Precise determination of quark masses

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Outline

1. Introduction and review of different methods
2. Extraction of quark masses from heavy-light meson masses
3. HISQ ensembles with (2+1+1)-flavors of dynamical quarks
4. Fit to lattice data and quark mass results
5. Comparison and conclusion

Introduction

- Six of the fundamental parameters of the Standard Model are quark masses
  - they cannot be measured directly (confined inside hadrons)
  - must be extracted indirectly from physical observables
- For observable particles such as electrons
  - the position of the pole in the propagator is the definition of its mass
  - the pole mass is the rest mass of an isolated particle
- The masses of quarks can be defined as theoretical parameters
  - renormalized, *e.g.*, in the $\overline{\mathrm{MS}}$ scheme at a given scale μ
- Precise values of quark masses are needed for precise calculations in SM/BSM
- In lattice QCD simulations, the bare quark masses can be tuned to obtain physical observables
- The resulting bare masses must be renormalized, but multiloop lattice-QCD calculations are difficult ($\Rightarrow$ limited accuracy)

- Methods that require only nonperturbative lattice-QCD calculations and continuum perturbative calculations yield better accuracy:

Nonperturbative calculation of quark mass renormalization constant

Quark masses are calculated in an intermediate scheme (variants of RI-MOM), and then converted to the $\overline{\text{MS}}$ scheme.
Employed by BMW, ETM, RBC/UKQCD, χQCD, HPQCD, $\cdots$
See D. Hatton's talk (July 26) for the most recent HPQCD work.

Heavy-quark correlator moments

By comparing moments calculated on lattice and QCD perturbation theory.
Employed by HPQCD, JLQCD, hotQCD, $\cdots$

Extraction based on dependence of meson masses on quark masses

A new method developed by Fermilab/MILC/TUMQCD collaborations to extract heavy quark masses from heavy-light meson masses (based on HQET):

$$\text{meson mass} \xLeftrightarrow{\text{quark pole mass}} \text{quark } \overline{\text{MS}} \text{ mass}$$

- Remarks on uncertainties:
  - Truncation in QCD perturbation theory might yield large uncertainties
  - The above methods involve different systematic errors

Extraction of quark masses from heavy-light meson masses

- HQET description of a HL meson mass in terms of its heavy quark mass

$$M_H = m_h + \bar{\Lambda} + \frac{\mu_\pi^2 - \mu_G^2(m_h)}{2m_h} + \mathcal{O}(1/m_h^2)$$

  - $\bar{\Lambda}$: energy of light quarks and gluons inside the system
  - $\mu_\pi^2/2m_h$: kinetic energy of the heavy quark inside the system
  - $\mu_G^2(m_h)/2m_h$: hyperfine energy due to heavy quark's spin (can be estimated from B^*-B splitting $\Rightarrow \mu_G^2(m_b) \approx 0.35\,\mathrm{GeV}^2$)
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- For the heavy quark mass, we use the minimal renormalon subtracted (MRS) scheme [PRD97, 034503 (2018)]
  - removes the leading infrared renormalon from the pole mass
  - has an asymptotic expansion identical to the perturbative pole mass
    (does not spoil the HQET power counting)
  - is a gauge- and scale-independent scheme;
    it does not introduce any factorization scale (unlike, *e.g.*, the RS or kinetic scheme)

- The MRS mass is defined as

$$m_{\mathrm{MRS}} = \overline{m}\left(1 + \sum_{n=0}^{\infty}\left[r_n - R_n\right]\alpha_s^{n+1}(\overline{m})\right) + \mathcal{J}_{\mathrm{MRS}}(\overline{m}) + \Delta m_{(c)}$$

$\overline{m}$:	$\overline{\text{MS}}$ mass at scale $\mu = \overline{m}$
r_n:	coefficients relating the $\overline{\text{MS}}$ mass to the perturbative pole mass
$-R_n$:	subtracting the leading renormalon from the perturb. series
$\mathcal{J}_{\mathrm{MRS}}$:	contribution from the leading renormalon (see backup slides)
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- For a theory with $n_l = 3$ massless quarks, and $R_0 = 0.535$:

$$r_n - R_n = (-0.1106,\ -0.0340,\ 0.0966,\ 0.0162, \ldots)$$

The smallness of $r_n - R_n$ reduces the truncation error in our work

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With the MRS mass for heavy quarks, we proceed to map bare quark masses to the MRS mass

Mapping bare quark masses to the $\overline{\mathsf{MS}}$ and MRS masses

- Introduce a "reference mass", and construct the identity (up to lattice artifacts)

$$m_{h,\mathrm{MRS}} = m_{r,\overline{\mathsf{MS}}}(\mu) \frac{\overline{m}_h}{m_{h,\overline{\mathsf{MS}}}(\mu)} \frac{m_{h,\mathrm{MRS}}}{\overline{m}_h} \frac{am_h}{am_r}$$

1) First factor: a fit parameter (we set $am_r = am_{p4s}$ and $\mu = 2\ \mathrm{GeV}$)
2) Second factor: running factor governed by the mass anomalous dimension (the five-loop result is known [JHEP 1410 (2014) 076])
3) Third factor:

$$m_{h,\mathrm{MRS}} = \overline{m}_h \left(1 + \sum_{n=0}^{3} \left[r_n - R_n\right] \alpha_s^{n+1}(\overline{m}_h) + \mathcal{O}(\alpha_s^5)\right) + \mathcal{J}_{\mathrm{MRS}}(\overline{m}_h) + \Delta m_{(c)}$$

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- The 2nd and 3rd factors require the strong coupling constant; we use

$$\alpha_{\overline{MS}}(5\,\text{GeV}; n_f = 4) = 0.2128(25) \quad \text{[HPQCD, arXiv:1408.4169]}$$

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- Discretization errors should be incorporated as powers of $(am_h)^2$ and $(a\Lambda)^2$

MILC ensembles with (2+1+1)-flavors of dynamical quarks

- Ensembles with physical mass for the strange quark:

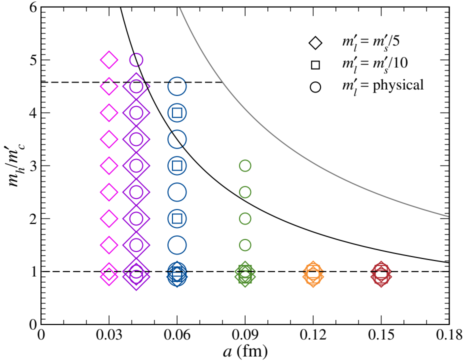
$\approx a$ (fm)	m_l/m_s	size	L (fm)	$M_\pi L$	M_π (MeV)
0.15	1/5	$16^3 \times 48$	2.38	3.8	314
0.15	1/10	$24^3 \times 48$	3.67	4.0	214
0.15	1/27	$32^3 \times 48$	4.83	3.2	130
0.12	1/5	$24^3 \times 64$	3.00	4.5	299
0.12	1/10	$24^3 \times 64$	2.89	3.2	221
0.12	1/10	$32^3 \times 64$	3.93	4.3	216
0.12	1/10	$40^3 \times 64$	4.95	5.4	214
0.12	1/27	$48^3 \times 64$	5.82	3.9	133
0.09	1/5	$32^3 \times 96$	2.95	4.5	301
0.09	1/10	$48^3 \times 96$	4.33	4.7	215
0.09	1/27	$64^3 \times 96$	5.62	3.7	130
0.06	1/5	$48^3 \times 144$	2.94	4.5	304
0.06	1/10	$64^3 \times 144$	3.79	4.3	224
0.06	1/27	$96^3 \times 192$	5.44	3.7	135
0.042	1/5	$64^3 \times 192$	2.91	4.34	294
0.042	1/27	$144^3 \times 288$	6.12	4.17	134
0.03	1/5	$96^3 \times 288$	3.25	4.84	294

- The fermion action is "highly improved staggered quark" (HISQ) action
- Physical-mass ensembles at most lattice spacings

Scale setting and calculating tuned quark masses

- Scale setting is done using f_{p4s}
  (the decay constant of a fiducial pseudoscalar meson with both valence masses equal to $m_{p4s} \equiv 0.4 m_s$)
- The physical value of f_{p4s} is set from f_π
- This method yields a simultaneous determination of both the lattice spacing a and the quark mass am_{p4s} (and in turn $m_s = 2.5 m_{p4s}$)
- The values of f_{p4s} and quark mass ratio m_s/m_l are determined by analyzing light-light data from the same ensembles
  $\Rightarrow$ Various systematic errors (such as FV, EM, continuum extrapolation, *etc.*) in estimate of f_{p4s} and tuned quark masses must be incorporated to our estimate of uncertainties

Heavy-light mesons with HISQ action



- We have 24 Ensembles:
  - 6 lattice spacings
  - several sea masses
- We calculate masses of pseudoscalar mesons for various light and heavy quarks with masses:
  - light valence: $m_{ud} \lesssim m_{\mathrm{v}} \lesssim m_s$
  - heavy valence: $m_c \lesssim m_h \lesssim m_b$
- We use only $am_h < 0.9$ to avoid large discretization errors

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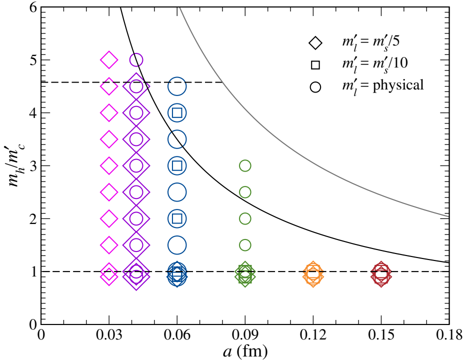

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EFT description of heavy-light meson masses

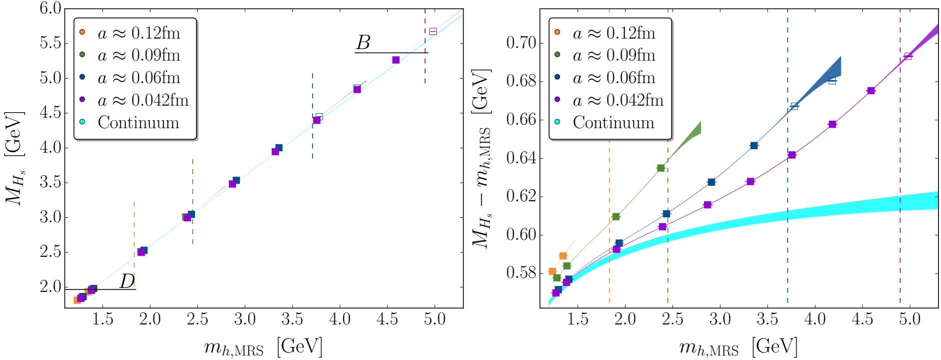
We employ HQET and heavy-meson staggered ChPT to describe the dependence of meson masses on both heavy and light quark masses and incorporate taste-breaking lattice artifacts

- Include HMrPQASχPT and higher order HQET terms

$$M_H = m_{h,\mathrm{MRS}} + \overline{\Lambda}_{\mathrm{MRS}} + \frac{\mu_\pi^2 - \mu_G^2(m_h)}{2m_{h,\mathrm{MRS}}} + \text{HMrPQAS}\chi\text{PT} + \text{higher order HQET}$$

- $m_{h,\mathrm{MRS}}$ is a function of am_h/am_{p4s} and $am_{p4s,\overline{\mathsf{MS}}}(2\ \mathrm{GeV})$
- The higher order terms are typically polynomials in dimensionless, "natural" expansion parameters:
  - Light-quark and gluon discretization: $(a\Lambda)^2$ with $\Lambda = 600\ \mathrm{MeV}$
  - Heavy-quark discretization: $(2am_h/\pi)^2$
  - Light valence and sea quark mass effects: $B_0 m_q/(4\pi^2 f_\pi^2)$
  - HQET: $\Lambda/m_{h,\mathrm{MRS}}$ with $\Lambda = 600\ \mathrm{MeV}$
- Our fit function has 77 parameters and 384 data points

A snapshot of the fit and data

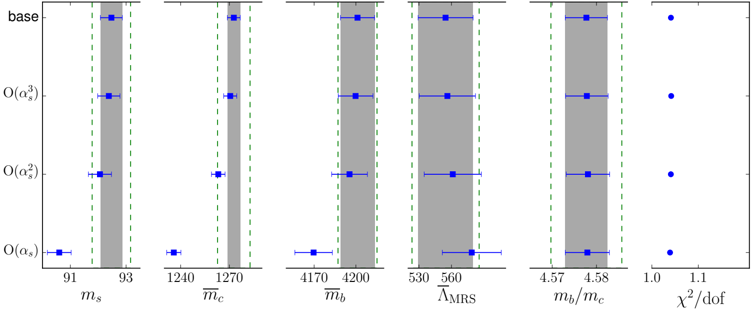


Dashed lines: $am_h \approx 0.9$; open symbols: data points omitted from fit
Vertical axis: heavy-strange meson masses
Horizontal axis: the fit values for the RS mass projected to continuum (no lattice artifacts)

- The combined-correlated fit gives $\chi^2/\text{d.o.f} \approx 1$, $p = 0.3$
- After extrapolating to continuum, experimental masses of D_s and B_s with EM effects subtracted are used to determine the charm- and bottom-quark masses

Stability of results under variation in number of loops

- We use
  - four-loop relation between the pole and $\overline{\text{MS}}$ mass
  - five-loop results for the quark mass anomalous dimension
  - five-loop results for beta function
- The plot shows the dependence of our final results on number of loops;



In the fits labeled by $O(\alpha_s^n)$, we keep n subleading orders;
the green dashed lines show the total errors.

- We do not introduce any systematic error associated with truncation in PT

Results for the strange, charm and bottom quarks

- The strange quark masses in a theory with 4 active flavors:

$$m_{s,\overline{\mathsf{MS}}}(2\ \mathrm{GeV}) = 92.52(40)_{\mathsf{stat}}(18)_{\mathsf{syst}}(52)_{\alpha_s}(12)_{f_{\pi,\mathsf{PDG}}}\ \mathrm{MeV}$$

- For quark mass ratios:

$$m_c/m_s = 11.784(11)_{\mathsf{stat}}(17)_{\mathsf{syst}}(00)_{\alpha_s}(08)_{f_{\pi,\mathsf{PDG}}}$$
$$m_b/m_s = 53.93(7)_{\mathsf{stat}}(8)_{\mathsf{syst}}(1)_{\alpha_s}(5)_{f_{\pi,\mathsf{PDG}}}$$
$$m_b/m_c = 4.577(5)_{\mathsf{stat}}(7)_{\mathsf{syst}}(0)_{\alpha_s}(1)_{f_{\pi,\mathsf{PDG}}}$$

- For heavy quarks:

$$\overline{m}_c = 1273(4)_{\mathsf{stat}}(1)_{\mathsf{syst}}(10)_{\alpha_s}(1)_{f_{\pi,\mathsf{PDG}}}\ \mathrm{MeV}$$
$$\overline{m}_b^{(n_f=5)} = 4197(12)_{\mathsf{stat}}(1)_{\mathsf{syst}}(8)_{\alpha_s}(1)_{f_{\pi,\mathsf{PDG}}}\ \mathrm{MeV}$$

where $\overline{m}_h = m_{h,\overline{\mathsf{MS}}}(m_{h,\overline{\mathsf{MS}}})$.

- Uncertainties:

"stat") Statistics and EFT fit

"syst") Various systematic uncertainties in inputs: FV, EM, topological charge freezing, contamination from higher order states...

α_s) Uncertainty in the strong coupling constant $\alpha_{s,\overline{\mathsf{MS}}}(5\,\mathrm{GeV}; n_f{=}4) = 0.2128(25)$ [HPQCD, arXiv:1408.4169]

$f_{\pi,\mathsf{PDG}}$) Uncertainty in the PDG value of $f_{\pi^\pm} = 130.50(13)$ MeV, which is used for scale setting

Results for HQET parameters

- For HQET parameters we have

$$\overline{\Lambda}_{\mathrm{MRS}} = 552(25)_{\mathsf{stat}}(6)_{\mathsf{syst}}(16)_{\alpha_s}(2)_{f_\pi,\mathsf{PDG}}\ \mathrm{MeV}$$
$$\mu_\pi^2 = 0.06(16)_{\mathsf{stat}}(14)_{\mathsf{syst}}(06)_{\alpha_s}(00)_{f_\pi,\mathsf{PDG}}\ \mathrm{GeV}^2$$
$$\mu_G^2(m_b) = 0.38(01)_{\mathsf{stat}}(01)_{\mathsf{syst}}(00)_{\alpha_s}(00)_{f_\pi,\mathsf{PDG}}\ \mathrm{GeV}^2$$

(Note that the prior value of $\mu_G^2(m_b)$ is set to $0.35(7)\,\mathrm{GeV}^2$ [Gambino and Schwanda, arXiv:1307.4551])

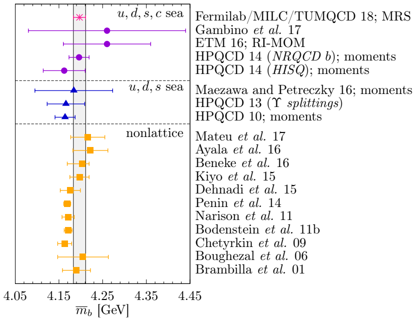
Results for the up and down quark masses

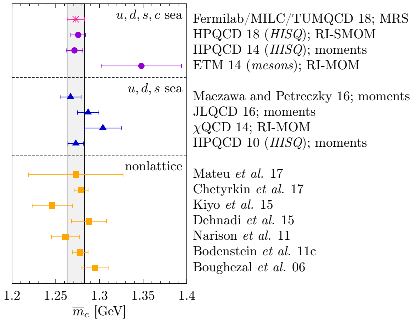
- To calculate the light quark masses we combine our determination of $m_{s,\overline{\mathsf{MS}}}(2\mathrm{GeV})$ and separate determination of mass ratios m_s/m_l and m_d/m_u

$$\begin{aligned} m_{l,\overline{\mathsf{MS}}}(2\ \mathrm{GeV}) &= 3.404(14)_{\mathsf{stat}}(08)_{\mathsf{syst}}(19)_{\alpha_s}(04)_{f_\pi,\mathsf{PDG}}\ \mathrm{MeV} \\ m_{u,\overline{\mathsf{MS}}}(2\ \mathrm{GeV}) &= 2.118(17)_{\mathsf{stat}}(32)_{\mathsf{syst}}(12)_{\alpha_s}(03)_{f_\pi,\mathsf{PDG}}\ \mathrm{MeV} \\ m_{d,\overline{\mathsf{MS}}}(2\ \mathrm{GeV}) &= 4.690(30)_{\mathsf{stat}}(36)_{\mathsf{syst}}(26)_{\alpha_s}(06)_{f_\pi,\mathsf{PDG}}\ \mathrm{MeV} \end{aligned}$$

- m_u and m_d values depend on separate calculation of EM effects on light-light mesons [MILC, arXiv:1807.05556]

Comparison





Our result is shown as a magenta burst, with the gray band showing how it compares directly with the other lattice and nonlattice results; see [arXiv:1802.04248 [hep-lat]] for details.

Recalling the three major methods used by lattice collaborations, we find very good agreement between different results.

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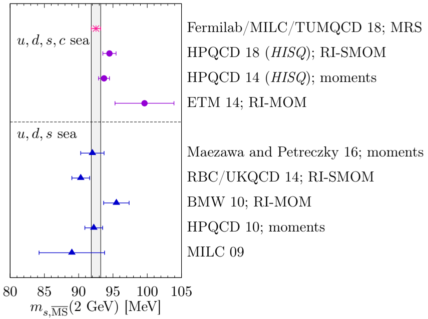

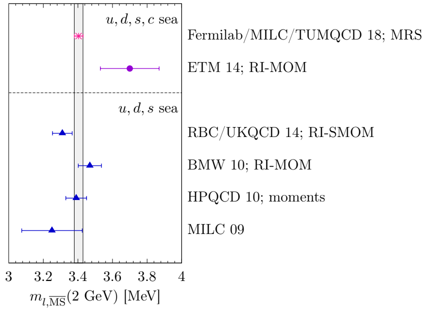

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Conclusion

- We reviewed three major methods used by lattice collaborations for precise determination of quark masses
- We presented results for up, down, strange, charm and bottom quark masses determined by Fermilab/MILC/TUMQCD collaborations
- Comparing these results and other lattice calculations, we find good agreement between quark masses obtained with different methods

Thanks for your attention!

back-up slides

Minimal renormalon subtracted mass

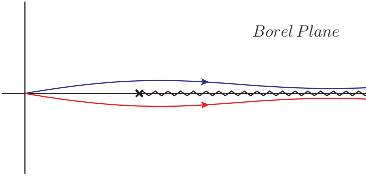
- The pole mass can be calculated at each order in perturbation theory

$$m_{\text{pole}} = \overline{m}\left(1 + \sum_{n=0}^{N} r_n \alpha_s^{n+1}(\overline{m}) + \mathcal{O}(\alpha_s^{N+2})\right)$$

  - $\overline{m}$ is the $\overline{\text{MS}}$ mass at scale $\mu = \overline{m}$
  - The series diverges because $r_n \propto (2\beta_0)^n \Gamma(n+b+1)$ as $n \to \infty$
- The divergent expression can be interpreted using the Borel transform

involves an integral of form $\int_0^\infty dz \, \frac{e^{-z/(2\beta_0\alpha_s)}}{(1-z)^{1+b}}$

with $b = \beta_1/(2\beta_0^2)$



- The idea in the MRS scheme is to divide the integral as

$$\int_0^1 dz \, \frac{e^{-z/(2\beta_0\alpha_s)}}{(1-z)^{1+b}} \quad \to \quad \mathcal{J}_{\text{MRS}}(\mu)$$

$$\int_1^\infty dz \, \frac{e^{-z/(2\beta_0\alpha_s)}}{(1-z)^{1+b}} \quad \to \quad \delta m \propto (-1)^b \Lambda_{\text{QCD}}$$

and subtract the ambiguous term δm from the pole mass

- $\mathcal{J}_{\mathrm{MRS}}(\mu)$ is defined as

$$\mathcal{J}_{\mathrm{MRS}}(\mu) = \frac{R_0}{2\beta_0}\mu e^{-1/[2\beta_0\alpha_{\mathrm{g}}(\mu)]}\sum_{n=0}^{\infty}\frac{1}{n!(n-b)}\left(\frac{1}{2\beta_0\alpha_{\mathrm{g}}(\mu)}\right)^n$$

where $b = \beta_1/(2\beta_0^2)$, R_0 is the overall normalization of the leading renormalon in the pole mass, and $\alpha_{\mathrm{g}}(\mu)$ is the coupling constant in the scheme with

$$\beta\left(\alpha_{\mathrm{g}}(\mu)\right) = -\frac{\beta_0\alpha_{\mathrm{g}}^2(\mu)}{1-(\beta_1/\beta_0)\alpha_{\mathrm{g}}(\mu)}$$

- For the relations between the RS and MRS schemes:

$$m_{\mathsf{RS}}(\nu_f) = m_{\mathrm{MRS}} - \mathcal{J}_{\mathrm{MRS}}(\nu_f)$$
$$\overline{\Lambda}_{\mathsf{RS}}(\nu_f) = \overline{\Lambda}_{\mathrm{MRS}} + \mathcal{J}_{\mathrm{MRS}}(\nu_f)$$

Discussion on smallness of truncation error

- In the MRS scheme, we use

$$m_{h,\mathrm{MRS}} = \overline{m}_h \left(1 + \sum_{n=0}^{\infty} [r_n - R_n]\, \alpha_s^{n+1}(\overline{m}_h)\right) + \mathcal{J}_{\mathrm{MRS}}(\overline{m}_h) + \Delta m_{(c)}$$

- $J_{\mathrm{MRS}}(\overline{m}_h)$ has a convergent expression in powers of $1/\alpha_s(\overline{m}_h)$
- Coefficients are small: $r_n - R_n = (-0.1106, -0.0340, 0.0966, 0.0162)$ for $n = (0, 1, 2, 3)$, three active flavors, and $R_0 = 0.535$.
  $\Rightarrow$ the errors from truncating perturbative QCD relations are negligible
- This is not necessarily the case when one uses other schemes
- Using the RS scheme [hep-ph/0105008] , which introduces a factorization scale $\nu \ll \overline{m}_h$ as

$$m_{h,\mathrm{RS}}(\nu) = \overline{m}_h \left(1 + \sum_{n=0}^{\infty} c_n(\nu, \overline{m}_h, \mu)\; \alpha_s^{n+1}(\mu)\right) + \Delta m_{(c)}$$

we then have $c_n(1\mathrm{GeV}, 4.2\mathrm{GeV}, 4.2\mathrm{GeV}) = (0.30, 0.52, 1.1, 2.2, \cdots)$ $c_n(1\mathrm{GeV}, 4.2\mathrm{GeV}, 3\mathrm{GeV}) = (0.30, 0.38, 0.59, 0.68, \cdots)$ the truncation error is expected to be of size $2.20\alpha_s^4(4.2\mathrm{GeV}) \times \overline{m}_h \approx 20$ MeV and $0.68\alpha_s^4(3\mathrm{GeV}) \times \overline{m}_h \approx 10$ MeV

Discussion on heavy quark discretization error

- In order to incorporate heavy quark discretization errors, in our fit function:

$$m_{h,\mathrm{MRS}} \quad \rightarrow \quad m_{h,\mathrm{MRS}} \times \left(1 + \alpha_{\overline{\mathsf{MS}}}(2\ \mathrm{GeV}) \sum_{n=1}^{4} k_n x_h^n\right) \quad \text{with } x_h = (2am_h/\pi)^2$$

- The prior values of the k_n are set to 0 ± 1, and the posterior values of k_n from our base fit:

$$k_n = (0.19, 0.07, -0.12, -0.46) \quad \text{for } n = (1, 2, 3, 4)$$

- When we include one more term:

$$k_n = (0.19, 0.06, -0.12, -0.37, -0.19) \quad \text{for } n = 1, 2, 3, 4, 5$$

  with extremely small change in our final results