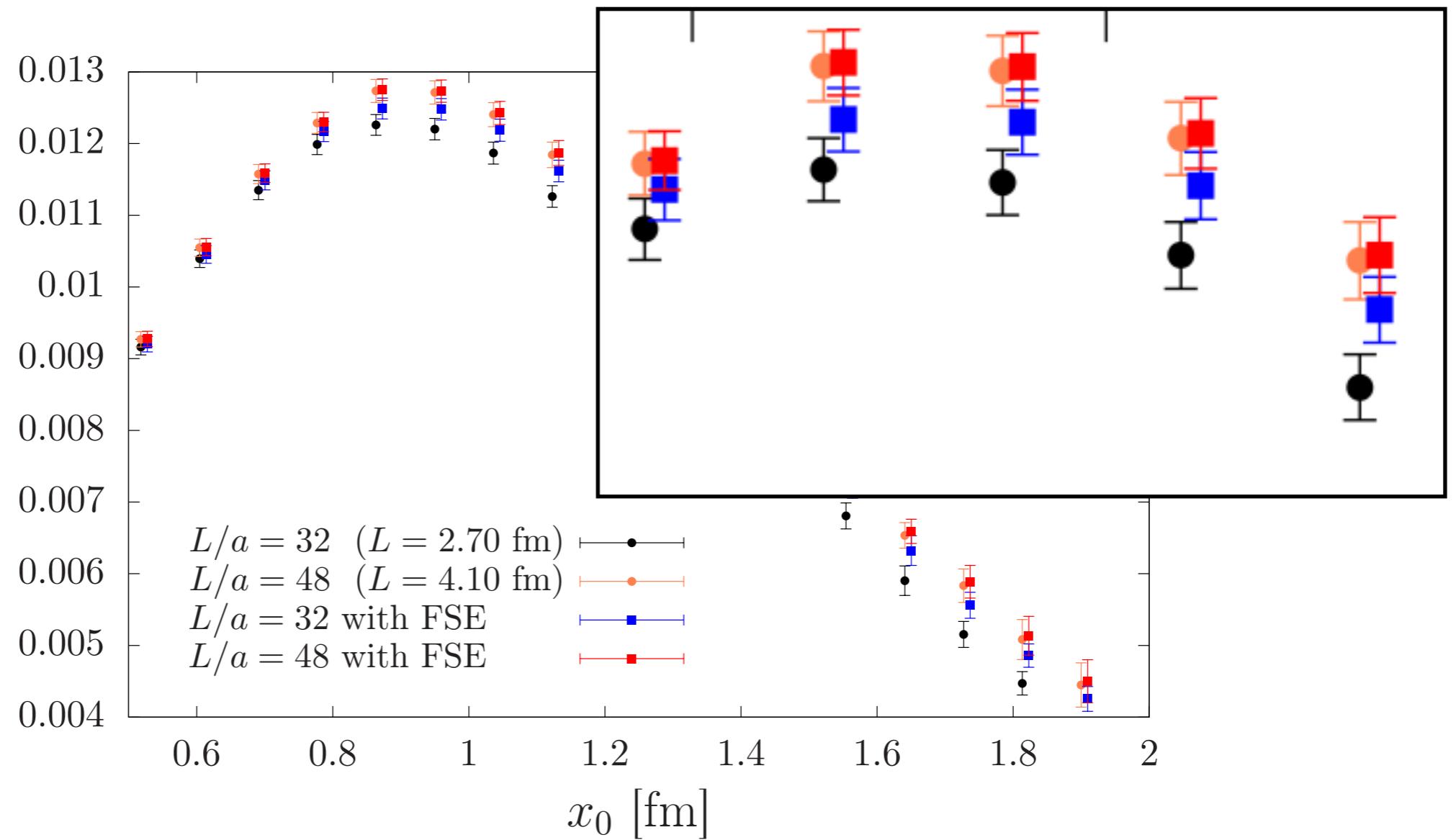
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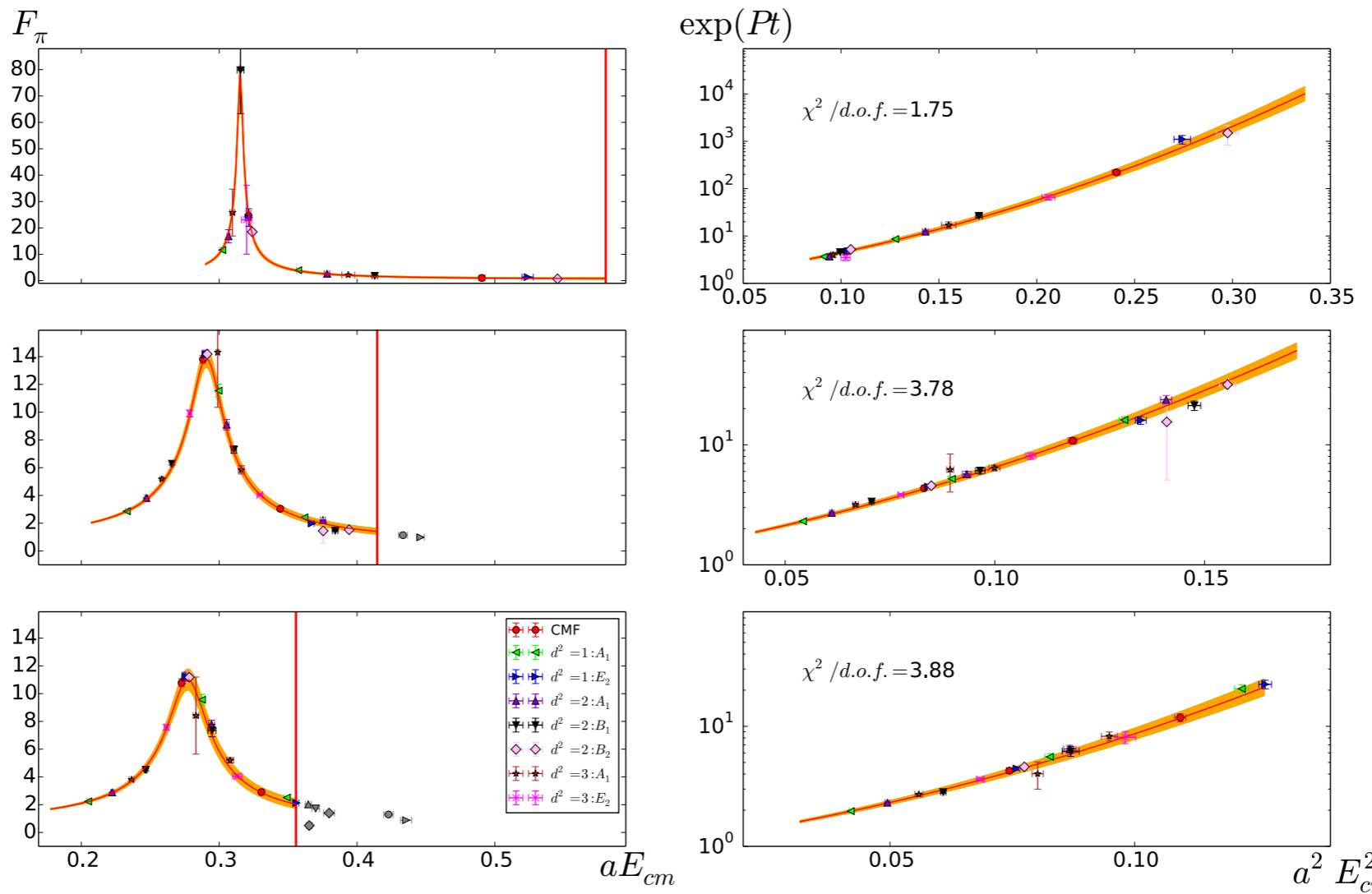


- \* Finite-size effects well described by GS parameterisation of  $F_\pi(\omega)$

# Preliminary results: light quarks

- \* Representation of  $F_\pi(\omega)$  via 3-subtracted Omnès relation

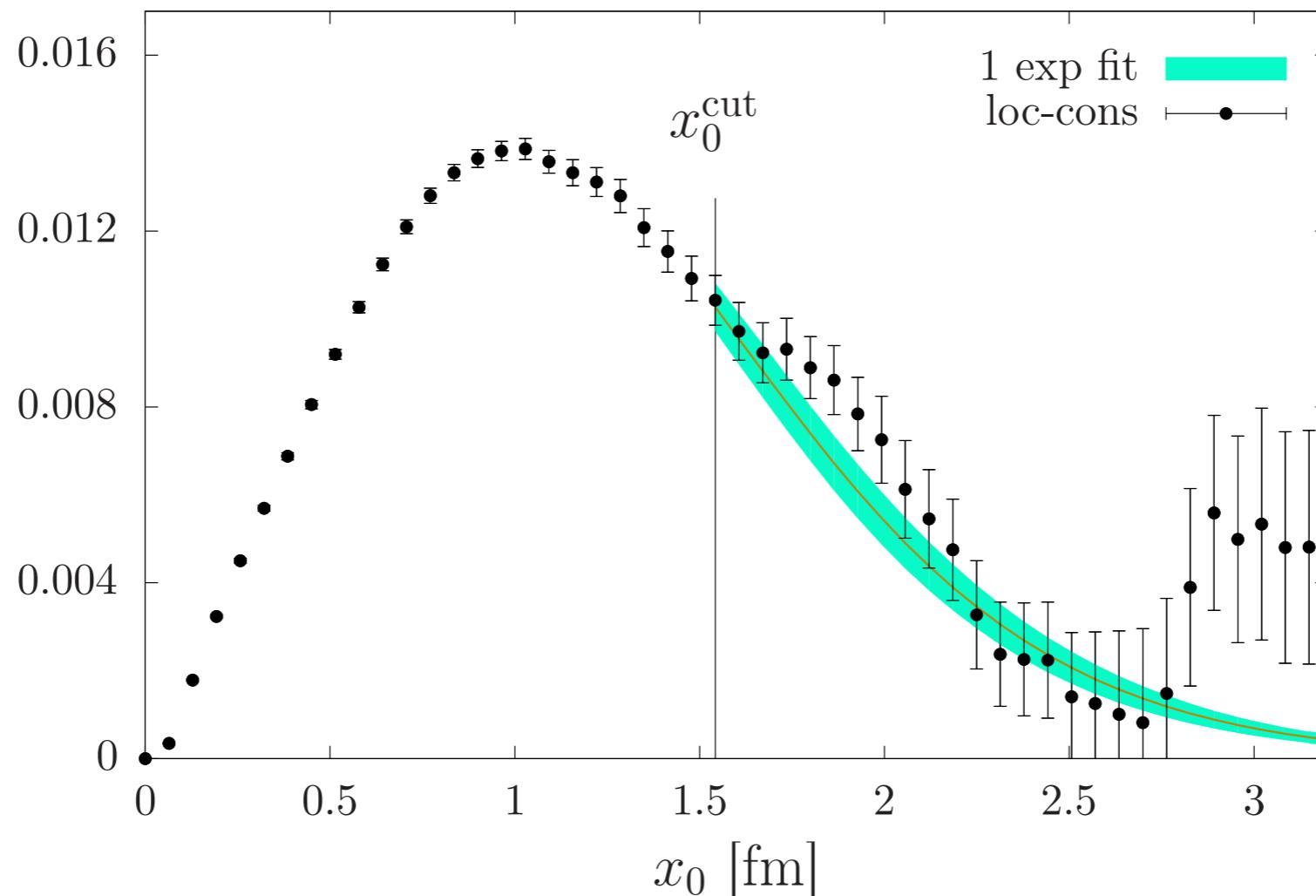
$$F_\pi(t) = \exp\left(P_{n-1}(t) + \frac{t^n}{\pi} \int_{4m_\pi^2}^{\infty} ds \frac{\delta_{11}(s)}{s^n(s-t-i\epsilon)}\right)$$



[Erben, Green, Mohler, HW, arXiv:1710.03529 and in prep.]

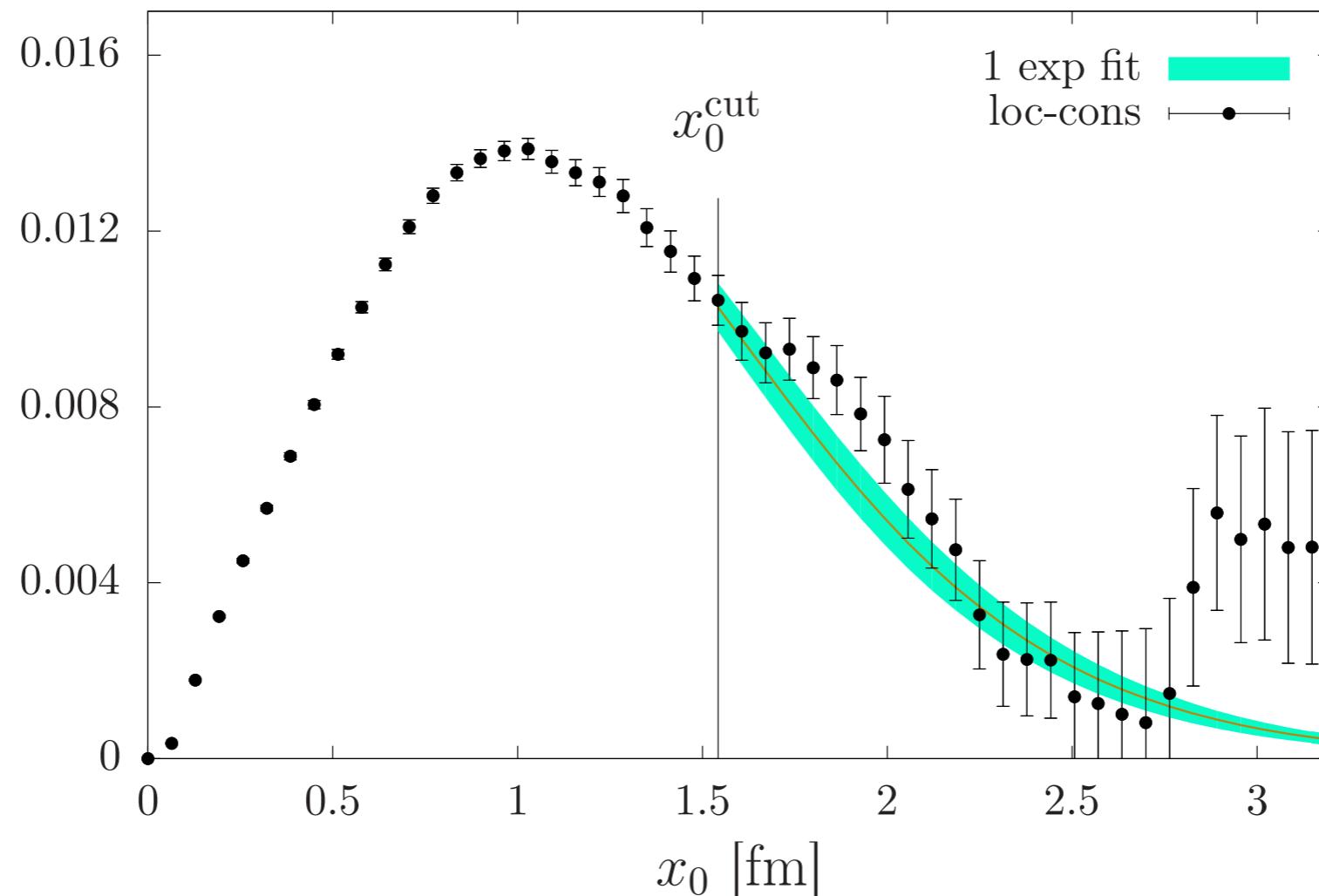
# Preliminary results: light quarks

- \* TMR integrand at the physical pion mass with current statistics



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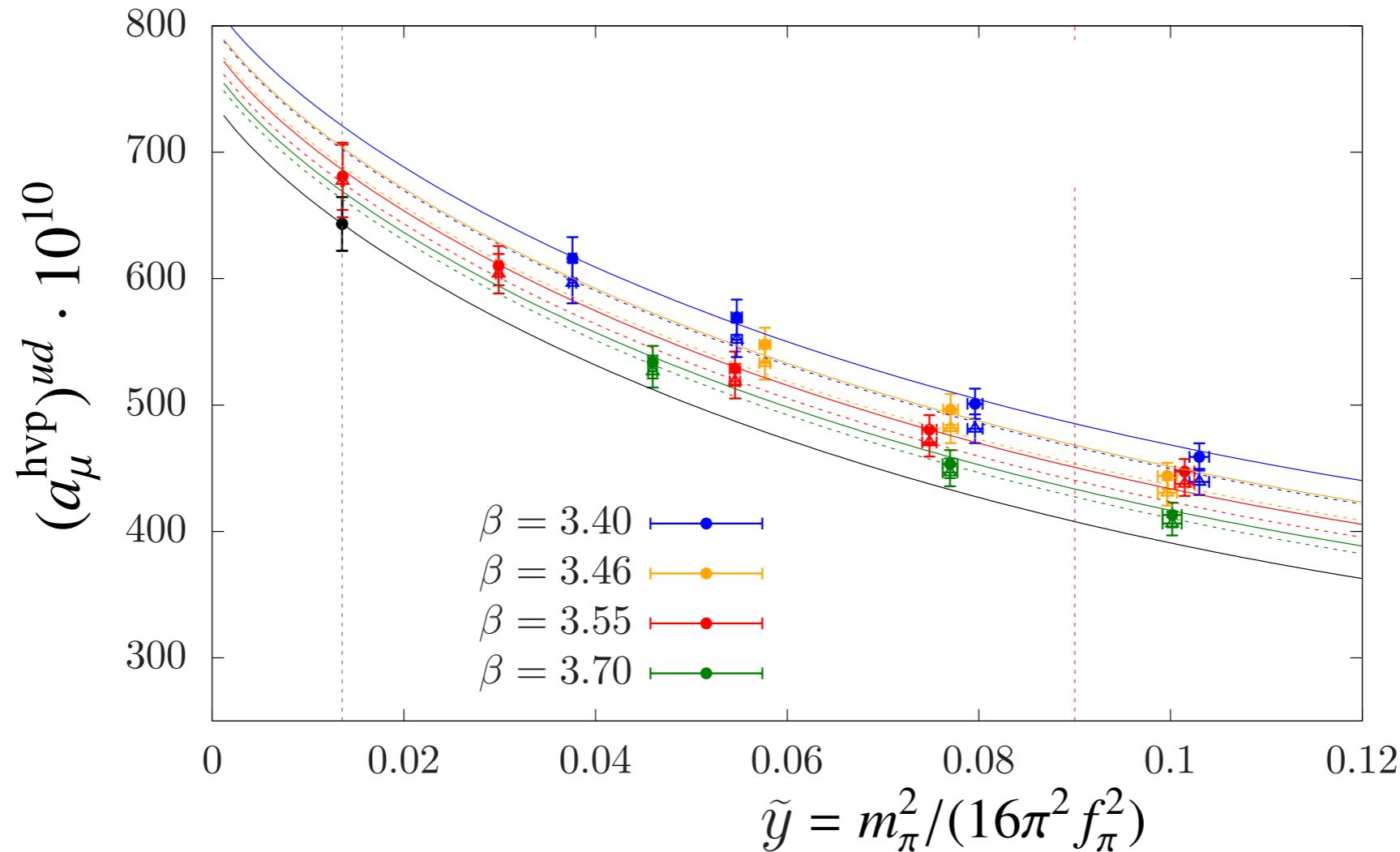
- \* TMR integrand at the physical pion mass with current statistics



- \* Estimated FV correction at  $m_\pi = 140$  MeV,  $m_\pi L = 4$ :  $(20.4 \pm 4.1) \cdot 10^{-10}$   
For  $m_\pi L = 6$  the correction is smaller by factor  $\approx 10$

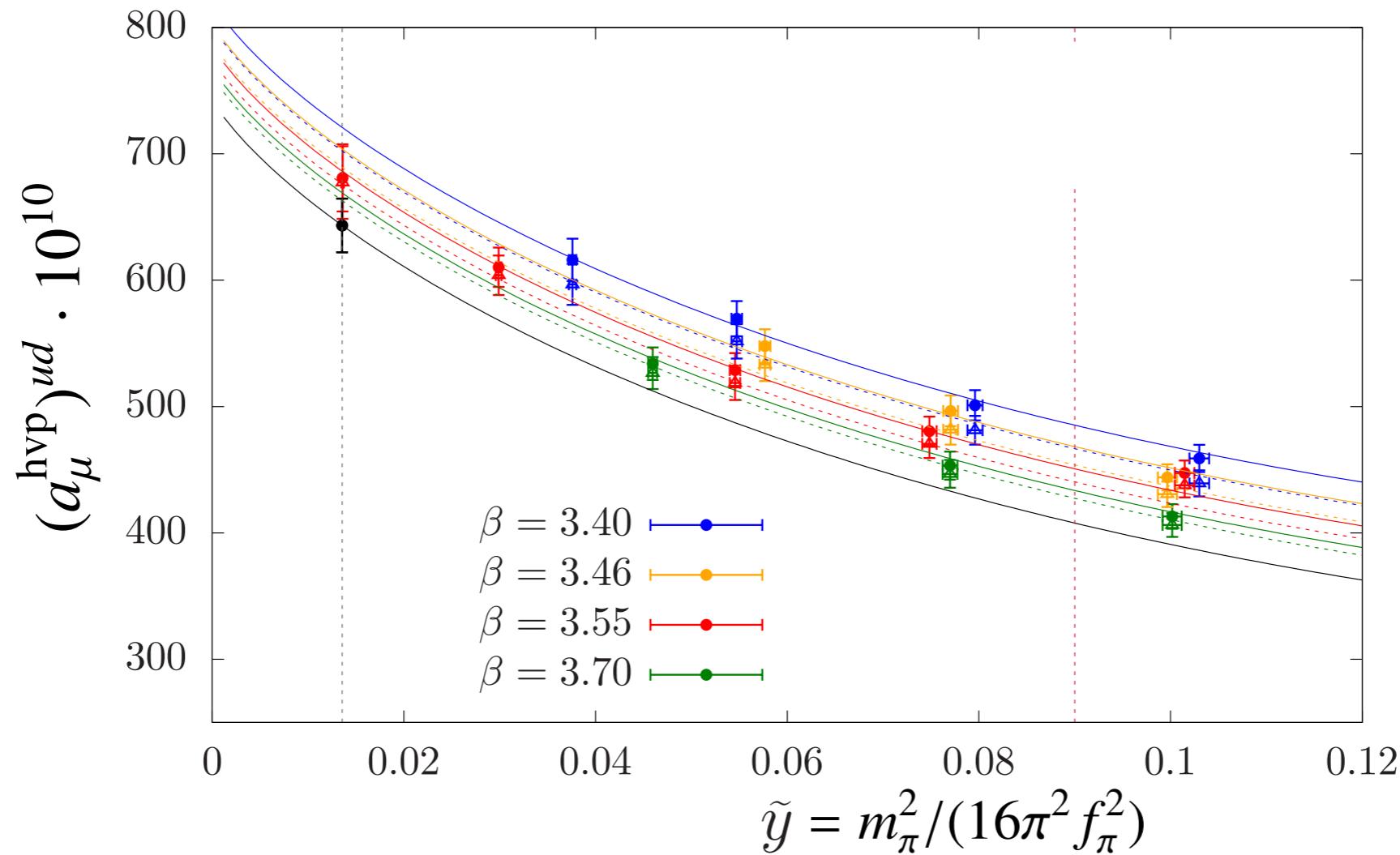
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# Preliminary results: light quarks

- \* Chiral and continuum extrapolation:



- \* Chiral fit in good agreement with direct calculation at physical pion mass

$$(a_\mu^{\text{hvp}})^{ud} = 643(21)_{\text{stat}}(xx)_{\text{syst}}$$

# Disconnected diagrams

- \* Stochastic noise cancellation: [Francis, GÜLPERS et al., 1411.7592]

$$G_{\text{disc}}(x_0) = -\frac{1}{9} \left\langle \left( \Delta^{ud}(x_0) - \Delta^s(x_0) \right) \left( \Delta^{ud}(0) - \Delta^s(0) \right) \right\rangle$$

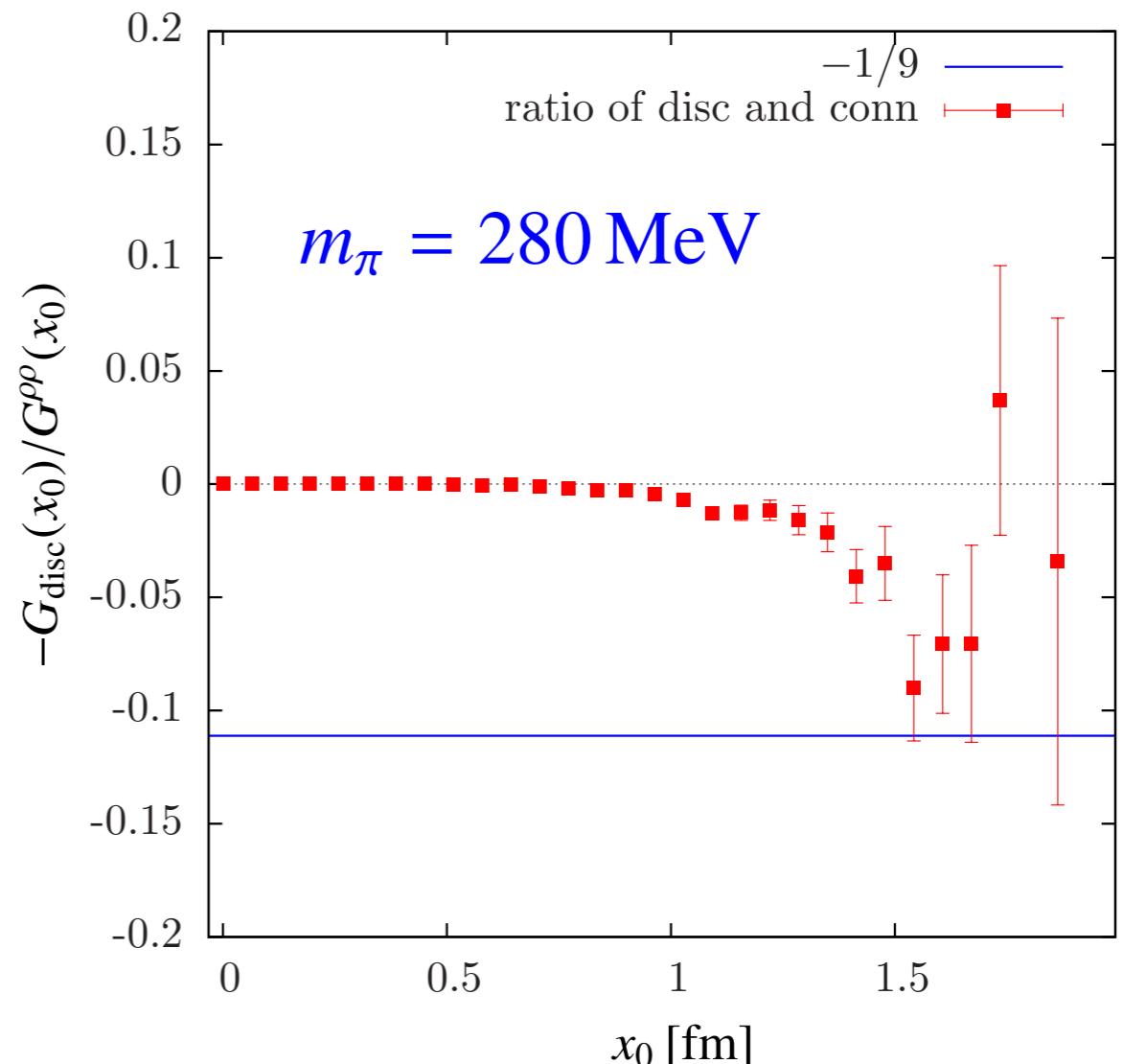
$$\Delta^f(x_0) = a^3 \sum_x \left\langle \text{Tr} (S^f(x, x) \gamma_k) \right\rangle$$

- \* Employ hierarchical probing,  
Hadamard vectors

[Statopoulos & Orginos, 1302.4077]

- \* Asymptotic behaviour:

$$-\frac{G_{\text{disc}}(x_0)}{G^{\rho\rho}(x_0)} \xrightarrow{x_0 \rightarrow \infty} -\frac{1}{9}$$



# Disconnected diagrams

- \* Lorentz-covariant coordinate space representation [H.B. Meyer, EPJC 77 (2017) 616]

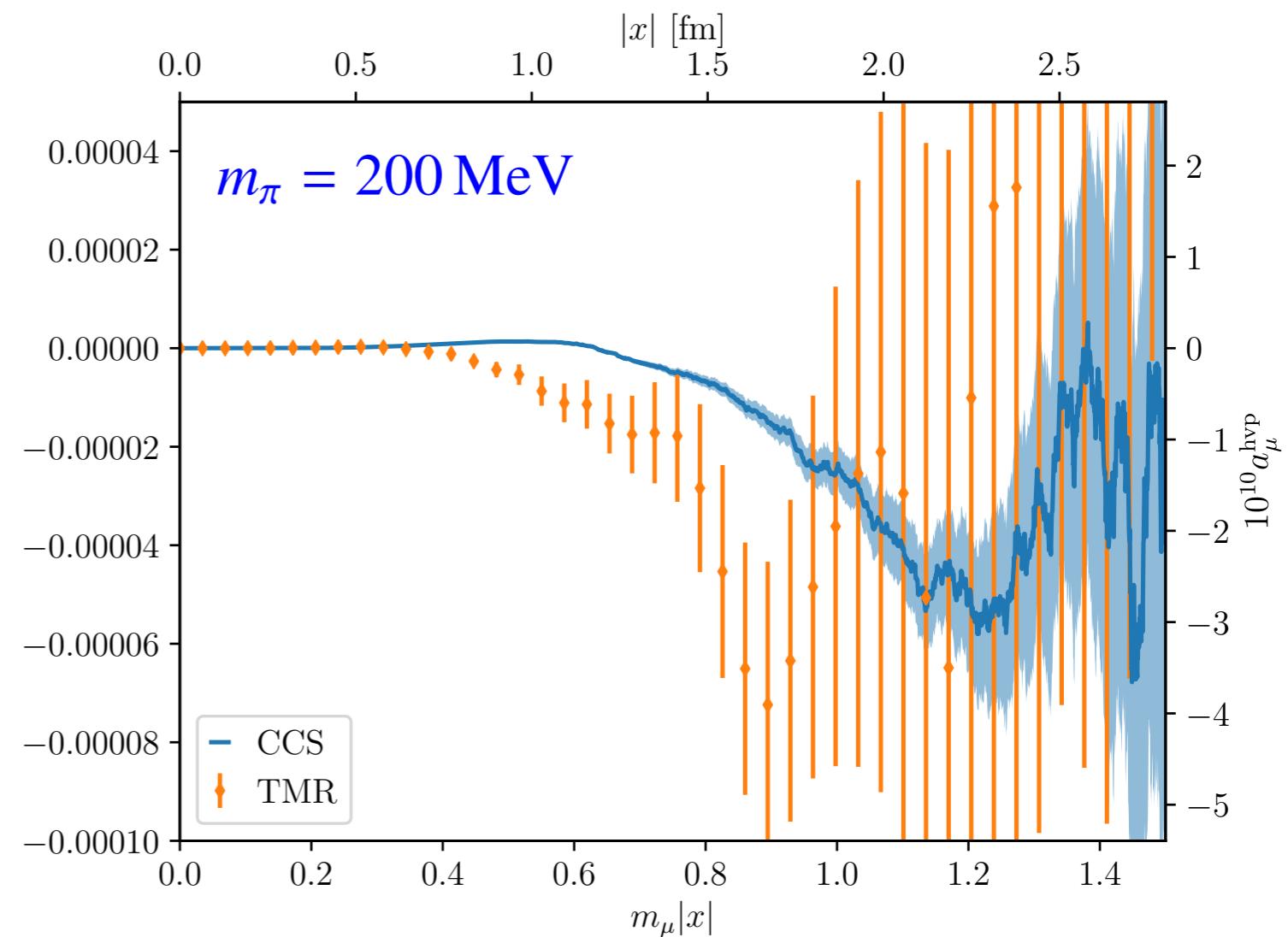
$$a_\mu^{\text{hyp}} = \lim_{R \rightarrow \infty} \int_{|x| < R} d^4x G_{\mu\nu}(x) H_{\mu\nu}(x), \quad G_{\mu\nu}(x) \equiv \langle J_\mu(x) J_\nu(0) \rangle$$

- \* Analytic expression for  $H_{\mu\nu}(x)$  in terms of Bessel functions

- \* Expect higher statistical precision

- \* Application to running of  $\sin^2 \theta_W$

[Marco Cè, FRI 15:40]



# Summary & Outlook

## Preliminary result for HVP contribution from CLS $N_f = 2+1$ ensembles

- Excellent control over long-distance regime: iso-vector correlator
- O(a) improved vector currents

$$a_\mu^{\text{hvp}} = 711 \cdot 10^{-10} \pm 3\%$$

- Error dominated by lattice scale

## Refinements

- Increase statistics
- Include quark-disconnected diagrams
- Include isospin-breaking corrections