$N\pi$ - state contamination in lattice calculations of the axial form factors of the nucleon

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Introduction

- Progress over the last years: Physical point simulations have become feasible

- Advantage
  No chiral extrapolation needed, i.e. one systematic error eliminated

- Problems
  - Numerically demanding
    need large volumes too
  - Signal-to-noise problem
  - Significant impact in correlation functions of multi-particle-states involving light pions

- ChPT can be used to estimate this multi-particle-state contamination
  - Nucleon axial, scalar, tensor charge; pdf moments

[OB, Lattice 2017]
$N\pi$ contamination in axial form factors of the nucleon

- In the following:
  \( N\pi \) contamination in axial form factors \( G_A(Q^2) \) and \( G_P(Q^2) \) of the nucleon

- Expectations: \( N\pi \) contamination is
  - in \( G_A(Q^2) \) of same order as in \( g_A = G_A(0) \)
  - significantly larger in \( G_P(Q^2) \)

- Calculational setup is the same as for \( N\pi \) contamination in axial charge \( g_A \)
  \( \rightarrow \) OB, Phys. Rev. D 94 (2016) 054505
The axial form factors $G_A(Q^2)$ and $G_P(Q^2)$

Matrix element of local isovector axial vector current
isospin symmetry assumed

$$
\langle N(p')|A^a_\mu(0)|N(p)\rangle = \bar{u}(p') \left( \gamma_\mu \gamma_5 G_A(Q^2) - i\gamma_5 \frac{Q_\mu}{2M_N} G_P(Q^2) \right) \frac{\sigma^a}{2} u(p)
$$

- axial ff
- induced pseudo scalar ff

Momentum transfer
- $Q_\mu = (iE_{\vec{p}'} - iE_{\vec{p}}, \vec{q})$
- $\vec{q} = \vec{p}' - \vec{p}$
- $\vec{p}' = 0$

Euclidean space time

chosen here
Lattice determination

Standard procedure:

- **Compute 3-pt function**
  \[ C_{3,A^3_\mu}(\vec{q}, t, t') = \sum_{\vec{x},\vec{y}} e^{i\vec{q}\cdot\vec{y}} \Gamma_{\beta\alpha} \langle N_\alpha(\vec{x}, t) A^3_\mu(\vec{y}, t') \bar{N}_\beta(0, 0) \rangle \]

  Axial vector current at \( t' \)
  Nucleon interpolating fields at \( t, 0 \)
  Projector \( \Gamma \)

- **Ratio with 2-pt function**
  \[ R_\mu(\vec{q}, t, t) = \frac{C_{3,A^3_\mu}(\vec{q}, t, t')}{C_2(0, t)} \sqrt{\frac{C_2(\vec{q}, t - t')}{C_2(0, t - t')}} \frac{C_2(\vec{0}, t)}{C_2(\vec{0}, t')} \frac{C_2(0, t)}{C_2(\vec{q}, t') C_2(\vec{q}, t)} \]

- **Consider asymptotically large time separations:** \( t, t', t-t' \rightarrow \infty \)

  \[ R_k(\vec{q}, t, t') \rightarrow \Pi_k(\vec{q}) = \frac{i}{\sqrt{2E_{N,\vec{q}}(M_N + E_{N,\vec{q}})}} \left( (M_N + E_{N,\vec{q}})G_A(Q^2)\delta_{3k} - \frac{G_P(Q^2)}{2M_N}q_3q_k \right) \]

- **Solve a linear system and extract the form factors**
  \[ \Pi_k(\vec{q}) \rightarrow G_A(Q^2), G_P(Q^2) \]
**Lattice determination**

- In practice: finite time separations $t$ and $t'$

\[ R_k(\vec{q}, t, t') \rightarrow G_{A}^{\text{eff}}(Q^2, t, t'), \ G_{P}^{\text{eff}}(Q^2, t, t') \]

- The effective form factors contain excited-state contributions and depend on $t, t'$

\[ G_{A,P}^{\text{eff}}(Q^2, t, t') = G_{A,P}(Q^2) \left[ 1 + \Delta G_{A,P}(Q^2, t, t') \right] \]

- Dominant excited state for physical pion mass and large time separations:

  2-particle $N\pi$ states
ChPT including nucleons

- SU(2) ChPT at LO
  - isospin symmetry, euclidean space time

contains the three pions and the nucleon doublet \( \Psi = \left( \begin{array}{c} p \\ n \end{array} \right) \)

\[
\mathcal{L}_{\text{int},1\pi}^{(1)} = \frac{ig_A}{2f} \bar{\Psi} \gamma_\mu \gamma_5 \sigma^a \Psi \partial_\mu \pi^a
\]

- \( g_A \) axial charge
- \( f \) pion decay constant

- Low energy constants at this order: \( g_A, f, M_N, M_\pi \)
  - experimentally well-known

- Also known: chiral expressions for
  - axial vector current
    - Gasser, Sainio, Švarc 1988, Fettes et al 2000
  - nucleon interpolating fields (local and smeared)
    - Nagata et al 2008; Wein, Bruns, Hemmert, Schäfer 2011; OB 2015
$N\pi$ contribution to the form factors

- To do: Compute 2-pt and 3-pt functions and the ratio $R_k$ in ChPT
- Example: Feynman diagrams for the 3-pt function

![Feynman diagrams for the 3-pt function](image)

Note: Tree diagrams are expected to give large $N\pi$ contribution
vanish for $Q^2 = 0 \rightarrow$ do not contribute to $g_A$

- Status: Leading contribution in $1/M_N$-expansion computed
- Work in progress: $1/M_N$-correction
$N\pi$ contribution to the form factors

Consider plateau estimates* for the form factors

$$G_{A}^{\text{plat}}(Q^2, t) = \min_{0 < t' < t} G_{A}^{\text{eff}}(Q^2, t, t')$$  \quad \text{N}\pi \text{ contamination leads to overestimation in } G_A$$

$$G_{P}^{\text{plat}}(Q^2, t) = \max_{0 < t' < t} G_{P}^{\text{eff}}(Q^2, t, t')$$  \quad \text{N}\pi \text{ contamination leads to underestimation in } G_P$$

Comment: $t' \approx t/2$. Midpoint estimate is equally good

ChPT requires $t \geq 2\text{fm}$ such that $t$ and $t-t' \geq 1\text{fm}$
(Experience from nucleon charges and pdf moments)

ChPT calculation in finite spatial volume, box length $L$, periodic BC
$\rightarrow$ discrete momenta $\vec{q}_n$ and momentum transfer $Q_n^2$

* I have nothing to say about the summation method
Overestimates by $\approx 5\%$ (no visible $Q^2$ dependence)

$G_{A}^{\text{plat}} \left( Q^2, t = 2 \text{fm} \right) / G_{A} \left( Q^2 \right) - 1$

$G_{P}^{\text{plat}} \left( Q^2, t = 2 \text{fm} \right) / G_{P} \left( Q^2 \right) - 1$

Underestimates by $\approx 10\% - 40\%$ depending on momentum transfer

$Q_n^2 / \text{GeV}^2$

$M_\pi L = 6$ (e.g. PACS coll.)
Results for $t = 2 \text{ fm}$

\[
\frac{G_A^{\text{plat}}(Q^2, t = 2\text{fm})}{G_A(Q^2)} - 1
\]

\[
\frac{G_P^{\text{plat}}(Q^2, t = 2\text{fm})}{G_P(Q^2)} - 1
\]

- $G_A^{\text{plat}}$ overestimates by $\approx 5\%$ (no visible $Q^2$ dependence)
- $G_A^{\text{plat}}$ agrees with result for $g_A$ in previous calculation
- $G_P^{\text{plat}}$ underestimates by $\approx 10\% - 40\%$ depending on momentum transfer
- Small FV effect for $M_\pi L \geq 3$

\[Q_n^2/\text{GeV}^2\]
Results
for $t = 2$ fm

$G_A^{\text{plat}}(Q^2, t = 2\text{fm}) - 1$

$G_P^{\text{plat}}(Q^2, t = 2\text{fm}) - 1$

- $G_A^{\text{plat}}$ overestimates by $\approx 5\%$ (no visible $Q^2$ dependence)
  - agrees with result for $g_A$ in previous calculation

- $G_P^{\text{plat}}$ underestimates by $\approx 10\% - 40\%$ depending on momentum transfer

- Small FV effect for $M_\pi L \geq 3$

- Increasing $t$ to 3 fm reduces $N\pi$ contribution roughly by a factor $1/2$
Observation 1

For some momentum transfer the extraction of the eff. form factors

\[ R_k(\vec{q}, t, t') \rightarrow G_A^{\text{eff}}(Q^2, t, t') , \ G_P^{\text{eff}}(Q^2, t, t') \]

can be done in various ways.

Example:
\[ \vec{q}_A = \frac{2\pi}{L} (1, 0, 1) \quad \vec{q}_B = \frac{2\pi}{L} (1, 1, 0) \]

lead to the same \( Q^2 \)

Extract effective form factors using

1. \( R_3(\vec{q}_A, t, t') \quad R_3(\vec{q}_B, t, t') \)
2. \( R_1(\vec{q}_A, t, t') \quad R_3(\vec{q}_B, t, t') \)
3. \( R_1(\vec{q}_A, t, t') \quad R_3(\vec{q}_A, t, t') \)

The three choices give practically the same effective form factors!

Deviations of \( O(10^{-4}) \)
Observation 2

Recall:

\[ G_{P}^{\text{eff}}(Q^2, t, t') = G_{P}(Q^2) \left[ 1 + \Delta G_{P}^{N\pi}(Q^2, t, t') \right] \]

Observation: The loop diagram contribution to \( \Delta G_{P}^{N\pi} \) is tiny
\( \Delta G_{P}^{N\pi} \) is dominated by one tree diagram!

Consequences

- ChPT is expected to work better for \( \Delta G_{P}^{N\pi} \) than for \( \Delta G_{A}^{N\pi} \)
  (no tower of narrowly spaced \( N\pi \) states)
  \( \Rightarrow \) result expected to be reliable for time separations much less than 2 fm (?)

- Excellent approximation:

\[ \Delta G_{P}^{N\pi}(q, t, t' = t/2) \approx - \exp \left[ -E_{\pi,q} \frac{t}{2} \right] \cosh \left[ \frac{q^2}{2M_N} \frac{t}{2} \right] \quad \xrightarrow{q \to 0} \quad -e^{-M_{\pi} \frac{t}{2}} \]

Note: Determined mainly by \( M_{\pi} \) and the source-sink separation \( t \)
Results

$N\pi$ contribution for small $Q^2$ is governed by

$$-e^{-M_{\pi}^2 \frac{t}{2}}$$

$$\frac{G_A^{\text{plat}}(Q^2, t)}{G_A(Q^2)} - 1$$

$$\frac{G_P^{\text{plat}}(Q^2, t)}{G_P(Q^2)} - 1$$

$\Delta G_P^{N\pi}$ approx.

$Q_n^2/\text{GeV}^2$
Impact on lattice calculations of $G_{P}(Q^{2})$

- Data underestimate experimental results and pion pole dominance model

Recent preprint by PACS coll.
arXiv:1807.03974

- $a \approx 0.085 \text{ fm}$
- $M_{\pi} \approx 146 \text{ MeV}$
- $M_{\pi}L \approx 6$
- $t \approx 1.3 \text{ fm}$

Data underestimate experimental results and pion pole dominance model
Impact on lattice calculations of $G_P(Q^2)$

- Data underestimate experimental results and pion pole dominance model
- Corrected data agree much better with pion pole dominance model
  
  To do: Check for various $t$, continuum limit, …

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Remove $N\pi$ contamination from the data by

$$G_P(Q^2) = \frac{G_P^{\text{plat}}(Q^2, t)}{1 + \epsilon_P^{\text{app}}(Q^2, t)}$$
Summary and outlook

- Presented here: LO ChPT results for the $N\pi$ excited-state contamination in the plateau estimates for the axial form factors of the nucleon
  - Overestimation for $G_A$, flat $Q^2$ dependence
  - Underestimation for $G_P$, strong $Q^2$ dependence for low $Q^2$
    ➡️ can qualitatively explain the distortion observed in lattice data for $G_P$

- Outlook: Analogous calculation for
  - pseudo scalar form factor
  - form factors for the vector current
Backup slides
Impact on lattice calculations of $G_P(Q^2)$

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Remove $N\pi$ contamination from the data by

$$G_P(Q^2) = \frac{G_P^{\text{plat}}(Q^2, t)}{1 + \epsilon_P^{\text{app}}(Q^2, t)}$$

Pole ansatz by PACS:

$$G_P(Q^2) = \frac{4M_N^2 G_A(Q^2)}{Q^2 + M^2_{\text{pole}}}$$

Fit to data:

$M_{\text{pole}} = 256(17)$ MeV
\( \epsilon_A(Q_2^2, t, t' = t/2), \epsilon_P(Q_2^2, t, t' = t/2) \)

\[ M_\pi L = 4 \]
\[ n_q = 2 \]
\[ n_{p,\text{max}} = 1 \ldots 12 \]

“tower of \( N\pi \) states”

\( \epsilon_P \) essentially independent of \( n_{p,\text{max}} \)
Tree contribution dominates \( \epsilon_P \)
$N\pi$ contamination in the correlation functions

3-pt function:

\[ C_{3,\mu}(\vec{q}, t, t') = C_{3,\mu}^{N}(\vec{q}, t, t') + C_{3,\mu}^{N\pi}(\vec{q}, t, t') \]

\[ = C_{3,\mu}^{N}(\vec{q}, t, t') \left( 1 + Z_{\mu}(\vec{q}, t, t') \right) \]

2-pt function: analogously

Ratios:

\[ R_{\mu}(\vec{q}, t, t') = \Pi_{\mu}(\vec{q}) \left( 1 + Z_{\mu}(\vec{q}, t, t') + \frac{1}{2} Y(\vec{q}, t, t') \right) \]

computable in ChPT

from 2-pt functions
\[ N\pi \text{ contamination in the correlation functions} \]

\[
Z_{\mu}(\vec{q}, t, t') = a_{\mu}(\vec{q})e^{-\Delta E(0, \vec{q})(t-t')} + \tilde{a}_{\mu}(\vec{q})e^{-\Delta E(\vec{q}, -\vec{q})t'}
\]

\[
+ \sum_{\vec{p}} b_{\mu}(\vec{q}, \vec{p})e^{-\Delta E(0, \vec{p})(t-t')} + \tilde{b}_{\mu}(\vec{q}, \vec{p})e^{-\Delta E(\vec{q}, \vec{p})t'}
\]

\[
+ \sum_{\vec{p}} c_{\mu}(\vec{q}, \vec{p})e^{-\Delta E(0, \vec{p})(t-t')}e^{-\Delta E(\vec{q}, \vec{p})t'}
\]

Energy gaps:

\[
\Delta E(0, \vec{q}) = E_{\pi, \vec{q}} + E_{N, q} - M_N
\]

\[
\Delta E(0, \vec{p}) = E_{\pi, \vec{p}} + E_{N, p} - M_N
\]

\[
\Delta E(\vec{q}, -\vec{q}) = E_{\pi, \vec{q}} + M - E_{N, q}
\]

Non-trivial results of the ChPT calculation: The coefficients in \( Z_\mu \)
\( N\pi \) contamination in the correlation functions

Example: Coefficients \( a_k \) from the tree-level diagrams

\[
a_k(\vec{q}) = a_k^\infty(\vec{q}) + \frac{E_{\pi,q}}{M_N} a_k^{\text{corr}}(\vec{q}) + O\left(\frac{1}{M_N^2}\right)
\]

NR Limit:

\[
a_k^\infty(\vec{q}) = \begin{cases} 
-\frac{1}{2} & \text{for } k = 1,2 \\
\frac{1}{2} \frac{q_3^2}{E_{\pi,q}^2 - q_3^2} & \text{for } k = 3
\end{cases}
\]

Relevant result for approximate \( \Delta G_{P}^{N\pi} \)

Correction:

\[
a_{k=1,2}^{\text{corr}}(\vec{q}) = -\frac{1}{4} \left( \frac{M_{\pi}^2}{E_{\pi,q}^2} - \frac{1}{g_A} \right) \\
a_{k=3}^{\text{corr}}(\vec{q}) = \frac{1}{4} \left( \frac{M_{\pi}^2}{E_{\pi,q}^2} - \frac{1}{g_A} \right) \frac{q_3^2}{E_{\pi,q}^2 - q_3^2}
\]
ChPT: Single nucleon contribution

\[ G_A(Q^2) = g_A \]
\[ G_P(Q^2) = 4M_N^2 \frac{g_A}{Q^2 + M_\pi^2} \]