Thermodynamics for SU(2) pure gauge theory using gradient flow

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arXiv:1805.07106 [hep-lat]
QCD in intermediate temperature

- Experimental data
  → small shear viscosity-to-thermal entropy ratio \((\eta/s)\)
  → perfect-liquid property rather than gas
- Large-\(N_c\) analysis based on AdS/CFT
  → lower bound for shear viscosity
- Lattice calculation
  - Shear viscosity \(\eta\)
    First step: Calculate correlation function of EMT
    - Bad signal-to-noise ratio
      (eg. 6-million Conf. needed in SU(3))
    - Definition of the correctly renormalization of EMT
    - Solving inverse-problem to obtain spectral function
  - Thermal entropy \(s\)
    Method: Integral method\(^1\), Gradient flow method\(^2\) etc.

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\(^2\) H. Suzuki, PTEP 2013, 083B03 (2013)
**SU(2) pure gauge theory**

- Focus on **SU(2) pure gauge theory**
  - Numerical cost is lower than SU(3) gauge theory
  - Provide larger signal of correction term of $1/N_c$
- Difference from **SU($N_c \geq 3$) gauge theory**

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**Graph**

- Our result
- SU(2) (EPJ C77 (2017) 821)
- SU(3)
- SU(4)
- SU(5) (PRL 103, 232001 (2009))
- SU(6)
- SU(8)
In this study

Thermodynamics for pure SU(2) using Gradient flow

1. Scale setting using $t_0$ reference scale
to determine relation between $\beta$ and lattice spacing

2. Measure thermodynamics quantities
   \[
   \left( \frac{s}{T^3}, \frac{\Delta}{T^4}, \frac{\varepsilon}{T^4}, \frac{P}{T^4} \right)
   \]
Gradient flow

- Yang-Mills gradient flow equation on lattice\(^3\)
  \[
  \partial_t V_t(x, \mu) - g_0^2 \left\{ \partial_{x, \mu} S_W \right\} V_t(x, \mu) = 0
  \]
  \[V_t(x, \mu) \bigg|_{t=0} = U(x, \mu)\]

  \(g_0\): bare coupling, \(t\): flow time, \(U(\mu, x)\): link variable, \(S_W\): Wilson-plaquette action.

- Renormalized EMT with gradient flow\(^4\)
  \[
  T^R_{\mu\nu} = \lim_{t \to 0} \left[ \frac{U_{\mu\nu}}{\alpha_U} + \frac{\delta_{\mu\nu}}{4\alpha_E} \left\{ E - \langle E \rangle_0 \right\} \right]
  \]
  \[U_{\mu\nu} = G_{\mu\rho} G_{\nu\rho} - \frac{\delta_{\mu\nu}}{4} G_{\rho\sigma} G_{\rho\sigma}, \quad E = \frac{1}{4} G_{\mu\nu} G_{\mu\nu}\]

  \(G_{\mu\nu}\): field strength consisting of \(V_t\)
  \(\alpha_U, \alpha_E\): calculated in 1-loop order of running coupling

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\(^3\)M. Luscher, JHEP 1008 (2010) 071.
\(^4\)H. Suzuki, PTEP 2013, 083B03 (2013).
Scale Setting

- Observable: \( t^2 \langle E \rangle \propto N_c^2 - 1 \)
- Reference scale: \( t^2 \langle E(t) \rangle \bigg|_{t=t_0} = 0.1 \)
  \( \rightarrow \) a natural scaling-down of the SU(3) case\(^5\)
- Configuration generation
  - Wilson-plaquette action, \( N_s = N_t = 32 \)
  - 1 sweep = 1 pseudo-heatbath + 20 over-relaxation
  - 100 sweep separation between measurements

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>2.42</th>
<th>2.50</th>
<th>2.60</th>
<th>2.70</th>
<th>2.80</th>
<th>2.85</th>
</tr>
</thead>
<tbody>
<tr>
<td># of Conf.</td>
<td>100</td>
<td>300</td>
<td>300</td>
<td>300</td>
<td>300</td>
<td>600</td>
</tr>
</tbody>
</table>

- Gradient flow
  - \( t/a^2 \in [0.00, 32.00] \), \( \Delta t/a^2 = 0.01 \)

**Scale Setting**

- Best fit function \( \frac{t_0}{a^2} \text{ vs. } \beta \): \( \beta \in [2.42, 2.85] \)
  \[
  \ln\left(\frac{t_0}{a^2}\right) = 1.258 + 6.409(\beta - 2.600) \\
  - 0.7411(\beta - 2.600)^2
  \]

- Compare with scale setting using string tension\(^6\)

Compare with “$r_0$ scale$^7$ ($r_c$ scale$^8$)”

\[
\frac{\sqrt{8t_0}}{r_0} = 0.6020(86)(40), \quad \frac{\sqrt{8t_0}}{r_c} = 1.126(7)(7),
\]

\[
\sqrt{8t_0} = 0.3010(43)(20)[\text{fm}].
\]

$r_0 = 0.5[\text{fm}], \quad r_c = 0.26[\text{fm}].$


Thermodynamics: \( T/T_c \) and Observables

- \( \beta \) vs. \( T/T_c \) for each \( N_t \)

<table>
<thead>
<tr>
<th>( T/T_c )</th>
<th>( N_T = 6 )</th>
<th>( N_T = 8 )</th>
<th>( N_T = 10 )</th>
<th>( N_T = 12 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.95</td>
<td>—</td>
<td>2.50</td>
<td>2.57</td>
<td>2.62</td>
</tr>
<tr>
<td>0.98</td>
<td>2.42</td>
<td>2.51</td>
<td>2.58</td>
<td>2.63</td>
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<tr>
<td>1.01</td>
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<td>2.52</td>
<td>2.59</td>
<td>2.64</td>
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<tr>
<td>1.04</td>
<td>2.44</td>
<td>2.53</td>
<td>2.60</td>
<td>2.65</td>
</tr>
<tr>
<td>1.08</td>
<td>2.45</td>
<td>2.54</td>
<td>2.61</td>
<td>2.66</td>
</tr>
<tr>
<td>1.12</td>
<td>2.46</td>
<td>2.55</td>
<td>2.62</td>
<td>2.66</td>
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<tr>
<td>1.28</td>
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<td>2.72</td>
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<tr>
<td>1.50</td>
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<td>1.76</td>
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<td>2.07</td>
<td>2.65</td>
<td>2.74</td>
<td>2.81</td>
<td>—</td>
</tr>
</tbody>
</table>

- Observables: entropy density \( (s) \), trace anomaly \( (\Delta) \), energy density \( (\varepsilon) \), pressure \( (P) \)

\[ sT = \varepsilon + P = T_{11}^{R} - T_{44}^{R}, \quad \Delta = \varepsilon - 3P = -\sum_{\mu=1}^{4} T_{\mu\mu}^{R} \]

\[ \text{the critical } \beta \text{ on } N_t = 6 \text{ from [J. Engels, J. Fingberg and D. E. Miller, Nucl. Phys. B 387 (1992) 501.]} \]
Procedure and Simulation setup

- Steps to calculate renormalized EMT\(^{10}\)
  1. Generate configuration at \( t = 0 \) on \( N_s^3 \times N_T \)
  2. Solve gradient flow eq. in \( a \ll \sqrt{8t} \ll R \)
  3. Construct renormalized EMT at each \( t \)
  4. Carry out an extrapolation, first \( a \to 0 \), next \( t \to 0 \)

- Simulation setup
  - Wilson-plaquette action
  - \( N_s/N_T = 4, \ N_T = 6, 8, 10, 12 \)
  - # of Conf. for each parameter: 200
  - 1 sweep = 1 pseudo-heatbath+\( N_t \) over-relaxation
  - 100 sweep separation between measurements

- Gradient flow
  - \( t/a^2 \in [0.00, 5.00] \), \( \Delta t/a^2 = 0.01 \)

\(^{10}\)M. Asakawa et al., Phys. Rev. D 90, no. 1, 011501 (2014)
Flow-time dependence of $s/T^3$ and $\Delta/T^4$

- left: $N_t$-dep. @ $T/T_c = 1.12$, right: $T$-dep. @ $N_T = 12$
- Fiducial window: $1/N_t \leq \sqrt{8tT} \leq 0.5$
- $(a, t) \rightarrow (0, 0)$ limit: constant- & linear-extrapolation
Result

- **Left panel:** $s/ T^4$ (black symbol), $\Delta/ T^4$ (red symbol)
  - $(a, t) \rightarrow (0, 0)$ sys. error of extrapolation (linear & constant) consistent in $T/T_c \geq 1.12$

- **Right panel:** $\varepsilon/ T^4$.vs. $P/ T^4$ (EOS) in $T > T_c$
  - Toward to SB limit ($\varepsilon/ T^4, P/ T^4 = (\pi^2/5, \pi^2/15)$)
  - $70 \sim 80\%$ of SB limit for $T \sim 2T_c$
    - NOT describe two-color QGP around $T \leq 2T_c$
Compare with HTL analysis

- Hard-Thermal-Loop (HTL) analysis\textsuperscript{11}
  
  ... 2-color case in NNLO

- Left panel: $\varepsilon/\varepsilon_{SB}$
  
  Our result is consistent with HTL in $T > T_c$

- Right panel: $(T/T_c)^2 \Delta/T^4$
  
  plateau and approaches to HTL result in $1.2 T_c \leq T$

\textsuperscript{11}J. O. Andersen \textit{et al.}, JHEP \textbf{1008} (2010) 113.
Summary

We investigate the thermodynamics of SU(2) pure gauge theory

1. Scale setting
   - \( t^2 \langle E(t) \rangle \bigg|_{t=t_0} = 0.1 \) for SU(2)
   - our scale-setting function is more precisely and cover wider \( \beta \) region

2. Obtaining \( s/T^3, \Delta/T^4, \text{EOS} \)
   - Confirm that the traceanomaly in the SU(2) pure gauge theory has a different scaling property from the \( N_c \geq 3 \) cases
   - Our results are more precisely than integral method
   - Consistent with integral method and HTL analysis in high temperature region
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Compare $t^2 \langle E \rangle$ with perturbative analysis

Compare with NLO result\(^{12}\)

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\(^{12}\)R. V. Harlander and T. Neumann, JHEP 1606 (2016) 161
Compare $t^2 \langle E \rangle$ with perturbative analysis

Compare with NNLO result\textsuperscript{13}

\begin{figure}
\centering
\includegraphics[width=\textwidth]{thermodynamics.png}
\caption{NNLO, $\beta=2.85$}
\end{figure}

\textsuperscript{13}R. V. Harlander and T. Neumann, JHEP \textbf{1606} (2016) 161
Scale Setting

- **Best fit function** \( t_0/a^2 \) vs. \( \beta \)

\[
\ln(t_0/a^2) = 1.258 + 6.409(\beta - 2.600) - 0.7411(\beta - 2.600)^2.
\]

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<tr>
<td>( t_0/a^2 )</td>
<td>1.083(2)</td>
<td>1.839(3)</td>
<td>3.522(10)</td>
</tr>
<tr>
<td>( \beta )</td>
<td>2.70</td>
<td>2.80</td>
<td>2.85</td>
</tr>
<tr>
<td>( t_0/a^2 )</td>
<td>6.628(36)</td>
<td>11.96(12)</td>
<td>16.95(17)</td>
</tr>
</tbody>
</table>
$a \to 0$ limit

$\frac{8t}{T} \in [0.25, 0.40], \quad \delta(\sqrt{8tT}) = 0.01$

- Each data is adopted closest to the fixed $\sqrt{8tT}$
- Constant extrapolation: to calculate the central value
- Linear extrapolation: to estimate the systematic error with constant extrapolation

Sys. error ... at most $3-\sigma \left( \frac{s}{T^3} \right)$ and $2-\sigma \left( \frac{\Delta}{T^4} \right)$
$t \rightarrow 0$ limit

- $\sqrt{8tT} \in [0.25, 0.40], \, \delta(\sqrt{8tT}) = 0.01$
- Carry out both constant- and linear-extrapolation
- We take the central result which is the better $\chi^2$/d.o.f
- Sys. error ... at most $2-\sigma$ ($s/T^3$) and $1-\sigma$ ($\Delta/T^4$)
Figure: Result at $\beta = 2.85$, $t/a^2 = 32$

Topological charge $Q$

$$Q = \frac{1}{32\pi^2} \int d^4 x \, \epsilon_{\mu\nu\rho\sigma} \text{Tr} G_{\mu\nu} G_{\rho\sigma}$$

$Q$ takes an almost integer-value
→ autocorrelation can be negligible in our data sets
Back Up: Renormalized Polyakov loop

- It is believed that universality class of pure SU(2) is same as that of 3-D Ising model
- Renormalization condition\(^{14}\) \(L_R(T = 1.76T_c) = 0.894\)

\[ L_R(T) \]

- Critical exponent (0.3265(3) in 3-D Ising model)
  - \(N_T = 10\): 0.159(3)
  - \(N_T = 12\): 0.242(3)

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