Computing $\hat{q}$ on a quenched SU(3) lattice

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Outline

- Quark and gluon plasma (QGP) produced in heavy-ion collision
  1) Importance of transport coefficient $\hat{q}$
  2) Need for an \textit{ab-initio} calculation of $\hat{q}$

- Formulating $\hat{q}$ on the lattice using Lattice gauge theory
  1) Previous study done on a quenched SU(2) lattice
  2) Extending calculations to a quenched SU(3) lattice

- Estimates of $\hat{q}$ on a quenched QGP plasma
- Sudden increase in $\epsilon$ near the transition region.
- Due to increase in the number of DOF
- Corresponds to a deconfined state of quarks and gluons
Creation of the Quark-Gluon Plasma (QGP)

Relativistic Heavy Ion Collider (RHIC) at BNL: **Collide two Au/Cu nuclei @ 20-200GeV per nucleon pair**

Large Hadron Collider (LHC) at CERN: **Collide two Pb nuclei @ 2.76TeV/5.5TeV per nucleon pair**
Creation of the Quark-Gluon Plasma (QGP)

- Initial state: two Lorentz-contracted nuclei approach each other
- Pre-equilibrium state: undergo hard collisions produce hard probes and drive the system to thermalization in the form of QGP matter
- QGP phase: the QGP expands hydrodynamically
- Hadronization: the QGP cools down and new hadrons are formed
- Freeze-out: hadron gas is so dilute that the interactions cease
Thermalized, non-perturbative, and strongly interacting plasma
Flow measurement: Evidence for strongly interacting QGP

Elliptic flow: \[ v_2 \equiv \left\langle \frac{p_x^2 - p_y^2}{p_x^2 + p_y^2} \right\rangle; \quad \frac{dN}{d\varphi} \propto N(1 + \sum_n 2v_n \cos(n\varphi)) \]

- Data indicate that \( v_2 \) is finite
- Hydrodynamical calculations describes the data
- Lattice QCD EOS is used as input
- QGP is thermalized and strongly interacting

Schenke et al., PRL 108 252301 (2012)
Evidence for strongly interacting QGP

**Proton Proton collision**

leading particle

hadrons

**Nucleus-Nucleus collision**

leading particle suppressed

hadrons

**Nuclear Modification Factor** \((R_{AA})\):

\[
R_{AA} \equiv \frac{d^2 N^{AA}/dydp_{\perp}}{d^2 N^{pp}/dydp_{\perp} \times \langle N^{AA}_{\text{coll}} \rangle}
\]

Transport parameter $\hat{q}$ and leading hadron suppression

**Transport parameter $\hat{q}$**: Average transverse momentum change per unit length

$$\hat{q}(\vec{r}, t) = \frac{< k_\perp >}{L}$$

$\hat{q}$ is Input parameter to full model calculation
Transport parameter $\hat{q}$ and leading hadron suppression

Transport parameter $\hat{q}$: Average transverse momentum change per unit length

$$\hat{q}(\vec{r}, t) = \frac{< k_\perp^2 >}{L}$$

$\hat{q}$ is Input parameter to full model calculation
Transport parameter $\hat{q}$

Based on fit to the experimental data
QGP is locally thermalized and highly non-perturbative

First principles calculation: Lattice QCD to compute $\hat{q}$
Light-cone coordinates

- **Minkowski coordinate**
  
  Four vector \( p = (p^0, p^1, p^2, p^3) \)

  Off-shellness \( p^2 = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 \)

- **Light-cone coordinate**
  
  Four vector \( p = (p^+, p^-, p_\perp) \)

  \[
  p^+ = \frac{(p^0 + p^3)}{\sqrt{2}} \\
  p^- = \frac{(p^0 - p^3)}{\sqrt{2}} \\
  p_\perp = (p_\perp^1, p_\perp^2)
  \]

  Off-shellness \( p^2 = 2p^+ p^- - p_\perp^2 \)

Examples: Particle traveling in \(+P_z\) direction : \( P^+ \gg P^-; \ P_\perp = 0 \)
Lattice formulation of $\hat{q}$

- Simplest process: A leading quark propagating through **hot plasma (gluons only)** at temperature $T$

$$q = \left( \frac{\mu^2}{2 q^-}, q^-, 0 \right) = (\lambda^2, 1, 0)Q; \quad \text{Hard scale} = Q; \quad \lambda \ll 1$$

$$k = (k^+, k^-, k_\perp 0) = (\lambda^2, \lambda^2, \lambda)Q; \quad \text{Glauber gluon}$$

- Life time of quark, $\tau \geq 4n_t a = \frac{4}{T}$

A. Majumder, PRC 87, 034905 (2013)
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- Life time of quark, $\tau \geq 4n_t a = \frac{4}{T}$

\[
\hat{q}(\vec{r}, t) = \sum_k k^2_\perp \frac{\text{Disc}[W(k)]}{L}
\]

\[
\hat{q} = \frac{4\pi^2 \alpha_s}{N_c} \int \frac{dy^- d^2y_\perp}{(2\pi)^3} \ d^2k_\perp e^{-i\frac{k^2_\perp}{2q^-} y^- + ik_\perp \vec{y}_\perp} 
\]

\[
x \langle M | F^+_{\perp \mu} (y^-, y_\perp) F^+_{\perp \mu}(0) | M \rangle
\]

A. Majumder, PRC 87, 034905 (2013)
Constructing a more general expression as $\hat{Q}$

- Physical form of $\hat{q}$
  \[
  \hat{q} = \frac{4\pi^2\alpha_s}{N_c} \int \frac{d^2y^- d^2y_\perp}{(2\pi)^3} \ e^{i\frac{k_\perp^2}{2q^-}y^- + ik_\perp y_\perp} \ \langle M | F_{\perp\mu}^+(y^-, y_\perp) F_{\perp\mu}^+(0) | M \rangle
  \]

- General form of $\hat{q}$: with $q^-$ is Fixed; $q_\perp = 0$; $q^+$ is variable
  \[
  \hat{Q}(q^+) = \frac{4\pi^2\alpha_s}{N_c} \int \frac{d^4y d^4k}{2\pi^4} \ e^{iky} \ 2q^- \langle M | F_{\perp\mu}^+(0) F_{\perp\mu}^+(y) | M \rangle \ \frac{1}{(q + k)^2 + i\epsilon}
  \]

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  $\hat{q} = \frac{4\pi^2\alpha_s}{N_c} \int \frac{dy^- d^2 y_\perp}{(2\pi)^3} d^2 k_\perp e^{-i\frac{k_\perp^2}{2q^-} y^- + ik_\perp \cdot \vec{y}_\perp} \langle M | F_{\perp \mu}^- (y^-, y_\perp) F_{\perp \mu}^+ (0) | M \rangle$

- General form of $\hat{q}$: with $q^-$ is Fixed; $q_\perp = 0$; $q^+$ is variable
  
  $\hat{Q}(q^+) = \frac{4\pi^2\alpha_s}{N_c} \int \frac{d^4 y d^4 k}{2\pi^4} e^{ik\cdot y} 2q^- \frac{\langle M | F_{\perp \mu}^+ (0) F_{\perp \mu}^+ (y) | M \rangle}{(q + k)^2 + i\epsilon}$

1) When $q^+ \sim T$

  $\text{Disc}[\hat{Q}(q^+)]_{at \ q^+ \sim T} = \hat{q}$

A. Majumder, PRC 87, 034905 (2013)
Constructing a more general expression as $\hat{Q}$

- **Physical form of $\hat{q}$**
  \[
  \hat{q} = \frac{4\pi^2 \alpha_s}{N_c} \int \frac{d^2 y^2 y_{\perp}}{(2\pi)^3} \, d^2 k_{\perp} \, e^{-i \frac{k_{\perp}^2}{2q_{\perp}} y_{\perp} + i k_{\perp} \cdot y_{\perp}} \langle M | F_{\perp \mu}^+(y_{\perp}) F_{\perp \mu}^+(0) | M \rangle
  \]

- **General form of $\hat{q}$**: with $q^-$ is Fixed; $q_{\perp} = 0$; $q^+$ is variable
  \[
  \hat{Q}(q^+) = \frac{4\pi^2 \alpha_s}{N_c} \int \frac{d^4 y \, d^4 k}{(2\pi)^4} \, e^{i k y} \, 2 q^- \frac{\langle M | F_{\perp \mu}^+(0) \, F_{\perp \mu}^+(y) | M \rangle}{(q + k)^2 + i \epsilon}
  \]

1) **When $q^+ \sim T$**

   \[
   \text{Disc}[\hat{Q}(q^+)]_{at \ q^+ \sim T} = \hat{q}
   \]

2) **When $q^+ = -q^-$**

   \[
   \frac{1}{(q + k)^2} \approx -2q^- q^- + 2q^-(k^+ - k^-) = -2q^- \left[ 1 - \left( \frac{k^+ - k^-}{q^-} \right) \right]^{-1} = -2q^- \left[ 1 - \left( \frac{\sqrt{2} k_z}{q^-} \right) \right]^{-1} = -2q^- \left[ \sum_{n=0}^{\infty} \left( \frac{\sqrt{2} k_z}{q^-} \right)^n \right]
   \]
Constructing a more general expression as $\hat{Q}$

- Physical form of $\hat{Q}$
  $$\hat{q} = \frac{4\pi^2 \alpha_s}{N_c} \int \frac{dy^- d^2 y_\perp}{(2\pi)^3} \, d^2 k_\perp e^{\frac{-i}{2q^-} y^- + ik_\perp \cdot y_\perp} \langle M | F_{\perp\mu} (y^-, y_\perp) F_{\perp\mu}^+(0) | M \rangle$$

- General form of $\hat{q}$: with $q^-$ is Fixed; $q_- = 0$; $q^+$ is variable
  $$\hat{Q}(q^+) = \frac{4\pi^2 \alpha_s}{N_c} \int \frac{d^4 y d^4 k}{2\pi^4} \, e^{ik \cdot y} \, 2q^- \frac{\langle M | F_{\perp\mu}^+(0) \, F_{\perp\mu}^+(y^-) | M \rangle}{(q + k)^2 + i\epsilon}$$

1) When $q^+ \sim T$
   $$\text{Disc} [\hat{Q}(q^+)]_{at \, q^+ \sim T} = \hat{q}$$

2) When $q^+ = -q^-$
   $$\frac{1}{(q + k)^2} \approx -2q^- q^- + 2q^- (k^+ - k^-) = -\frac{1}{2q^-} \left[ 1 - \left( \frac{k^+ - k^-}{q^-} \right) \right]^{-1} = -\frac{1}{2q^-} \left[ 1 - \left( \frac{\sqrt{2k_z}}{q^-} \right)^n \right]^{-1} = -\frac{1}{2q^-} \left[ \sum_{n=0}^{\infty} \left( \frac{\sqrt{2k_z}}{q^-} \right)^n \right]$$

$$\hat{Q}(q^+ = -q^-) = \frac{4\pi^2 \alpha_s}{N_c} \frac{1}{q^-} \langle M | F_{\perp\mu}^+(0) \sum_{n=0}^{\infty} \left( \frac{i\sqrt{2D_z}}{q^-} \right)^n F_{\perp\mu}^+(0) | M \rangle$$

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A. Majumder, PRC 87, 034905 (2013)
Extracting $\hat{q}$ through analytic structure of $\hat{Q}(q^+)$

When $q^+ \in \#[-T, T]\$ 

$q^2 \approx 0$ (in-medium scattering)
Extracting $\hat{q}$ through analytic structure of $\hat{Q}(q^+)$

$\hat{Q}(q^+) = \frac{4\pi^2\alpha_s}{N_c} \int \frac{d^4y d^4k}{2\pi^4} e^{i k y} 2q^- \frac{\langle M | F_{\perp}^{+\mu}(0) F_{\perp}^{+\mu}(y) | M \rangle}{(q + k)^2 + i\epsilon}$

1) When $q^+ \in [-#T, #T]$
   
   $q^2 \approx 0$ (in-medium scattering)

2) When $q^+ \in [#T, +\infty)$
   
   $q^2 \gg 0$ (Bremsstrahlung radiation)
Extracting $\hat{q}$ through analytic structure of $\hat{Q}(q^+)$

\[ \hat{Q}(q^+) = \frac{4\pi^2\alpha_s}{N_c} \int \frac{d^4y}{2\pi^4} e^{iky} 2q^- \begin{vmatrix} M^{+\mu}(0)F^{+\mu}(y) \end{vmatrix} \frac{1}{(q+k)^2+i\epsilon} \]

1) When $q^+ \in [-#T, #T]$
   
   $q^2 \approx 0$ (in-medium scattering)

2) When $q^+ \in [#T, +\infty)$
   
   $q^2 \gg 0$ (Bremsstrahlung radiation)

3) When $q^+ \in (-\infty, -#T]$
   
   $q^2 \ll 0$ (Space-like); \( \lim_{q^+ \to -\infty} \text{Disc}[\hat{Q}(q^+)] = 0 \)
Extracting $\hat{q}$ through analytic structure of $\hat{Q}(q^+)$

$$I_1 = \oint dq^+ \frac{\hat{Q}(q^+)}{2\pi i (q^+ + q^-)} = \hat{Q}(q^+ = -q^-)$$

Contour C1

$q^+ = -q^-$
Extracting $\hat{q}$ through analytic structure of $\hat{Q}(q^+)$

Contour $C_1$

$$I_1 = \oint \frac{dq^+}{2\pi i} \frac{\hat{Q}(q^+)}{q^+ + q^-} = \hat{Q}(q^+ = -q^-)$$

Contour $C_2$: On stretching it to infinity

$q^+ = -q^-$
Extracting \( \hat{q} \) through analytic structure of \( \hat{Q}(q^+) \)

Contour C1

\[
I_1 = \oint \frac{dq^+}{2\pi i} \frac{\hat{Q}(q^+)}{q^+ + q^-} = \hat{Q}(q^+) = -q^-
\]

Contour C2: On stretching it to infinity

\[
I_1 = \int_{-\#T}^{\#T} dq^+ \frac{\hat{q}(q^+)}{q^+ + q^-} + \int_{0}^{\infty} dq^+ \frac{\text{Disc}[\hat{Q}(q^+)]}{q^+ + q^-}
\]

Pure thermal part  Pure Vacuum part
\[ \hat{q} \text{ as a series of local operators} \]

- Physical form of \( \hat{q} \) at LO:

\[
\hat{q} = \frac{4\sqrt{2}\pi^2\alpha_s}{N_cT} \langle M|F^+_{\perp\mu}(0) \sum_{n=0}^{\infty} \left( \frac{i\sqrt{2}D_z}{q^-} \right)^n F^+_{\perp\mu}(0) |M\rangle_{\text{Thermal-Vacuum}}
\]
\( \hat{q} \) as a series of local operators

- **Physical form of \( \hat{q} \) at LO:**

\[
\hat{q} = \frac{4\sqrt{2}\pi^2\alpha_s}{N_c T} \langle M | F_{\perp\mu}^+(0) \sum_{n=0}^{\infty} \left( \frac{i\sqrt{2}D_z}{q^-} \right)^n F_{\perp\mu}^+(0) | M \rangle (\text{Thermal–Vacuum})
\]

Xiangdong Ji, PRL 110, 262002 (2013)
Parton PDF operator product expansion with \( D_z \) derivatives
\( \hat{q} \) as a series of local operators

- Physical form of \( \hat{q} \) at LO:

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\hat{q} = \frac{4\sqrt{2}\pi^2 \alpha_s}{N_c T} \langle M | F_+^{\mu} (0) \sum_{n=0}^{\infty} \left( \frac{i\sqrt{2}D_z}{q^-} \right)^n F_+^{\mu} (0) | M \rangle_{\text{ Thermal-Vacuum}}
\]

Rotating to Euclidean space:

\[
x^0 \rightarrow -ix^4; \quad A^0 \rightarrow iA^4
\]

\[\Rightarrow F^{0i} \rightarrow iF^{4i}\]

LO operators:

\[
\sum_{i=1}^{2} \text{Trace} [F^{3i}F^{3i} - F^{4i}F^{4i}] + 2i \sum_{i=1}^{2} \text{Trace} [F^{3i}F^{4i}]
\]

Uncrossed operator

Crossed operator

LO operators with \( D_z \) derivative:

\[
\sum_{i=1}^{2} \text{Trace} [F^{3i}D_zF^{3i} - F^{4i}D_zF^{4i}] + i \sum_{i=1}^{2} \text{Trace} [F^{3i}D_zF^{4i} + F^{4i}D_zF^{3i}]
\]

Xiangdong Ji, PRL 110, 262002 (2013)
Parton PDF operator product expansion with \( D_z \) derivatives
Operators in quenched SU(2) plasma

- Average over 5000 configuration
- Transition temperature $T_c \in [170, 350]$ MeV
- Crossed correlator is small for $T \sim 400$ MeV

A. Majumder, PRC 87, 034905 (2013)
Operators in quenched SU(3) plasma

Preliminary (in collaboration with Chiho Nonaka)
Operators in quenched SU(3) plasma

Preliminary (in collaboration with Chiho Nonaka)
Operators in quenched SU(3) plasma

- Operators looks similar to SU(2)
- Need to set the lattice spacing ($a_L$) i.e. relation between $g$ and $a_L$
Scale setting on the lattice using Polyakov loop

- Expectation value of Polyakov loop:

\[
P = \frac{1}{n_x n_y n_z} \text{tr} \left[ \sum_{n=0}^{n_t-1} \prod_{\vec{r}} U_A(na, \vec{r}) \right]
\]

- Two loop beta function

\[
a_L = \frac{1}{\Lambda_L} \left( \frac{11}{16\pi^2 g^2} \right)^{\frac{-51}{121}} \exp \left( -\frac{8\pi^2}{11g^2} \right)
\]

Temperature, \(T = \frac{1}{n_t a_L}\) (Pure SU(3))

- Nonperturbative correction

\[
\text{Tune} \quad \frac{T_c}{\Lambda_L} \text{ is independent of } g
\]
Real part of FF correlator in quenched SU(3)

- Uncrossed correlator is dominant at high temperature
- Crossed correlator goes to zero at high temperature
- Correlator with Dz derivative are suppressed
Real part of FF correlator in quenched SU(3)

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̂q in quenched SU(3) plasma
$\hat{q}$ in quenched SU(3) plasma

![Graph showing $\hat{q}/T^3$ vs. T for Au+Au at RHIC and Pb+Pb at LHC with different lattice sizes: 2x8^3, 4x16^3, 6x24^3.](image)
Summary and Future work

- First calculation of $\hat{q}$ on SU(3) quenched plasma
  - Analytic continuation to deep Euclidean space and expressed as local operators
  - Scale setting using perturbative loop beta function with non-perturbative correction using Polyakov loop.

- Temperature dependence of $\hat{q}$
  - Real part of $\hat{q}$ goes as $T^3$ for $T > 400$ MeV
  - Real part of $\hat{q}$ shows scaling behavior (Nt=2,4 and 6)
  - Imaginary part goes to 0 for $T > 400$ MeV

- Future work
  - Extend calculation using Improved Action and bigger lattice size
  - Include radiation diagram contributions
  - Extend to unquenched plasma (QGP)
Thanks to group members and colleagues

- Abhijit Majumder
- Chiho Nonaka (Nagoya University)
- ShanShan Cao, Yasuki Tachibana and Chathuranga Sirimana
Imaginary part of FF correlator in quenched SU(3)

- Imaginary part of FF correlator does not contribute