# $Z_2$ gauge theory with tensor renormalization

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# Introduction

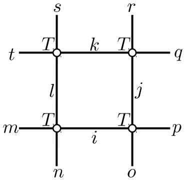
- Tensor renormalization
  - A numerical renormalization method like the density matrix renormalizarion.
  - Completely free of the sign problem.
- $Z_2$ gauge theory
  - Application of the tensor renormalization to LGT is a step to reach calculations in the QCD with the sign problem.
  - The pure $Z_2$ gauge theory is suitable the first test bed.

    $\Rightarrow$ **(2+1) finite temperature $Z_2$ gauge theory**

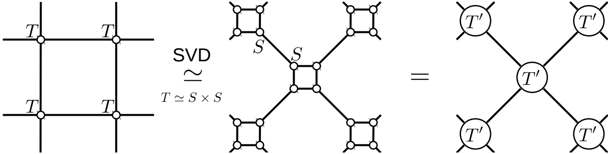
# Tensor renormalization methods

- Tensor network formulation

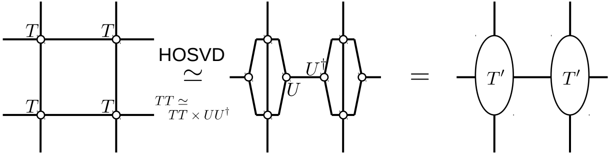
$$Z = \sum_{\cdots,i,j,k,l,\cdots} \cdots T_{ilmn} T_{pjio} T_{qrkj} T_{kstl} \cdots$$




- dim.=2: Tensor Renormalization Group (TRG) Levin and Nave, PRL 99 120601(2007)




- dim.≥2: Higher Order TRG (HOTRG) Xie et al. PRB 86, 045139(2012)



# $Z_2$ gauge theory

- The partition function:

$$Z = 2^{-3N} \sum_{\{\sigma=\pm 1\}} \prod_{n,\mu>\nu} e^{-\beta\sigma_{n,\mu\nu}},$$

$$\sigma_{n,\mu\nu} = \sigma_{n,\mu}\sigma_{n+\hat{\mu},\nu}\sigma_{n+\hat{\nu},\mu}\sigma_{n,\nu}$$

  - $\sigma_{n,\mu}$ is defined on the link labeled by a site $n$ with a direction $\mu$.
  - $N$ is the system size.
- (2+1) finite temperature gauge theory
  - We take $\mu = 0(1,2)$ as the temporal(spatial) direction(s).
  - $n = (n_0, n_1, n_2)$, $0 \leq n_0 < N_T, 0 \leq n_{1,2} < N_S\}$.
  - $N_T$ corresponds the temperature.
  - At fixed $N_T$, It belongs to the two-dimentional Ising universality class.

# Tensor network for $Z_2$ gauge theory Liu et al. PRD 88, 056005(2013)

- Expansion of the Boltzmann factors

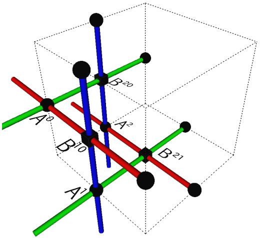
$$e^{\beta \sigma_{n,\mu\nu}} = \cosh \beta \sum_{i,j,k,l=0,1} B^{(n,\mu\nu)}_{ijkl} \sigma^{i}_{n,\mu} \sigma^{j}_{n+\hat{\mu},\nu} \sigma^{k}_{n+\hat{\nu},\mu} \sigma^{l}_{n,\nu},$$

$$B^{(n,\mu\nu)}_{ijkl} = (\tanh \beta)^{(i+j+k+l)/4} \delta_{i,j} \delta_{j,k} \delta_{k,l}$$

- Summation of the $\sigma_{n,\mu}$

$$\sum_{\sigma_{n,\mu}} \sigma^{i+j+k+l}_{n,\mu} = 2A^{(n,\mu)}_{ijkl},$$

$$A^{(n,\mu)}_{ijkl} = \delta_{i+j+k+l \bmod 2,0}.$$




- The partition function is rewritten by

$$Z = (\cosh \beta)^{3N} \sum_{\{i\}} \prod_{n} \left( \prod_{\mu} A^{(n,\mu)} \right) \left( \prod_{\mu > \nu} B^{(n,\mu\nu)} \right).$$

# Reconstruction

We classify the sites as spatial even $(n_1 + n_2 \bmod 2 = 0)$ and odd sites, and gather the tensors around each even site.

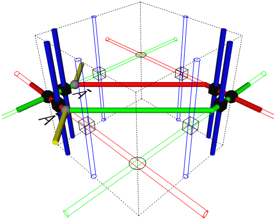
- For temporal links on spatial odd sites,
  - $(n_0 > 0)$ we take the gauge fixing

    $$\sigma_{n,0} = 1$$

    and so can omit $A^{(n,0)}$.
  - $(n_0 = 0)$

    $$A^{(n,0)}_{ijkl} = \sum_{p=0,1} \bar{A}_{ijp}\bar{A}_{klp},$$

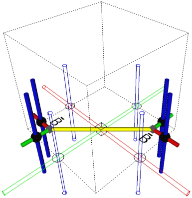
    $$\bar{A}_{ijp} = \delta_{i+j+p \bmod 2,0}$$



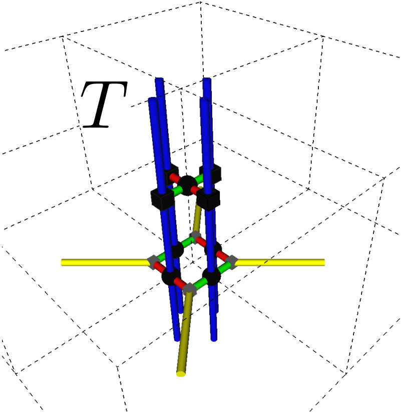
- For spatial planes,

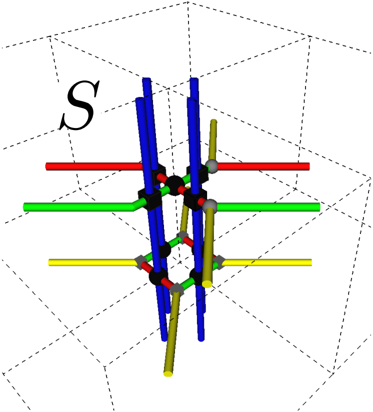
  $$B^{(n,21)}_{ijkl} = \bar{B}_{ijp}\bar{B}_{klp},$$

  $$\bar{B}_{ijp} = (\tanh\beta)^{(i+j)/4}\delta_{i,j}\delta_{j,p}$$



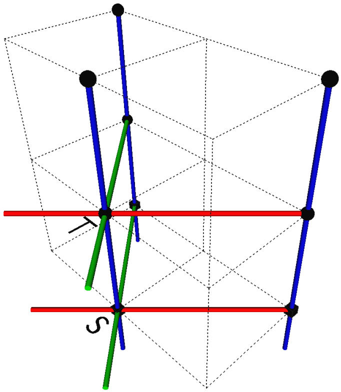
- For spatial even sites,
  we define new tensors $T$ on $n_0 > 0$ and $S$ on $n_0 = 0$
  by summing out the internal indices.



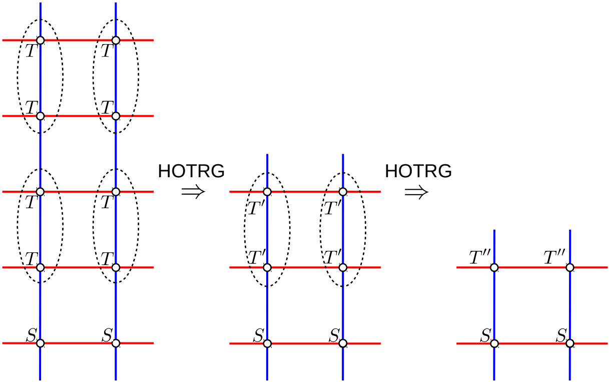



- The partition function is written by

$$Z = (\cosh \beta)^{3N} \sum_{\{i\}} \left( \prod_{n:n_0>0} T^{(n)} \right) \left( \prod_{n:n_0=0} S^{(n)} \right)$$

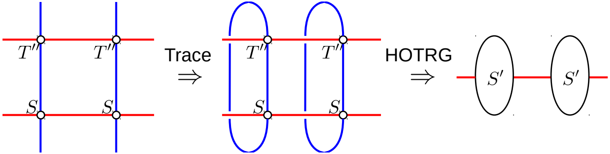


1. Until the temporal size of $T$ part is one,
   repeatably coarse-grain $T$ by HOTRG for the temporal directon.




- The d.o.f. retained in the HOTRG is denoted $D_1$.

2 By trace of the temporal indices and
HOTRG for the temporal direction,
reduce the tensor network to 2 dimensions.



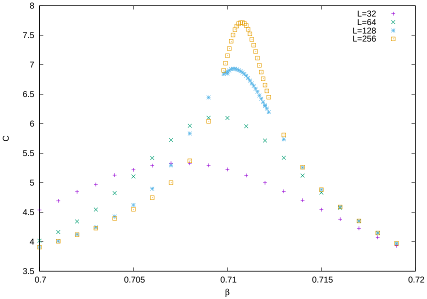

- The d.o.f. retained in the HOTRG is denoted $D_2$.

3. By TRG, coarse-grain the 2 dimensions network.
   - The d.o.f. retained in the TRG is denoted $D_3$.

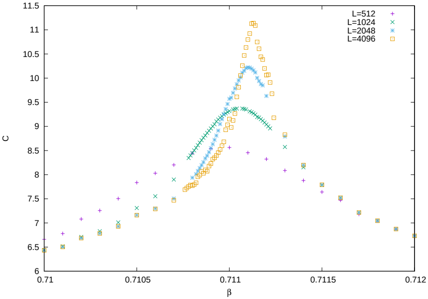
- Settings: $D_1 = 16, D_2 = 128, D_3 = 128.$
- Result of

$$C = \beta^2 \frac{d^2}{d\beta^2} \ln Z$$

by numerical difference of obtained $\ln Z$.




$N_T = 3, N_S = 32, 64, 128, 256$



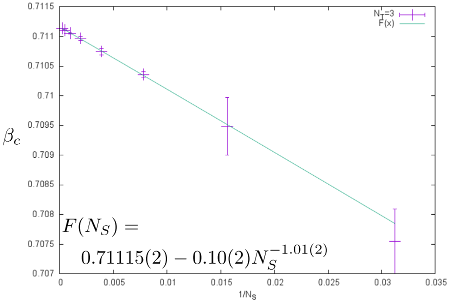

$N_S = 512, 1024, 2048, 4096$

# Finite size scaling

- FSS of $\beta_c$:

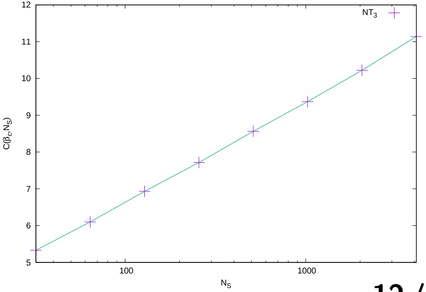
$$\beta_c(N_S) = \beta_c(\infty) + cN_S^{-1/\nu}$$

The expected value $\nu = 1$




- FSS of $C(\beta_c)$:

$$C(\beta_c, N_S) \propto \log(N_S)$$



# Comparison with Monte Carlo calculation

| $N_T$ | $N_S$(TN) | $\beta_c$(TN) | $\nu$(TN) | $N_S$(MC) | $\beta_c$(MC) |
|---|---|---|---|---|---|
| 2 | [32,4096] | 0.65610(2) | 1.00(2) | 4,8,16,32 | 0.65608(5) |
| 3 | [32,4096] | 0.71116(2) | 1.01(3) | 24 | 0.71102(8) |
| 5 | [64,4096] | 0.74072(6) | 1.07(7) | 40 | 0.74057(3) |

MC: M.Caselle and M.Hasenbusch, Nuclear Physics B 470 [FS] (1996)

- $\nu$(TN) consistent $\nu = 1$.
- $\beta_c$(TN) are little larger than $\beta_c$(MC). It may be brecause $N_S$ of TN are bigger than MC.

# Summary

- We have fomulated the tensor network of the (2+1) $Z_2$ FTGT.
- We have obtained the numerical results of FSS which are consistent with previous studies or theoriticaly expected values.
- In future,
  1. Generalization for SU(2)
     - The tensors $A, B$ for SU(2) have already been formalized by Liu et al. PRD 88, 056005(2013).
  2. Caluculation in $N_T > 5$
  3. Formulation of Polyakov loop
     - Impure tensor formalization S.Morita and N.Kawashima, arXiv:1806.10275