Linked Cluster Expansions for the Functional Renormalization Group

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Outline

1. Functional Renormalization Group
2. Linked Cluster Expansions and the Functional Renormalization Group
3. Critical Behavior of φ^4 Theory in Four Dimensions
4. Spatial Linked Cluster Expansions in Friedmann-Lemaître spacetimes
5. Conclusions and Outlook

Functional Renormalization Group (FRG)

The FRG is a reformulation of QFT; study non-linear response of functionals to scale dependent mode modulation – in functional integral replace action $s[\varphi] \to s[\varphi] + \frac{1}{2}\varphi \cdot R_k \cdot \varphi$. R_k suppresses low energy modes.

Modern formulations focus on the Legendre transform of the Polchinski equation, determining the Legendre Effective (aka Effective Average) Action.

Legendre Effective Action Method

Wetterich, Christof.
"Exact evolution equation for the effective potential."
Physics Letters B 301.1 (1993): 90-94.

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} Tr \left(\frac{\partial_k R_k}{\Gamma_k^{(2)}[\phi] + R_k} \right)$$

Successes of FRG

- New approximation schemes: no expansion in conventional coupling constants.
- Consistent with known results: ϵ expansion, large N expansion, ...
- Excellent effort to outcome ratio: relatively little effort yields fixed points, critical exponents, Wilson-type β-functions, some access to momentum-dependent correlation functions.
- Computations feasible for any spacetime dimension D.

Weaknesses of FRG

1. Wetterich equation solved via truncation Ansätze

$$\Gamma_k[\phi] = \sum_n c_{n,k} \sigma_n[\phi]$$

   However, exact $\Gamma[\phi]$ is highly non-local, no structural characterization known. In particular, solution of $\Gamma_k[\phi]$ flow eq. via (non-series) truncations is ad-hoc, no clear ordering principle. Non-local terms, *e.g.* $\phi\partial^{-2}\phi$, $(\partial\phi)^2\phi^5\partial^{-10}\phi$?

2. To solve Wetterich equation, need initial condition(s) typically at $k = \Lambda_{UV}$ (it may be ill-posed at $k = \Lambda_{UV}$).

   With standard choice: $\Gamma_{k=\Lambda_{UV}}[\phi] = s_{bare}[\phi]$, one makes implicit reference to perturbation theory.

3. No statement about asymptotic correctness or convergence of truncations is known.

Remedying the Weaknesses

- Fix Weakness 2: use ultralocal+linking split of action in lattice formulation

$$s[\varphi] = \sum_x \underbrace{s_0(\varphi_x)}_{\text{ultralocal}} + \frac{1}{2}\underbrace{\varphi \cdot \ell \cdot \varphi}_{\text{linking}},$$

  and specify ultralocal initial data at some $k = k_0$ via exact single site integrals depending on $s_0(\varphi)$ only (choose R_k s.t. $R_{k=k_0} = -\ell$) [Dupuis-Machado, 2010].

- We address Weakness 1 via linked cluster expansion of $\Gamma_k[\phi]$ via $\ell \to \ell + R_k$ (potentially long ranged).

- Perspective on Weakness 3: rigorous proofs for convergence of linked cluster expansion known in many other cases.

Linked Cluster Expansion (LCE) and the FRG

On lattice write action $s[\varphi] = \sum_x s_0(\varphi_x) + \frac{\kappa}{2}\varphi \cdot \ell \cdot \varphi$.
ultralocal *linking*

LCE is expansion of quantities in powers of κ, in particular

$$\Gamma_\kappa[\phi] = \sum_{l=0}^{\infty} \kappa^l \, \Gamma_l[\phi].$$

FRGs entail closed recursion relations for Γ_ls.

Obtain solution to Wetterich eq. from solution to LCE recursion:

$$\Gamma_k[\phi] = \Gamma_\kappa[\phi]\Big|_{\ell \to \ell + R_k}$$

However, direct iteration of recursion impractical beyond $O(\kappa^6)$

Solve recursions with GRAPHICAL METHODS instead.

Γ_κ LCE Graph Rules

Goal: Convert known LCE graph rules for Generating Functional $W_\kappa[J]$ [Wortis, 1974] into ones applicable to $\Gamma_\kappa[\phi]$ LCE.

$\Gamma_\kappa[\phi]$ related to $W_\kappa[J]$ by modified Legendre transform:

$$\Gamma_\kappa[\phi] := \phi \cdot J_\kappa[\phi] - W_\kappa[J_\kappa[\phi]] - \frac{\kappa}{2}\phi \cdot \ell \cdot \phi\,, \quad \frac{\delta W_\kappa}{\delta J}\big(J_\kappa[\phi]\big) = \phi\,.$$

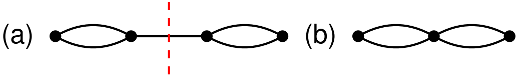
Insert κ-series expansions for Γ_κ, W_κ, and J_κ, get mixed Γ_m ($m < l$), W_m ($m \leq l$) recursion ($*$) for Γ_l.

Our result: exact graph solution of the recursion.

Connected and One-Line-Irreducible Graphs

$W_\kappa[J]$ LCE graph expansion$\rightarrow$ **Connected** graphs.

$\Gamma_\kappa[\phi]$ LCE graph expansion$\rightarrow$ **One-Line-Irreducible** (or 1PI) graphs.



Analogous to perturbation theory.

Considerable net computational gain:

l	$\lvert\mathcal{C}_l\rvert$	$\lvert\mathcal{L}_l\rvert$
2	2	1
3	5	2
4	12	4
5	33	8
6	100	22

Table 1: Number of connected, one-line irreducible graphs with l edges.

Theorem

For any $l \geq 2$ the solution of the recursion ($*$) is given by

$$\Gamma_l[\phi] = \sum_{L=(V,E)\in\mathcal{L}} \frac{(-)^{l+1}}{Sym(L)} \prod_{e\in E} \ell_{s(e),t(e)} \prod_{v\in V} \mu^{\Gamma}(v|L)$$

$$\mu^{\Gamma}(v|L) = \sum_{n=1}^{|I(v)|} \sum_{T\in\mathcal{T}(B(v),n)} (-)^{s(T)} \frac{|Perm(B(v))|}{Sym(T)} \mu(T)\,.$$

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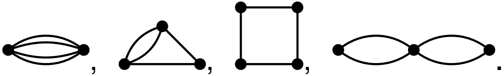
- At order l draw all topologically distinct 1PI graphs with l edges.
  E.g. The following graphs contribute to Γ_4:



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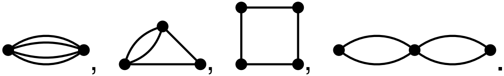

- Divide by the symmetry factor Sym(L) of the graph.

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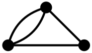
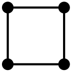
  E.g. The following graphs contribute to Γ_4:

$\frac{1}{48}$ , $\frac{1}{4}$ , $\frac{1}{8}$ , $\frac{1}{8}$ .

- Divide by the symmetry factor Sym(L) of the graph.

- In a graph L, for each edge connecting vertices v, v' write $-\ell_{v,v'}$, and for each vertex v a vertex weight $\mu^{\Gamma}(v|L)$.

  $\mu^{\Gamma}(v|L)$ is a finite sum of products of **exactly computable** single site functions $\varpi_n(\phi)$, $\gamma_n(\phi)$, determined by the single site action action $s_0(\varphi)$.

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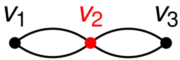

$$\mu^{\Gamma}(v_1|L) = \varpi_2(\phi_{v_1}),$$

$$\mu^{\Gamma}(v_2|L) = \varpi_4(\phi_{v_2}) - \gamma_2(\phi_{v_2})\varpi_3(\phi_{v_2})^2,$$

$$\mu^{\Gamma}(v_3|L) = \varpi_2(\phi_{v_3}).$$

$\mu^{\Gamma}(v|L)$ can be obtained as a sum over labeled tree graphs.

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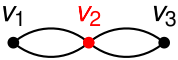

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  $\mu^{\Gamma}(v|L)$ can be obtained as a sum over labeled tree graphs.
- The $\mu^{\Gamma}(v)$ data can be stored in a look-up table.

Proof $\approx$ 40 pages, R.B. and M.N. under review.

Critical Behavior of φ^4 Theory in Four Dimensions

Reparametrize φ^4 action on lattice:

$$s[\varphi] = \sum_x \left(\varphi_x^2 + \lambda(\varphi_x^2 - 1)^2 - \lambda\right) - \frac{\kappa}{2} \sum_{x,y} \varphi_x \ell_{xy} \varphi_y$$

ultralocal *hopping*

- Critical line $\kappa_c(\lambda)$ yields continuum limit:

  correlation length $\xi \to \infty \Longleftrightarrow m_R = 1/\xi \to 0.$

- $\kappa_c(\lambda)$ obtained by Lüscher-Weisz [Lüscher-Weisz, 1987] using LCE of generalized susceptibilities, e.g.

$$\chi_2 := \sum_x < \varphi_x \varphi_0 >^c = \sum_{l \geq 0} \kappa^l \chi_{2,l}.$$

Considerable effort required.

FRG to LCE correspondence yields dramatic simplification.

FRG Perspective

Wetterich eq. can be solved on lattice by emulating LCE

$$\Gamma_k[\phi] = \Gamma_\kappa[\phi]\big|_{\ell\to\ell+R_k} = \sum_{l=0}^{\infty} \kappa^l \Gamma_{k,l}[\phi]$$

- Critical line determined by bulk quantities: use

  $$\Gamma_k[\phi]\big|_{\phi=\varphi=\mathrm{const}} = U_k(\varphi) = \sum_{l=0}^{\infty} \kappa^l U_{k,l}(\varphi)$$

  itself as bulk quantity.
- Expansion and resummation of κ-series commutes with homogenization in ϕ.
- Use homogenized FRG, i.e. the Local Potential Approximation (LPA) to resum κ-series.

Critical line from LPA

Rescale field and potential to obtain dimensionless LPA eq.

$$k\partial_k V_k(\varphi) = -4V_k(\varphi) + \varphi V_k'(\varphi) + \frac{vol(k)}{1 + V_k''(\varphi)}.$$

Base continuum limit **directly** on Gaussian fixed point.

Expand $V_k(\varphi) = \sum_{i=0}^{N} \frac{g_{2i}(k)}{(2i)!}\varphi^{2i}$, get closed system of N coupled ODEs.

In 4 dim. find only Gaussian fixed point with $g_{2i}^* = 0$ as $k \to 0$.

Inject bare data (λ, κ) at ultra-local scale $k = k_0$, numerically integrate to $k = 0$ to reach fixed point.

Shooting to the Fixed Point

Inject bare data (λ, κ) via ultralocal intial conditions $g_{2i}(k = k_0)$, employ shooting technique for ODEs to reach fixed point.

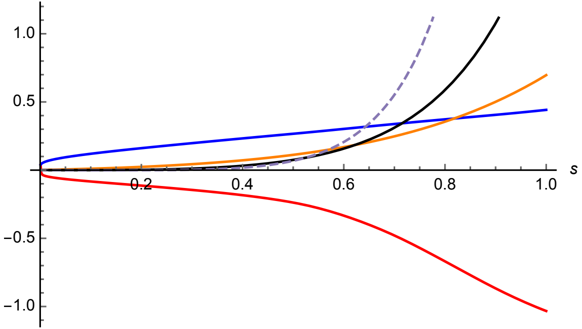


Figure 1: Flow of $g_2(s), g_4(s), g_8(s), g_{10}(s)$ for $(\lambda, \kappa) = (4.3303, 0.091693)$. Red: g_2, Blue: g_4, Orange: g_6, Black: g_8, Dashed: g_{10}. $s := k/k_0$

$\kappa_c(\lambda)$ Results and Comparsion

Compare our results to Lüscher-Weisz benchmark:

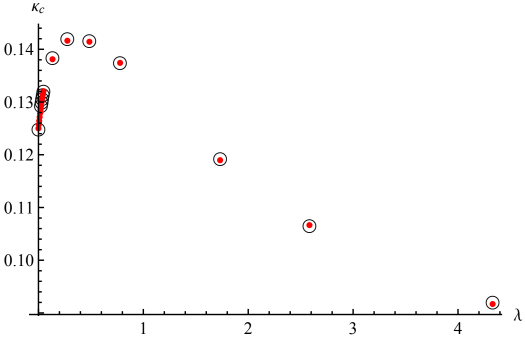


Figure 2: Critical line $\kappa_c(\lambda)$ computed from LPA (Red) compared to the benchmark [Lüscher-Weisz, 1987] (Black).

$\kappa_c(\lambda)$ Results and Comparsion

λ	$\kappa_{c,LW}$	κ_c	$\Delta\kappa_c$
0	0.1250(1)	0.1250	0
2.4841×10^{-2}	0.1294(1)	0.12928(3)	9.66×10^{-4}
3.5562×10^{-2}	0.1308(1)	0.13068(3)	9.48×10^{-4}
1.3418×10^{-1}	0.1385(1)	0.1381(4)	2.82×10^{-3}
2.7538×10^{-1}	0.1421(1)	0.1416(4)	3.36×10^{-3}
4.8548×10^{-1}	0.1418(1)	0.1414(4)	2.64×10^{-3}
7.7841×10^{-1}	0.1376(1)	0.1374(4)	1.30×10^{-3}
1.7320	0.1194(1)	0.1190(5)	3.61×10^{-3}
2.5836	0.1067(1)	0.1066(5)	3.94×10^{-3}
4.3303	0.09220(9)	0.0917(7)	5.51×10^{-3}
∞ (LW) or 100 (here)	0.07475(7)	0.07225(9)	3.34×10^{-2}

Table 2: Critical values for ϕ_4^4 theory in $D=4$. Left, $\kappa_{c,LW}$ from Lüscher-Weisz [3]. Right κ_c from LPA. The relative deviation is defined as $\Delta\kappa_c = (\kappa_{c,LW} - \kappa_c)/\kappa_{c,LW}$.

Remarks on the interplay between LCE and FRG

There is a fruitful interplay between:

LCE for $\Gamma_\kappa[\phi]$ with exact graph sum formula for l^{th} order.	Solution $\Gamma_k[\phi]$ of Wetterich eq. with ultralocal initial data.

LHS is amenable to convergence proofs, yields correlation functions, & new types of approximations via subsums.

RHS governs partial resummations, e.g. can obtain contributions at fixed order in $\hbar$. Resumming polygons gives:

$$\Gamma_\kappa^{\mathcal{O}(\hbar)} = \frac{1}{2}\mathrm{Tr}\Big[\ln(1 + \kappa\ell\varpi_2)\Big]$$

Spatial LCE in Friedmann-Lemaître spacetimes

Consider flat FL spacetimes $ds^2 = -N(t)^2 dt^2 + a(t)^2 \delta_{ab} dx^a dx^b$.

- Keep t real and continuous to avoid issues with Wick rotation and discretization of $a(t)$.
- Discretize d-dim. space on hypercubical lattice with spacing a_s.

Decompose scalar field action into spatially ultralocal plus linking term:

$$S[\phi] = \sum_{x \in \Sigma} s[a_s^{d/2} \phi(\cdot, x)] + \check{\kappa} \mathcal{V}[\phi],$$

$$s[\varphi] = \int_{t_1}^{t_2} dt \left\{ \frac{a^d}{2N} (\partial_t \varphi)^2 - N a^{d-2} \frac{d}{a_s^2} \varphi^2 - N a^d U(\varphi) \right\}(t) ,$$

$$\mathcal{V}[\phi] = \frac{a_s^{d-2}}{2} \int_{t_1}^{t_2} dt\, N(t) a(t)^{d-2} \sum_{x,y} \phi(t, x) \ell_{xy} \phi(t, y) .$$

Spatial LCE for Generating Functional W[J]

Postpone solution of real time QM on FL. Assume generating functional $\omega[\jmath]$ of QM and its moments to be known.

Set

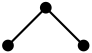
$$\omega_n(t_n, \ldots, t_1|x) := \frac{\delta^n \omega[\jmath]}{\delta\jmath(t_n)\ldots\delta\jmath(t_1)}\Big|_{\jmath(\cdot)\mapsto a_s^{d/2}J(\cdot,x)},$$

and $W_0[J] = \sum_x \omega[a_s^{d/2}J(\cdot,x)]$.

Can formulate graph rules for LCE of $W[J] = W_0[J] + \sum_{l\geq 1}\check{\kappa}^l W_l[J]$ in QFT in terms of ω_n's.

Result: $W_l[J]$ is a sum of contributions over connected graphs with temporal measure $d\nu(t) = N(t)a(t)^{2d-2}dt$.

E.g.

$W_2[J] =$  $+$  ,

$$= a_s^{-4} \int d\nu(t_1)d\nu(t_2) \left\{ \frac{i\hbar}{2} \sum_{x_1,x_2} (\ell_{x_1 x_2})^2 \, \omega_2(t_1, t_2|x_1)\omega_2(t_2, t_1|x_2) \right.$$

$$\left. + \sum_{x_1,x_2,x_3} \ell_{x_1 x_2}\ell_{x_2 x_3} \, \omega_1(t_1|x_1)\omega_2(t_1, t_2|x_2)\omega_1(t_2|x_3) \right\}.$$

Spatial LCE for Legendre Effective Action

Aim at analogous expansion for Legendre effective action $\Gamma[\phi] = \Gamma_0[\phi] + \sum_{l\geq 1} \check{\kappa}^l \Gamma_l[\phi]$ in terms of (fewer) 1PI graphs.

Write $\gamma[\varphi]$ for Legendre transform of $\omega[\jmath]$ and $\Gamma_0[\phi] := \sum_x \gamma[a_s^{d/2}\phi(\cdot, x)]$. Set $\varpi_n(t_n, \ldots, t_1|x) = \omega_n(t_n, \ldots, t_1|x)\big|_{\jmath(\cdot)\mapsto a_s^{d/2}(\partial\gamma/\partial\varphi)(a_s^{d/2}\phi(\cdot,x))}$.

In the absence of temporal discretization Legendre transform of $W[J]$ could be ill-defined due to ϖ_n at coinciding t's.

Corollary to main Theorem: $\Gamma_l[\phi]$ in spatial LCE are well-defined for all $l \geq 1$, only integrability of ϖ_n's short t singularities wrt $d\nu(t)$ is required.

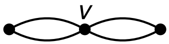
Graph rules for $\Gamma_l[\phi]$ in Spatial LCE

The graph rules for the covariant Euclidean case carry over with the following changes:

- A vertex v of degree n is attributed a factor $\mu^{\Gamma}(e_n, \ldots, e_1|v)$ where $e_1, \ldots, e_n$ are the edges incident on v.
- Embed the 1PI graph into $\Lambda^{|V|} \times \mathbb{R}^{|E|}$ by associating each vertex with a unique spatial lattice point, $i \mapsto x_i \in \Lambda$, $i = 1, \ldots, |V|$. Associate to each edge label a unique real time variable, $e \mapsto t \in \mathbb{R}$, $e = 1, \ldots, l = |E|$. Perform an unconstrained sum $x_1, x_2, \ldots, x_{|V|}$ and an unconstrained integration $d\nu(t_1), \ldots, d\nu(t_l)$.

Modified weights in spatial LCE for $\Gamma_l[\phi]$

Recall the vertex v in the pair of glasses graph at $l = 4$:



The weight of the labeled vertex v now is

$$\varpi_4(t_1, t_2, t_3, t_4|v) - \int ds_1 ds_2 \varpi_3(t_1, t_2, s_1|v)\gamma_2(s_1, s_2|v)\varpi_3(s_2, t_3, t_4|v),$$

where t_1, t_2, t_3, t_4 are the time variables associated to the edges.

Combinatorics of tree graph formula in Theorem is **unchanged.**

Conclusions and Outlook

Spatial LCE on Friedmann-Lemaître spacetimes brings many cosmological issues into the realm of non-perturbative lattice techniques, e.g.

- The dynamical status of spatial homogeneity in the early universe. In perturbation theory only small deviations from the assumed spatially homogeneous initial state can be explored, while in the present setting any ultralocal state can be dynamically evolved.
- Interacting ground states (as opposed to Bunch-Davies), their VEVs like $\Gamma[\phi = \mathit{const}.]$, and their relation to the cosmological constant problem.
- (Non-)triviality of scalar QFTs in Friedmann-Lemaître spacetimes.

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