Constraint HMC Algorithms for gauge-Higgs models

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Gauge-Higgs Unification

- Origin of the potential responsible for the Brout-Englert-Higgs mechanism is unknown.
- In Gauge-Higgs Unification (GHU) models the Higgs field is associated with some extra-dimensional components of a gauge field [Manton, 1979].
- In the case of one extra dimension the Higgs potential is zero at tree level and is generated only through quantum effects.
- What happens non-perturbatively? → lattice study.
- Measure the effective potential with respect to the Higgs field using constraint HMC algorithms.
Wilson Gauge action for bulk gauge group $SU(2)$:

$$S_{W}^{orb} = \frac{\beta_4}{2} \sum_{P_4} w \cdot \text{tr} \{1 - P_4\} + \frac{\beta_5}{2} \sum_{P_5} \text{tr} \{1 - P_5\}$$

Boundary links satisfy $gUg^{-1} = U$ with $g = -i\sigma^3$

$$w = \begin{cases} \frac{1}{2} & \text{plaquette on boundary} \\ 1 & \text{otherwise} \end{cases}$$

**5D Orbifold**

- bare anisotropy is $\gamma = \sqrt{\beta_5/\beta_4} \simeq a_4/a_5$
- stick-symmetry $g_s = -i(\cos \theta \sigma_1 + \sin \theta \sigma_2)$
- $N_5 a_5 = \pi R$, $R$ radius of extra dimension.
5D Orbifold Model Higgs operators

Scalar Polyakov loop (defined at \( n = (n_\mu, 0) \))

\[ p(n) = l(n) g \, l^\dagger(n) \, g^\dagger \]

Higgs field

\[ h(n) = [p(n) - p^\dagger(n), g]/(4N_5) \sim A_5^1 \sigma^1 + A_5^2 \sigma^2 \]

Higgs operators

\[ \mathcal{H}(n_0) = \sum_{n_1,n_2,n_3} \text{tr} [hh^\dagger] \quad , \quad P(n_0) = \sum_{n_1,n_2,n_3} \text{tr} [p] \]
5D Orbifold Model Phase Diagram

**Notes**

On the torus

- confined and Coulomb phase.

On the orbifold

- One more phase: $U(1)$ gauge links deconfine separately.
- No compactification observed at $\gamma > 1$.
- Interesting physics is found at $\gamma < 1$.

**DR to 4D Abelian Higgs** [Alberti, Irge, Knechtli and Moir, 1506.06035]
Non-perturbative Gauge-Higgs Unification

- $SU(2)$ pure gauge theory on a 5D orbifold has a Higgs phase with the Higgs mechanism realized as a quantum and bosonic effect.
- A Standard-Model like spectrum can be reproduced.
- Localization on the 4D boundaries is observed.
- Cut-off effects appear to be small. Excited state energies are $2-5$ times larger than the ground states.
- Review on lattice works on extra dimensions
  

- ...
4D Abelian Higgs Unitary Gauge

\[ \phi(x) = \rho(x) \exp i\varphi(x) \rightarrow \phi_1 = \rho \cos \varphi, \ \phi_2 = \rho \sin \varphi \]

\[ S_\rho[V, \rho] = \sum_x \rho_x^2 + \lambda(\rho_x^2 - 1)^2 \]

\[ -2K \rho_x \sum_\mu \rho_{x+\mu} \text{tr} \left( e^{-i\varphi_x} U_{x,\mu} e^{i\varphi_{x+\mu}} \right) = V_{x,\mu} \]

\[ H[V, \rho] = S_\rho[V, \rho] + \frac{1}{2} \sum_x \pi(x)^2 + \mu \left( \frac{1}{\Omega} \sum_x \rho(x) - \Phi \right) \]

constraint EOMs: [Fodor, Holland, Kuti, Nogradi, Schroeder, 0710.3151]
4D Abelian Higgs Constraint HMC

time derivatives of the constraint to ensure it’s unchanged

\[ \frac{\partial}{\partial t} \left( \frac{1}{\Omega} \sum_x \rho(x) - \Phi \right) = \sum_x \dot{\rho}(x) = \sum_x \pi(x) = 0 \]

\[ \frac{\partial}{\partial t} \sum_x \pi(x) = \sum_x \dot{\pi}(x) = 0 \Rightarrow \mu = -\sum_x \frac{\partial S_{\rho}}{\partial \rho(x)} \]

to measure the effective potential we use the derivative with respect to

\[ \frac{d}{d\Phi} = \frac{1}{\Omega} \sum_x \frac{d \rho_x}{d\Phi} \frac{d}{d\rho_x} \text{ of } S_{\rho}[V, \rho] \]

\[ \frac{dU_{\Omega}}{d\Phi} = 2\Phi + 4\lambda \left\langle \frac{1}{\Omega} \sum_x (\rho(x)^2 - 1)\rho(x) \right\rangle_{\Phi} \]

\[ + 2\kappa \left\langle \frac{1}{\Omega} \sum_{x,\mu} (\rho(x) + \rho(x + \hat{\mu}))V_{\mu}(x) \right\rangle_{\Phi} \]
4D Abelian Higgs Effective Potential

\[ \frac{dU_\Omega}{d\Phi} \approx 2c_1 \Phi + 3c_2 \Phi^2 + 4c_3 \Phi^3 \]

[ Irges, Koutroulis, 1703.10369 ]
first step towards 5D orbifold model we work on the torus

\[ S_{tor}^W = \frac{\beta_4}{2} \sum_{P_4} \text{tr} \{1 - P_4\} + \frac{\beta_5}{2} \sum_{P_5} \text{tr} \{1 - P_5\} \]

to simplify the Higgs operator (Polyakov loop) we use the axial gauge along the fifth dimension \[ V_5 = \prod_{n=0}^{N_5} [U_5(x, n)] \]

\[
\begin{align*}
H[U] &= S_{tor}^W[U] + \frac{1}{2} \sum_{x, \mu} \text{tr} \left[ \pi^2_\mu(x) \right] + \lambda \left( \frac{1}{\Omega} \sum_x \text{tr} V_5(x) - \Phi \right) \\
\dot{V}_5(x) &= \pi_5(x) V_5(x) \\
\dot{\pi}_5(x) &= -\frac{\partial S[V_5]}{\partial V_5(x)} - \frac{\lambda}{4\Omega} \text{tr} \left[ \sigma_i V_5(x) \right] \sigma^i 
\end{align*}
\]
time derivatives of the constraint to ensure it’s unchanged

\[
\frac{\partial}{\partial t} \left( \frac{1}{\Omega} \sum V_5 - \Phi \right) = \sum \dot{V}_5 = \sum \text{tr} \pi_5 V_5
\]

\[
\frac{\partial}{\partial t} \sum \text{tr} \pi_5 V_5 = \sum \text{tr} \dot{\pi}_5 V_5 + \sum \text{tr} \pi_5^2 V_5
\]

\[
\frac{\lambda}{4\Omega} = \left( \sum \text{tr} \pi_5^2 V_5 - \sum \text{tr} \frac{\partial S[V_5]}{\partial V_5} V_5 \right) / \sum \text{tr} \{ \text{tr} [\sigma_i V_5] \sigma^i V_5 \}
\]

\[
\Rightarrow \sum \text{tr} \pi_5^2 V_5 \text{ makes the Hamiltonian non-separable}
\]

\[
H(U, \pi) \neq H_1(\pi) + H_2(U)
\]

as a consequence we cannot use standard leap-frog, and need to work with a new symplectic algorithm
Rattle Integration Scheme

\[
\begin{align*}
\pi_{n+1/2} &= \pi_n - \frac{h}{2} \left( \frac{\partial S}{\partial V_n} + \frac{\lambda}{4\Omega} \text{tr} [\sigma_i V_n] \sigma^i \right) \\
V_{n+1} &= e^{h\pi_{n+1/2}} V_n, \quad 0 = \frac{1}{\Omega} \sum \text{tr} V_{n+1} - \nu \\
\pi_{n+1} &= \pi_{n+1/2} - \frac{h}{2} \left( \frac{\partial S}{\partial V_{n+1}} + \frac{\mu}{4\Omega} \text{tr} [\sigma_i V_{n+1}] \sigma^i \right) \\
0 &= \frac{1}{\Omega} \sum \text{tr} \{ \text{tr} [\sigma_i V_{n+1}] \sigma^i \pi_{n+1} \} = -\frac{2}{\Omega} \sum \text{tr} V_{n+1} \pi_{n+1}
\end{align*}
\]

The first three equations determine \((\pi_{n+1/2}, V_{n+1}, \lambda)\), whereas the remaining two give \((\pi_{n+1}, \mu)\)...

\[
\begin{align*}
\frac{\mu}{4\Omega} &= \left( 2 \sum \text{tr} \pi_{n+1/2} V_{n+1}/h - \sum \text{tr} \frac{\partial S}{\partial V_{n+1}} V_{n+1} \right) / \sum \text{tr} \{ \text{tr} [\sigma_i V_{n+1}] \sigma^i V_{n+1} \}
\end{align*}
\]
Effective Higgs Potential of 5D Torus Model - no SSB!

evaluated Jacobian \[ J = \frac{\partial (V_{n+1}, \pi_{n+1})}{\partial (V_n, \pi_n)} \Rightarrow \text{det} J = 1! \]
Conclusions

Constraint HMC Algorithms for gauge-Higgs models

- implemented the constraint HMC for 4D Abelian gauge-Higgs model and computed the effective Higgs potential in the spontaneously broken phase
- implemented a symplectic constraint HMC algorithm for 5D torus and found no SSB in accordance with the absence of stick symmetry

Outlook

- test constraint HMC algorithm for 5D orbifold
- study connection to 4D Abelian Higgs model via effective potential derived from constraint HMCs
- other applications: finite temperature QCD