Exploring the phase diagram of finite density QCD at low temperature by the complex Langevin method

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based on collaboration with  
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Introduction

- QCD phase diagram (conjectured)
Introduction

- QCD phase diagram

- Silver-Blaze phenomenon, neutron star, the quark matter phase, color SC

\[
\begin{align*}
T & \quad \text{QGP phase} \\
\text{Hadron phase} & \quad \text{low temperature} \\
\mu & \quad \text{Color Superconductor etc.}
\end{align*}
\]
Introduction

- QCD phase diagram

- QGP phase
- Color Superconductor etc.

- Hadron phase

- difficult to study due to Sign problem

- low temperature

- Silver-Blaze phenomenon, neutron star, the quark matter phase, color SC
Finite density QCD

Partition function

\[ Z = \int dU \, \text{det} \, M[U, \mu] e^{-S_g[U]} \]

- \( S_g[U] \) : gauge action
- \( \text{det} \, M[U, \mu] \) : fermion determinant

- When \( \mu \neq 0 \), the fermion determinant becomes complex, which causes the sign problem.

- Complex Langevin method (CLM) is a promising approach to solve the sign problem.
CLM for lattice QCD

- complexify the link variables

\[ U_{x\mu} \in SU(3) \rightarrow U_{x\mu} \in SL(3, \mathbb{C}) \]

- consider holomorphic extension of the action

\[ S[U] \rightarrow S[U] \]

- update the link variables according to the complex Langevin equation

\[ U_{x\mu}(t + \epsilon) = \exp \left[ i(\epsilon v_{x\mu}(U) + \sqrt{\epsilon} \eta_{x\mu}(t)) \right] U_{x\mu}(t) \]

\[ t : \text{Langevin time} \]
\[ \eta_{x\mu} : \text{Gaussian noise} \]
\[ v_{x\mu} = -D_{x\mu} S[U] : \text{drift term} \]
Correctness of the results of CLM

The two causes for failure of the CLM.

- excursion problem
  large deviation of $U_{x \mu}$ from SU(3)
- singular drift problem
  appearance of near zero eigenvalues of the Dirac operator

Gauge cooling
[Seiler, Sexty, Stamatescu '13]

Solution

Deformation of the Dirac op.
[YI, Nishimura '16]
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In this work, we don’t need to use this technique.
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To judge whether the CLM works or not, we consider

- probability distribution of the magnitude of drift term $v_{x\mu} = -\mathcal{D}_{x\mu}S[\mathcal{U}]$

\[
p(u) = \frac{1}{4NV} \left\langle \sum_{x\mu} \delta(u - u_{x\mu}) \right\rangle
\]

\[
u_{x\mu} \equiv \sqrt{\frac{1}{N_c^2 - 1} \text{tr}(v_{x\mu}v_{x\mu}^\dagger)}
\]

- asymptotic behavior of $p(u)$ at large $u$
  - exponential fall-off $\rightarrow$ reliable results.
  - power law fall-off $\rightarrow$ unreliable results.

Gauge cooling
[Seiler, Sexty, Stamatescu ’13]

deformation of the Dirac op.
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Setup

Lattice setup:
- Lattice size: $8^3 \times 16$
- Staggered fermion with $N_f = 4$
- $\beta = 5.7$
- Quark mass: $m_q a = 0.01, 0.05$
- Chemical potential: $\mu a = 0.1, \ldots, 0.5$

Setup for CLM:
- Langevin step size: $\epsilon = 10^{-4}$, adaptive step size employed
- Total number of Langevin steps: $5 \times 10^5 \sim 15 \times 10^5$
- Gauge cooling at every Langevin step
The aim of this work

• We measure the baryon number density

\[ \langle n \rangle = \frac{1}{N_V N_c} \frac{\partial}{\partial(\mu a)} \log Z \]

• We focus on the **low temperature region**.

Silver-Blaze phenomenon?
transition to nuclear matter/quark matter phase?

cf.) Tsutsui’s talk: analysis at high temperature region

• We compare the results with RHMC results of the phase
quenched model.
Histogram of the drift ($m_q a = 0.05$)

The CLM successfully works even at the intermediate region of $\mu$. 

- Power law fall-off: $\mu = 0.3, 0.325$
- Exponential fall-off: $\mu = 0.225, 0.25, 0.275$
- Reliable region: $0.35 \leq \mu \leq 0.47$
Baryon number density ($m_q a = 0.05$)

\[ \langle n \rangle \]
Comparison with phase quenced simulation ($m_q a = 0.05$)

Not much difference between CLM and PQ results within the reliable region.
Baryon number density ($m_q a = 0.01$)
Comparison with phase quenched simulation \((m_qa=0.01)\)

\[
\langle n \rangle
\]

\[
\mu
\]
$m_q a = 0.01 \text{ vs } 0.05$

For $\mu \lesssim 0.3$

**PQ**

- $\langle n \rangle$ starts to grow earlier for $m_q a = 0.01$ than $m_q a = 0.05$.

**CLM**

- The region of $\mu$ in which the CLM works depends on the quark mass.
- Singular drift problem becomes severer for small $m_q$.
- In the Silver-Blaze region, the CLM is not reliable.
\[ m_q a = 0.01 \text{ vs } 0.05 \]

For \( 0.3 \lesssim \mu \lesssim 0.45 \)

**PQ**
- \( \langle n \rangle \) is sensitive to quark mass.
- \( \mu \) consistent with the formation of pion condensate.

**CLM**
- Not sensitive to quark mass.
- Nucleon condensate (not pion)
- The plateau region w.r.t. \( \mu \) is observed.
- The finite box is already filled with nucleons.
$m_qa=0.01$ vs 0.05

For $\mu \gtrsim 0.45$

**CLM**

- Singular drift problem occurs at large $\mu$.
- $\langle n \rangle$ starts to grow again within the reliable region.

To confirm this, we need to simulate with larger volume.
Summary

• Using the CLM, we have explored finite density QCD at low temperature with $N_f = 4$ staggered fermions.

• Even at the intermediate region of $\mu$, the CLM works without the singular drift problem, where $\langle n \rangle$ has a plateau region suggesting nuclear matter phase.

• $\langle n \rangle$ starts to grow rapidly at the end of the plateau region, which may be a trend toward some transition.
Future work

• measure the pion mass and the nucleon mass at the simulated parameter points, to understand better the critical behavior w.r.t. $\mu$.

• Understand how the singular drift problem is avoided in the nuclear matter phase by measuring the eigenvalue distribution of the Dirac operator.

• Perform simulations with $N_f = 2$ Wilson fermions. (The CL simulation with $N_f = 2$ staggered fermions is problematic even at small $\mu$ due to the large unitarity norm. [Kogut, Sinclair ’17] We have also confirmed this.)
Histogram of the drift ($m_q a = 0.01$)

$\rho(u) * d\tau_{ad}$

$\mu \leq 0.25$

$\mu = 0.10$
$\mu = 0.15$
$\mu = 0.20$
$\mu = 0.25$

Reliable
$\mu = 0.1$

Unreliable
$\mu = 0.15-0.25$

power law fall-off

exponential fall-off
Histogram of the drift ($m_q a = 0.01$)

- $\mu \geq 0.3$
- Reliable: $\mu = 0.325 - 0.45$
- Unreliable: $\mu = 0.3, 0.5$

- Power law fall-off
- Exponential fall-off

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<th>$\mu$</th>
<th>0.3</th>
<th>0.325</th>
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