Low $T$ condensation and scattering data

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Motivation: Particle Number vs. $\mu$

- Typical scenario at low $T$ (fixed) and finite $V$ ($= L^{d-1}$):

- Nature of condensation steps?
- Condensation thresholds $\mu_c^i \Leftrightarrow$ low energy parameters
Our approach

- We study the low temperature regime of a QFT at finite density

- For small spatial lattices particle sectors are separated by finite energy steps $\Rightarrow$ particle condensation

- Critical chemical potentials $\mu^i_c$ of these transitions are related to the minimal $N$-particle energies $W_N$

- At finite volume scattering information is encoded in $W_N$

- We measure the $\mu^i_c$ with worldline simulations and extract low energy scattering parameters
The framework

► Model: complex scalar field with $\phi^4$ interaction in 2D and 4D

$$S = \sum_{n \in \Lambda} \left( (2d + m_0^2) |\phi_n|^2 + \lambda |\phi_n|^4 - \sum_{\nu=1}^{d} \left[ e^{\mu \delta_{\nu,d}} \phi_n^* \phi_{n+\hat{\nu}} + e^{-\mu \delta_{\nu,d}} \phi_n^* \phi_{n-\hat{\nu}} \right] \right)$$

► Sign problem $\Rightarrow$ MC not possible in conventional representation

► Worldline representation with real and positive weights solves sign problem

Measuring the condensation thresholds (2D case)

Fit the steps to a logistic function \( \mu_i^c \) is inflection point

\[
N(\mu) = (i - 1) + \left[ 1 + e^{-k(\mu - \mu_i^c)} \right]^{-1}
\]

Simulation data

Fit: \( k = 398(23), \mu_c^1 = 0.2602(12) \)
Interpretation of the condensation steps

- Condensation thresholds are related to the minimal $N$-particle energies:
  Bruckmann et al., PRL 115, 231601 (2015)

$$W_1 = \mu_c^1 \equiv m$$
$$W_2 = \mu_c^1 + \mu_c^2$$
$$W_3 = \mu_c^1 + \mu_c^2 + \mu_c^3$$

\[ \vdots \]

- $W_N$ depend on low energy parameters (LEP)

- Describe condensation thresholds in terms of LEP
Important cross-check with conventional spectroscopy

- Compute $N$-particle energies with connected $2N$-point functions:

\[
\langle (\tilde{\phi}_t)^N (\tilde{\phi}_0^*)^N \rangle_c \propto e^{-tE}\n\]

with $E_1 = m$, $E_2 = W_2$, $E_3 = W_3$, \ldots

$\Rightarrow$ fields are projected to zero momentum

- $\mu$ is absent $\Rightarrow$ MC in conventional representation

- Extract the $N$-particle energies from the exponential decay of correlators
Spectroscopy versus WL simulations (2D case)

⇒ Interpretation of condensation steps as $m$, $W_2$, $W_3$ confirmed!
**L-dependence of N-particle energies (4D case)**

\[ m = m_\infty + \frac{A}{L^{3/2}} e^{-Lm_\infty} \]

Rummukainen & Gottlieb 1995

\[ W_2 = 2m + \frac{4\pi a}{mL^3} \left[ 1 - \frac{aI}{L\pi} + \left(\frac{a}{L}\right)^2 \frac{I^2 - J}{\pi^2} + O\left(\frac{a}{L}\right)^3 \right] \]

Huang & Yang 1957, Lüscher 1986

\[ W_3 = 3m + \frac{12\pi a}{mL^3} \left[ 1 - \frac{aI}{L\pi} + \left(\frac{a}{L}\right)^2 \frac{I^2 + J}{\pi^2} + O\left(\frac{a}{L}\right)^3 \right] \]

Beane et al. 2007, Hansen & Sharpe 2014, 15, 16, Sharpe 2017

- Infinite-volume mass \( m_\infty \)
- Scattering length \( a \quad \left[ \delta(k) = -ak + O(k^2) \right] \)
- Numerical constants \( I = -8.914 \) and \( J = 16.532 \)
Results for 4D

$m_\infty = 0.168(1)$ and $a = -0.078(7)$

Good "prediction" of $W_3$ except for very small $L (\equiv N_s)$

C. Gattringer, M. Giuliani, O. Orasch, PRL 120, 241601 (2018)
Scattering data in 2D

In 2D the full scattering phase shift can be determined from the periodic boundary condition: M. Lüscher, U. Wolff, Nucl. Phys. B 339, 222 (1990)

\[ e^{2i\delta(k)} = e^{-ikL} \]

For short-range interaction:

\[ W_2 = 2\sqrt{m^2 + k^2} \quad \Rightarrow \quad \delta(k) = -\frac{L}{2} \sqrt{\left(\frac{W_2}{2}\right)^2 - m^2} \]
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From 4D: Up to leading order \( W_3 \) should be described by \( \delta(k) \)

\[ W_3 = \sqrt{m^2 + (k_1 + k_2)^2} + \sqrt{m^2 + k_1^2} + \sqrt{m^2 + k_2^2} \]
Results for 2D

Very good agreement of spectroscopy and worldline results
Results for 2D

- Good prediction of $W_3$ with $\delta(k)$ from $W_2$
- Condensation steps are determined by scattering phase shift $\delta(k)$
Summary

- Low temperature study of the $\phi^4$ model at finite density
- Sign problem is evaded by using a worldline representation
- For low $T$ particle sectors are separated by finite energy steps $\Rightarrow$ critical chemical potential/condensation thresholds $\mu^i_c$
- The $N$-particle energy is the sum of the $\mu^i_c, i = 1, \ldots, N$
- Cross-check of condensation steps with spectroscopy calculations
- Scattering parameters can be extracted from $W_N$
  - 4D: scattering length $a$
  - 2D: full scattering phase shift $\delta(k)$
- Condensation thresholds are determined by scattering data
Thank you for listening!
Backup slides
Simulation parameters

- Parameters for 4D simulations:
  \[ \eta = 2d + m_0^2 = 7.44, \lambda = 1.0, N_T = 320, 640, N_s = 3, 4, \ldots, 10 \]

- Parameters for 2D simulations:
  \[ \eta = 2d + m_0^2 = 2.6, \lambda = 1.0, N_T = 400, N_s = 2, 4, \ldots, 16 \]
Worldline representation for the charged scalar $\phi^4$ field

- In the worldline approach the grand canonical partition sum is exactly rewritten in terms of dual link variables $k_{n,\nu} \in \mathbb{Z}$

$$Z = \sum_{\{k\}} e^{\mu\beta W_t[k]} B[k] C[k]$$

- $W_t[k] = \text{temporal winding number of the worldlines}$

- Real and positive weights $B[k]$

- Constraints $C[k] = \prod_n \delta(\vec{\nabla} \cdot \vec{k}_n)$

$$\Rightarrow \quad \vec{\nabla} \cdot \vec{k}_n = \sum_{\nu} (k_{n,\nu} - k_{n-\hat{\nu},\nu}) = 0 \quad \forall n$$
Interpretation of the condensation steps

Grand canonical partition sum

\[ Z = \text{tr} e^{-\beta (H - \mu \hat{Q})} = e^{-\beta \Omega(\mu)} \]

Low T: Z will be governed by the minimal grand potential \( \Omega(\mu) \) in each particle sector

\[
\Omega(\mu) \xrightarrow{T \to 0} \begin{cases}
\Omega_{\text{min}}^{N=0} = 0, & \mu \in [0, \mu_c^1) \\
\Omega_{\text{min}}^{N=1} = W^{(1)} - 1\mu, & \mu \in (\mu_c^1, \mu_c^2) \\
\Omega_{\text{min}}^{N=2} = W^{(2)} - 2\mu, & \mu \in (\mu_c^2, \mu_c^3) \\
\vdots,
\end{cases}
\]

with renormalized mass \( W^{(1)} \equiv m \), minimal 2-particle energy \( W^{(2)} \), …