Lattice QCD Calculation of the Proton Charge Radius

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Proton Charge Radius

Since the proton is comprised of three quarks that are separated in space, it has an extended charge distribution. The **proton charge radius** is the root mean square distance of a charge carrier from the charge center of the proton:

$$(r_{\text{rms}})^2 = \langle r^2 \rangle = \int \mathbf{r}^2 \rho(r) d^3\mathbf{r} = 4\pi \int_0^\infty r^4 \rho(r) dr$$

where $\rho(r)$ is the charge density as a function of distance from the proton center.

The proton charge radius has been measured experimentally using both electrons and muons as the probes. The measured radii [1] are

- Electrons: 0.8751 ± 0.0061 fm
- Muons: 0.84087 ± 0.00039 fm

The electronic and muonic measurements are shown in red and blue, respectively.

The 5σ discrepancy is currently unresolved and is called the **proton radius puzzle**.

Proton Form Factors

One method of measuring the proton charge radius is through scattering experiments, where a beam of leptons is fired at a proton target and the energies and angles of the scattered leptons are measured. The data are fitted to electric and magnetic **form factors** G_E and G_M. In the non-relativistic limit, the electric form factor is related to the charge distribution by Fourier transform [2]:

$$G_E(\mathbf{q}^2) = \frac{4\pi}{|\mathbf{q}|} \int_0^\infty r \sin(|\mathbf{q}|r)\rho(r) dr$$

The charge radius is then given by

$$\langle r^2 \rangle = -6 \frac{dG_E(\mathbf{q}^2)}{d|\mathbf{q}^2|}\bigg|_{\mathbf{q}^2=0} = -6 \frac{dG_E(Q^2)}{dQ^2}\bigg|_{Q^2=0}$$

Computing the Proton Charge Radius

On the lattice, we can simulate lepton-proton scattering experiments and calculate the amplitude

$$\langle B_\alpha(t, \mathbf{p}') \mathcal{O}_\mu(\tau) \bar{B}_\beta(0, \mathbf{p}) \rangle$$

of a proton being scattered from a state with momentum $\mathbf{p}$ to momentum $\mathbf{p}'$. $\mathcal{O}_\mu(\tau)$ is an operator corresponding to interaction with a photon at Euclidean time $\tau < t$. When sandwiched between proton spinors, $\mathcal{O}_\mu$ can be expanded as [3, 4]

$$\gamma_\mu F_1(Q^2) + \frac{\sigma_{\mu\nu} q^\nu}{2m_p} F_2(Q^2)$$

where the Dirac and Pauli form factors F_1, F_2 are related to the electromagnetic form factors [4] by

$$G_E(Q^2) = F_1(Q^2) - \frac{Q^2}{4m_p^2} F_2(Q^2)$$
$$G_M(Q^2) = F_1(Q^2) + F_2(Q^2)$$

By adjusting $\mathbf{p}$ and $\mathbf{p}'$, we can obtain a plot of G_E at various values of Q^2, from which we would like to determine the slope at 0.

We used the lattice QCD software Chroma and related packages [5, 6, 7] for our calculations.

Our Approach

Since $\mathbf{p}, \mathbf{p}'$ are quantized on the lattice, it is not possible to use values of Q^2 arbitrarily close to 0. When computing the slope, this introduces unknown systematic uncertainties since the shape of the form factor graph near 0 is unknown.

If the final and initial states were different in mass, we could have $\Delta\mathbf{p} = 0$ but have $\Delta E \neq 0$ and $Q^2 < 0$. In the limit where the mass splitting vanishes, we could have Q^2 arbitrarily close to 0. This would allow us to extract the slope at zero without large systematic uncertainties from extrapolation in Q^2.

We can introduce this mass splitting by performing the calculations with nondegenerate u and d quark masses and then looking at the current $\mathcal{O}_\mu = \bar{u}\gamma_\mu d$, which changes a proton into a neutron. (This also introduces a third term $\frac{iq_\mu}{m_1+m_2} F_3(Q^2)$ to our expansion of $\mathcal{O}_\mu$ in terms of form factors.) We will exaggerate this mass splitting for calculation purposes and then extrapolate to the physical values. This **isovector current**, while not directly corresponding to a physical process, gives form factors related to the proton and neutron form factors:

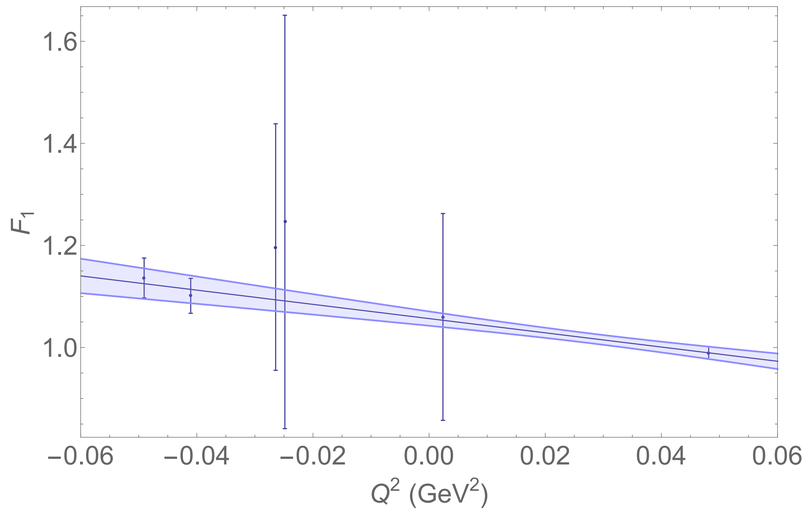
$$F_i^{\text{IV}} = F_i^p - F_i^n$$

The neutron form factors are well determined by experiments scattering neutron beams off the electron cloud of lead atoms [8], so a determination of the isovector current is sufficient to calculate the proton form factors. Furthermore, the value of F_2^{IV} at $Q^2 = 0$ is known very precisely from measurements of nucleon magnetic moments, so the only parameter we need to determine is the slope of F_1^{IV}.

Results

We used this method on an ensemble of 750 $24^3 \times 64$ configurations with lattice spacing $a = 0.12$ fm and sea pion mass 440 MeV. We took the lighter valence quark (the one we call u) to be equal to the sea quark mass and varied the heavier valence quark mass to cover a range of nucleon mass splittings from 0 to 0.44 GeV.

This is a plot of the Dirac form factor $F_1^{\text{IV}}(Q^2)$ plotted against Q^2 for $m_p = 1.52$ GeV, $m_n = 1.75$ GeV. The various values of Q^2 arise from various combinations of $\mathbf{p}$ and $\mathbf{p}'$ with $|\mathbf{p} - \mathbf{p}'| \leq 2\pi/L$, the smallest nonzero change in momentum allowed on a lattice of length L.



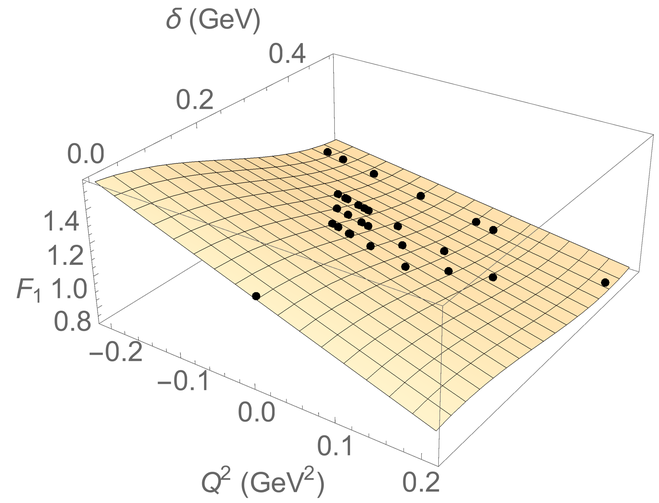
By repeating this at various nucleon mass splittings $\delta = m_n - m_p$, we can fit F_1 to the data in the (Q^2, δ) plane using the fit function

$$F_1(\delta, Q^2) = A(1 + C\delta^2 + DQ^2 + E\delta Q^2 + F\delta^3)$$

where A is a renormalization constant and the coefficient of the δ term is set to 0 by the Ademollo-Gatto theorem [9]. The value of $\frac{1}{F_1(0)} \frac{\partial F_1}{\partial Q^2}\big|_{\delta=Q^2=0}$ that we are interested in is just given by the coefficient D.

Results (cont.)

Below, we show the fit function and the locations of all our data points in the (Q^2, δ) plane. Our fit corresponds to a charge radius of 0.320 ± 0.074 fm^2 (not counting systematic uncertainty from the fit) at a pion mass of 440 MeV.



Comparison to Other Work

Some attempts to compute the charge radius on the lattice have often used the so-called dipole fit, where $F_1(Q^2)$ is assumed to have the form $1/(1 + Q^2/\Lambda^2)^2$. Such an assumption gives rise to uncontrolled systematics.

A more general approach, called the z expansion, involves transforming the momentum transfer $t = -Q^2$ by

$$z = \frac{\sqrt{t_{\text{cut}} - t} - \sqrt{t_{\text{cut}} - t_0}}{\sqrt{t_{\text{cut}} - t} + \sqrt{t_{\text{cut}} - t_0}}$$

(where $t_{\text{cut}} = 4m_\pi^2$ and t_0 can be chosen arbitrarily) and then Taylor expanding $F_1(z)$. A small model dependence is introduced by truncating the series, but this can be controlled by comparing fits with different numbers of terms. [10] used the z expansion at the physical point to estimate the proton charge radius as $\langle r_E^2 \rangle^{\text{IV}} = 0.787 \pm 0.087$ fm^2.

Another approach is the so-called Rome method, which introduces twisted boundary conditions to modify the momentum quantization condition and then differentiates with respect to the twist angle. This eliminates extrapolation-related systematic uncertainties but has a noisy signal. A physical-point calculation using the Rome method [10] obtained a charge radius of $\langle r_E^2 \rangle^{\text{IV}} = 0.753 \pm 0.273$ fm^2, a much larger uncertainty than the z expansion provided.

In contrast, our work has potentially large systematics from the $\delta \to 0$ limit. In principle, we could reduce these by using smaller values of δ, but since $-Q^2 \leq \delta^2$, it is difficult to fit the Q^2 dependence at small δ. The difficulty of controlling the systematics resulting from the $\delta \to 0$ limit makes our method impractical compared to other approaches.

References

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