Progress on relativistic three-particle quantization condition

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Outline

• Motivation
• Status
• Completing the formalism: including resonant subchannels
• Numerical results from the isotropic approximation
• Numerical results including higher partial waves
Motivation

• Studying resonances with three particle decay channels
  • $\omega(782, I^GJ^{PC} = 0^{-1}--) \to 3\pi$  (no resonant subchannels)
  • $a_2(1320, I^GJ^{PC} = 1^{-2}++) \to \rho\pi \to 3\pi$
  • $N(1440) \to \Delta\pi \to N\pi\pi$
  • $X(3872) \to J/\Psi\pi\pi$

• Calculating weak decay amplitudes involving 3 or more particles, e.g. $K \to 3\pi$, $D \to 2\pi, 4\pi, \ldots$

• Determining NNN interactions
Methodology & Status

2 & 3 particle spectrum from LQCD

Quantization conditions
\[ \det [F_2^{-1} + \mathcal{K}_2] \]
\[ \det [F_3^{-1} + \mathcal{K}_{df,3}] \]

Intermediate scattering quantities

Integral equations in infinite volume

Scattering amplitudes
\[ \mathcal{M}_2, \mathcal{M}_3, \mathcal{M}_{23}, \ldots \]
Methodology & Status

Quantization conditions
\[
\det \left( F_2^{-1} + \mathcal{K}_2 \right)
\]
\[
\det \left( F_3^{-1} + \mathcal{K}_{df,3} \right)
\]

Intermediate scattering quantities

Three approaches
- Relativistic [Briceño, Hansen, SRS]
- NREFT [Hammer, Pang, Rusetsky]
- Finite-volume Khuri-Treiman [Döring, Mai]

Each have pros and cons
- Intermediate scattering quantities differ
- All require partial-wave truncation
- Similar challenges for numerical implementation

Integral equations in infinite volume
Status of relativistic approach

- Original work applied to scalars with G-parity & no subchannel resonances [Hansen, SRS: 1408.5933 & 1504.04248]

\[
\det \left[ F_3^{-1} + \mathcal{H}_{df,3} \right]
\]

- Second major step: removing G-parity constraint, allowing 2↔3 processes [Briceño, Hansen, SRS: 1701.07465]

\[
\det \begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix}^{-1} + \begin{pmatrix} \mathcal{H}_{22} & \mathcal{H}_{23} \\ \mathcal{H}_{32} & \mathcal{H}_{df,33} \end{pmatrix} = 0
\]
Completing the formalism

- Second major step: removing G-parity constraint, allowing $2 \leftrightarrow 3$ processes [Briceño, Hansen, SRS: 1701.07465]

\[
\det \left[ \begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix}^{-1} + \begin{pmatrix} \mathcal{H}_{22} & \mathcal{H}_{23} \\ \mathcal{H}_{32} & \mathcal{H}_{df,33} \end{pmatrix} \right] = 0
\]

- Final major step: allowing subchannel resonance (i.e. pole in $\mathcal{K}_2$) [Briceño, Hansen, SRS: 1808.XXXXX]

\[
\det \left[ \begin{pmatrix} F_{\tilde{2}\tilde{2}} & F_{\tilde{2}3} \\ F_{3\tilde{2}} & F_{33} \end{pmatrix}^{-1} + \begin{pmatrix} \mathcal{K}_{df,2\tilde{2}} & \mathcal{K}_{df,23} \\ \mathcal{K}_{df,3\tilde{2}} & \mathcal{K}_{df,33} \end{pmatrix} \right] = 0
\]
Completing the formalism

- **Second major step:** removing G-parity constraint, allowing $2 \leftrightarrow 3$ processes \[\text{Briceño, Hansen, SRS: 1701.07465}\]

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det \left[ \begin{pmatrix} F_2 & 0 \\ 0 & F_3 \end{pmatrix}^{-1} + \begin{pmatrix} \mathcal{K}_{22} & \mathcal{K}_{23} \\ \mathcal{K}_{32} & \mathcal{K}_{df,33} \end{pmatrix} \right] = 0
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- **Final major step:** allowing subchannel resonance (i.e. pole in $\mathcal{K}_2$) \[\text{Briceño, Hansen, SRS: 1808.XXXXX}\]

\[
det \left[ \begin{pmatrix} F_{\tilde{2}\tilde{2}} & F_{\tilde{2}3} \\ F_{3\tilde{2}} & F_{33} \end{pmatrix}^{-1} + \begin{pmatrix} \mathcal{K}_{df,\tilde{2}\tilde{2}} & \mathcal{K}_{df,\tilde{2}3} \\ \mathcal{K}_{df,3\tilde{2}} & \mathcal{K}_{df,33} \end{pmatrix} \right] = 0
\]

Determined by $K_2$ & Lüscher finite-volume zeta functions

Infinite-volume quantities related to $M_2$ & $M_3$ by known integral equations

resonance + particle channel (not physical)
Formalism to-do list

- Multiple poles in $K_2$
- Nondegenerate particles with spin
- Connecting formalism for resonances to that for stable particles (e.g. raising $m_q$ stabilizes $\rho$)
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• Multiple poles in $K_2$
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All are straightforward!
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Isotropic low-energy approximation

[Briceño, Hansen & SRS, 1803.04169]

• Scalar particles with G parity so no $2 \leftrightarrow 3$ transitions and no subchannel resonances (e.g. $3 \pi^+$)

• 2-particle interactions are purely s-wave, and determined by the scattering length alone (which can be arbitrarily negative, $a \to -\infty$)

• Point-like three-particle interaction $K_{df,3}$, independent of momenta

• Reduces problem to 1-dim. quantization condition, although intermediate matrices involve finite-volume momenta up to cutoff $|k| \sim m$

• Analog in our formalism of the approximations used in other approaches: [Hammer, Pang, Rusetsky, 1706.07700; Mai & Döring, 1709.08222; Döring et al., 1802.03362; Mai & Döring, 1807.04746]
Impact of $K_{df,3}$ on spectrum

$am = -10$ (strong attractive interaction)

Local 3-particle interaction has significant effect on energies, especially in region of simulations ($mL < 5$), and thus can be determined
Volume-dependence of 3-body bound state

\[ a_m = -10^4 \quad \text{and} \quad m^2 K_{df,3}^{iso} = 2500 \quad \text{(unitary regime)} \]

**FIG. 6.** Finite-volume energy dependence for the bound state that arises for \( m^2 K_{iso}^{df,3} = 2500 \) and \( a_m = 10^{-4} \). In all three figures the solutions to the quantization condition are marked in orange, as points in (a) and (b) and as the curved solid line in (c). The curving (turquoise) line in panel (a) is a fit of Eq. (35) (neglecting the higher-order corrections) to the data in this panel. The same fit line is shown in panel (b) for lower values of \( mL \), along with a horizontal, solid (red) line showing the infinite-volume energy of the bound state \( E_B(\infty) \). The horizontal dashed (black) line shows the threshold energy \( E = 3m \).

Panel (c) displays \( E_B(L) \) for smaller \( mL \), along with the same two horizontal lines as in (b) and the asymptotic prediction.

**D. Volume-dependence of the threshold-state energy**

In this section we investigate in detail the energy of the threshold state. We have already shown examples of this energy for various values of \( a \) in Fig. 3, and our aim here is to provide a detailed comparison with the predicted large-volume behavior. The analytic prediction is

\[
E_B(L) = c_3 L^3 + c_4 L^4 + c_5 L^5 + \tilde{c}_6 L^6 + O(L^7),
\]

(36)

Need quantization condition to determine finite-volume effects for realistic values of \( mL \).
Bound state wave-function

- Work in unitary regime \((ma=-10^4)\) and tune \(\mathcal{K}_{df,3}\) so 3-body bound state at \(E_B=2.98858\) m

- Solve integral equations numerically to determine \(\mathcal{M}_{df,3}\) from \(\mathcal{K}_{df,3}\)

- Determine wavefunction from residue at bound-state pole

\[
\mathcal{M}^{(u,u)}_{df,3}(k,p) \sim -\frac{\Gamma^{(u)}(k)\Gamma^{(u)}(p)}{E^2 - E_B^2}
\]

- Compare to analytic prediction from NRQM in unitary limit \([\text{Hansen & SRS, 1609.04317}]\)

\[
|\Gamma^{(u)}(k)_{NR}|^2 = |c||A|^2 \frac{256\pi^{5/2}}{3^{1/4}} \frac{m^2\kappa^2}{k^2(\kappa^2 + 3k^2/4)} \frac{\sin^2 \left( s_0 \sin^{-1} \frac{\sqrt{3}k}{2\kappa} \right)}{\sinh^2 \frac{\pi s_0}{2}}
\]

Known constant

Determined by fit to volume-dependence of bound-state energy

Known constant
Bound state wave-function

\[
|\Gamma^{(u)}(k)_{NR}|^2 = |c||A|^2 \frac{256\pi^{5/2}}{3^{1/4}} \frac{m^2 \kappa^2}{k^2 (\kappa^2 + 3k^2/4)} \sin^2 \left( s_0 \sinh^{-1} \frac{\sqrt{3}k}{2\kappa} \right) \frac{\sinh^2 \frac{\pi s_0}{2}}{\sinh^2 \frac{\pi s_0}{2}}
\]

Works over many orders of magnitude to expected accuracy
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Beyond the isotropic approximation

[Tyler Blanton, Fernando Romero-Lopez & SRS, in progress]

- In 2-particle case, assume s-wave dominance at low energies, then systematically add in higher waves (suppressed by $q^{2l}$)

- We are implementing the same general approach for $\mathcal{K}_{df,3}$, making use of the facts that it is relativistically invariant and completely symmetric under initial- & final-state permutations, and expanding about threshold

- We work in the G-parity invariant theory with 3 identical scalars, so the first channel beyond s-wave has $l=2$ (d-wave)
Beyond the isotropic approximation

\[ \mathcal{K}_{df,3} = \mathcal{K}_{iso}^{df,3}(E) + c_A \mathcal{K}_{3A} + c_B \mathcal{K}_{3B} + \mathcal{O}(\Delta^3) \]

\[ \mathcal{K}_{iso}^{df,3} = c_0 + c_1 \Delta + c_2 \Delta^2 \]

\[ \mathcal{K}_{3A} = \sum_{i=1}^{3} (\Delta_i^2 + \Delta_i')^2 \]

\[ \mathcal{K}_{3B} = \sum_{i,j=1}^{3} t_{ij}^2 \]

\[ \Delta = s - 9m^2 \]

\[ \Delta_1 = (p_2 + p_3)^2 - 4m^2 \text{ etc.} \]

\[ \Delta_1' = (p_2' + p_3')^2 - 4m^2 \text{ etc.} \]

\[ t_{ij} = (p_i - p_j)^2 \]

\[ c_0 \text{ is the leading term—only term kept in isotropic approx} \]

\[ c_1 \text{ is coefficient of the only linear term} \]

Only three coefficients needed at quadratic order:

\[ c_2, c_A \text{ & } c_B \]

Many fewer than the 7 angular variables + s dependence present at arbitrary energy!
Decomposing into spectator/dimer basis

\[ \mathcal{K}_{df,3} \]

\{ \begin{align*}
    p'_1 & \quad l',m' \\
    p'_2 & \\
    p'_3 & \\
\end{align*} \}

Decompose into harmonics in dimer CM frame: \( l,m \)

spectator momentum

\[ \mathcal{K}_{3A}, \mathcal{K}_{3B} \Rightarrow l'=0,2 \& l=0,2 \]

For consistency, need \( \mathcal{K}_{2(0)} \sim 1+q^2+q^4 \& \mathcal{K}_{2(2)} \sim q^4 \)

\[
\frac{1}{\mathcal{K}_{2(0)}^{(0)}} = \frac{1}{16\pi E_2} \left[ \frac{1}{a_0} + r_0 \frac{q^2}{2} + P_0 r_0^3 q^4 \right] \\
\frac{1}{\mathcal{K}_{2(2)}^{(2)}} = \frac{1}{16\pi E_2} \frac{1}{q^4} \frac{1}{a_5^2} 
\]

Implemented quantization condition through quadratic order, for \( P=0 \), including projection onto overall cubic group irreps
First results including \( l=2 \)

\[
\mathcal{H}_{\text{df},3} = 0, \ a_0 = -10, \ r_0 = 0.5, \ P_0 = 0.5, \ -1.5 \leq a_2 \leq 0.1
\]

More in progress!
First results including $l=2$

$$\mathcal{H}_{df,3} = 0, \ a_0 = -10, \ r_0 = 0.5, \ P_0 = 0.5, \ -1.5 \leq a_2 \leq 0.1$$

Spectrum for mL=5

More in progress!
Any questions?