Axial U(1) symmetry and Dirac spectra in high-temperature phase of $N_f=2$ lattice QCD

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QCD phase diagram
(for $u, d, s$ quarks)

Phase transition
(crossover)

Chiral condensate
(chiral symmetry breaking)
$\textbf{U}(1)_A$ symmetry (in vacuum, broken by anomaly) is restored above $T_c$?

- Above $T_c$, chiral symmetry breaking by $\langle \bar{q}q \rangle$ is restored
  \[ \Rightarrow \text{How about } U(1)_A \text{ symmetry?} \]

\[ \Delta_{\pi-\delta} = \int_0^\infty d^4x \left[ \pi^a(x)\pi^a(x) - \delta^a(x)\delta^a(x) \right] \]
If $U(1)_A$ is restored...

**Colombia plot is modified?**

$m_s \rightarrow \infty$  \hspace{0.5cm} \hspace{0.5cm} m^{\text{cri}}(?) \hspace{0.5cm} N_f = 2 \text{ world}$

Conventionally, at $m_{u,d} \rightarrow 0$, 2\textsuperscript{nd} with $O(4)$

1\textsuperscript{st} order? 2\textsuperscript{nd} order, not $O(4)$?

$N_f = 1 \text{ world}$

$m_{u,d,s} \rightarrow \infty$  \hspace{0.5cm} \hspace{0.5cm} m_{u,d} \rightarrow \infty$
\[\text{U}(1)_A \text{ symmetry above } T_c \Rightarrow \text{Long-standing problem in QCD}\]

- Gross-Pisarski-Yaffe (Dilute instanton gas model, 1981) restored at enough high T
- Cohen (1996) w/o zero mode (or instanton) ⇒ restored
- Aoki-Fukaya-Taniguchi (2012) zero mode suppressed, restored in chiral limit at \(N_f = 2\)
- HotQCD (DW, 2012) broken
- JLQCD (topology fixed overlap, 2013) restored
- TWQCD (optimal DW, 2013) restored
- LLNL/RBC (DW, 2014) broken (restored at higher T?)
- Dick et al. (overlap on HISQ, 2015) broken
- Sharma et al. (overlap on DW, 2015,2016,2018) broken
- Brandt et al. (Wilson, 2016) restored
- Ishikawa et al. (Wilson, 2017) restored
- JLQCD (reweighted overlap on DW, 2017) restored
- Rohrhofer et al. (DW, 2017) restored

⇒ Many suggestions from lattice QCD (and models)...
**U(1)$_A$ symmetry restoration** by JLQCD Collaboration

⇒ overlap fermion (exact chiral symmetry on the lattice)

<table>
<thead>
<tr>
<th></th>
<th>valence/sea quark</th>
<th>Setup</th>
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</thead>
<tbody>
<tr>
<td>G. Cossu et al. PRD87 (2013)</td>
<td>OV on OV (Topology fixed sector)</td>
<td></td>
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<tr>
<td>A. Tomiya et al. PRD96 (2017)</td>
<td>DW on DW OV on DW OV on (rewighted) OV</td>
<td>$1/a=1.7\text{GeV}$ (a=0.11fm)</td>
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<tr>
<td></td>
<td>In progress</td>
<td></td>
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<tr>
<td></td>
<td>OV on DW OV on (rewighted) OV</td>
<td>$1/a=2.6\text{GeV}$ (a=0.076fm) (Finer lattice)</td>
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Outline

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2. U(1)$_A$ susceptibility from Dirac spectra (zero mode and ultraviolet divergence)
3. Results
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   3-2: Cutoff and volume dependences
4. Summary
Chiral condensate and Dirac spectra

Banks-Casher relation:

\[
\langle \bar{q} q \rangle = \lim_{m \to 0} \int_0^\infty d\lambda \, \rho(\lambda) \left( \frac{2m}{\lambda^2 + m^2} \right)
\]

\[
\rho(\lambda) \equiv \lim_{V \to \infty} \frac{1}{V} \Sigma_{\lambda'} < \delta(\lambda - \lambda') >
\]

\[
\rho(0) = -\langle \bar{q} q \rangle / \pi
\]

with interaction

Chiral condensate induced by low modes
T-dependence of Dirac spectra

Low $T$:

$\rho(0) \neq 0$  
$\Rightarrow$ Spontaneous chiral symmetry breaking

$\langle \bar{q}q \rangle$

High $T$:

$\rho(0) = 0$  
$\Rightarrow$ Chiral symmetry restoration

Critical Temp.
$U(1)_A$ susceptibility and low modes of Dirac spectra

$$\Delta_{\pi-\delta} = \int_0^\infty d\lambda \, \rho(\lambda) \frac{2m^2}{(\lambda^2 + m^2)^2}$$

Cf.) Banks-Casher relation: $\langle \bar{q}q \rangle = \lim_{m \to 0} \int_0^\infty d\lambda \, \rho(\lambda) \frac{2m}{\lambda^2 + m^2}$

Low mode contribution is enhanced by the factor of $1/\lambda^4$
Note 1:

$U(1)_A$ susc. = Low modes + **Zero mode**?

\[
\Delta_{\pi-\delta} = \int_0^\infty d\lambda \rho(\lambda) \frac{2m^2}{(\lambda^2 + m^2)^2} \quad \Rightarrow \quad \Delta_{\pi-\delta}^{ov} \equiv \frac{1}{V(1-m^2)^2} \sum_i \frac{2m^2(1 - \lambda_{ov}^{(i)} \lambda_{ov}^{(i)})^2}{\lambda_{ov}^{(i)} \lambda_{ov}^{(i)}}
\]

The factor of $1/\lambda^4$ enhances zero-mode contribution?

In $V \to \infty$ limit, we know zero-mode contribution is suppressed:

\[
\Delta_{0\text{-mode}}^{ov} = \frac{2N_0}{V m^2} (\propto 1/\sqrt{V})
\]

New order parameter: we subtract zero mode

\[
\overline{\Delta}_{\pi-\delta}^{ov} \equiv \Delta_{\pi-\delta}^{ov} - \frac{2N_0}{V m^2}
\]
Note 2: \( \text{U(1)}_A \text{ susc.} = \text{Physics} + \text{Ultraviolet divergence?} \)

\[ \Delta_{\pi-\delta} = \int_0^\infty d\lambda \rho(\lambda) \frac{2m^2}{(\lambda^2 + m^2)^2} \]

\[ \rho(\lambda) \sim \lambda^3 \]

\[ \sim \frac{1}{\lambda^4} \]

\[ \Delta_{\pi-\delta} \propto m^2 \ln \Lambda + \ldots \]

The term depends on cutoff \( \Lambda \) and valence quark mass \( m \)

We assume valence quark mass dependence of \( \Delta_{\pi-\delta} \) (for small \( m \)):

\[ \Delta_{\pi-\delta}(m) = \frac{a}{m^2} + b + cm^2 + O(m^4) \]

\( \approx \) From 3 eqs. for \( \Delta_{\pi-\delta}(m_1), \Delta_{\pi-\delta}(m_2), \Delta_{\pi-\delta}(m_3) \), \( a \) and \( c \) are eliminated

\[ \Rightarrow \Delta_{\pi-\delta} \sim b + O(m^4) \] (that depends on sea quark mass)
Overlap Dirac spectra at $T = 220\text{MeV}$

$m_q = 2.6\text{MeV}$

$m_q = 26\text{MeV}$
$U(1)_A$ susceptibility at $T = 220\text{MeV}$

⇒ At $m=2.6\text{MeV}$, we found suppression of $10^{-4}\text{GeV}^2$

$\Delta_{\pi-\delta}$ is almost zero

⇒ In the chiral limit, $U(1)_A$ will be restored.
Small mass region \( \Rightarrow \) small \( \Delta_{\pi-\delta} \) by low mode suppression

Large mass region \( \Rightarrow \) large \( \Delta_{\pi-\delta} \) by low mode enhancement
$U(1)_A$ susceptibility \((\text{UV-subt. before/after})\)

⇒ Ultraviolet divergence \(\sim m^2 \ln \Lambda\) is subtracted from $\Delta_{\pi-\delta}$
$U(1)_A$ susceptibility (UV-subt. before/after)

$\Rightarrow$ Ultraviolet divergence ($\sim m^2 \ln \Lambda$) is subtracted from $\Delta_{\pi-\delta}$

$32^3 \times 12, \beta = 4.3, T = 220\,\text{MeV}$
Did we really remove the ultraviolet contribution?

Check of cutoff dependence

\[ \Delta_{\pi-\delta} = \int_0^\infty d\lambda \rho(\lambda) \frac{2m^2}{(\lambda^2 + m^2)^2} \quad \Rightarrow \quad \Delta_{\pi-\delta}^{ov} \equiv \frac{1}{V(1 - m^2)^2} \sum_i \frac{2m^2(1 - \lambda_{ov}^{(i)})^2}{\lambda_{ov}^{(i)4}} \]

\[ \rho(\lambda_{ov}) \]

40th low modes

To evaluate \( \Delta_{\pi-\delta} \), we sum up 40 lowest modes

⇒ Cutoff dependence by the number of low modes
$U(1)_A$ susceptibility (cutoff dependence)

⇒ No cutoff dependence (saturated by a few low modes)
U(1)\(_A\) susceptibility (volume effect)

\[ \Rightarrow \text{For small } m, \text{ V-dependence seems to be small} \]
$U(1)_A$ susceptibility ($T=220, 330\text{MeV}$)

⇒ With increasing $T$, $U(1)_A$ is more restored
Summary and outlook

• In high-temperature phase \((T > T_c)\) at \(N_f = 2\), we studied \(\text{U}(1)_A\) susceptibility

• **Strong suppression in the chiral limit** (for \(T = 220\text{-}330\text{MeV}\))

• Checked volume and cutoff dependences

• Topological susceptibility \(\Rightarrow \) **talk by Y. Aoki**

• Parametrization as function of \(m_q\) (larger than \(m_q^2\)?)

• Near \(T_c\) \((N_t = 14\)?, chiral transition?)

• \(N_f = 2 + 1\) sector
Backup
Note 1:

\[ U(1)_A \text{ susc.} = \text{Low modes} + \text{Zero mode?} \]

\[
\Delta_{\pi-\delta} \equiv \int_0^\infty d\lambda \rho(\lambda) \frac{2m^2}{(\lambda^2 + m^2)^2}
\]

\[
\rho_{0\text{-mode}}(\lambda) = \frac{1}{V} \sum_{0\text{-mode}} \delta(\lambda)
\]

\[
\Delta_{\text{zero}} = \int_0^\infty d\lambda \frac{1}{V} \sum_{0\text{-mode}} \delta(\lambda) \frac{2m^2}{(\lambda^2 + m^2)^2}
\]

\[
= \frac{1}{V} \sum_{0\text{-mode}} \frac{2m^2}{m^4}
\]

\[
= \frac{1}{V} \sum_{0\text{-mode}} \frac{2}{m^2} = \frac{2N_0}{Vm^2}
\]

\[
\lim_{V \to \infty} \Delta_{\text{zero}} = 0
\]

Zero mode contributions in \( \Delta_{\pi-\delta} \) will be suppressed in \( V \to \infty \) limit
U(1)$_A$ susceptibility (DW/OV reweighting)

⇒ DW/OV reweighting is crucial in small m region
Histogram of topological charge at $T = 220\text{MeV}$

$m_q = 2.6\text{MeV}$

$m_q = 10\text{MeV}$

Small $m_q$: all conf. are $Q=0$ sector

Large $m_q$: $Q\neq 0$ sectors appear

Using $\chi_t \equiv \frac{\langle Q_t^2 \rangle}{V}$, we plot $\chi_t$
Topological susceptibility at $T = 220\text{MeV}$

⇒ In small $m_q$ region, $\chi_t=0$?
⇒ Around $m_q \sim 10\text{MeV}$, we found a jump (critical mass?)
Topological susceptibility (Temperature dependence)

\[ \chi_t [\text{MeV}^4] \]

- With increasing $T$, $\chi_t$ becomes small

$\Rightarrow$ With increasing $T$, $\chi_t$ becomes small
Topological susceptibility (Volume dependence)