Scattering length from BS wave function inside the interaction range


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Main messages

- Lattice QCD can provide on-shell scattering amplitude using Bethe-Salpeter (BS) wave function not only ”outside” the interaction range but also ”inside” the interaction range
  
  ♦ Our result of the scattering length agrees with the value of Lüscher’s formula

- Lattice QCD can provide not only on-shell scattering amplitude but also half-off-shell scattering amplitude
  
  ♦ Half-off-shell scattering amplitude can be an additional input for effective theories and models of hadrons, as a supplement to experiments
1 Introduction

Hadron interactions can be studied by lattice QCD

- Direct approach:
  - The standard method is Lüscher formula, which utilizes BS wave function outside the interaction range of two hadrons Lüscher(1986,1990),... cf. many talks in Lattice 2018
  - Related issue: a relation between on-shell scattering amplitude and BS wave function inside the interaction range was discussed in the infinite volume Lin et al.(2001),CP-PACS(2005),Yamazaki and Kura-mashi(2017) cf. talk by Yamazaki-san
  → We explore this relation by a finite volume simulation

- Indirect approach: ex. HAL QCD method (indirect method through a potential from BS wave function) cf. talks by Doi-san, Iritani-san
Formulation (in brief)

Scattering amplitude $H(p; k)$ is obtained by BS wave function $\phi(x; k)$

- Integration range for $H(p; k)$ can be changed from $\infty$ to finite value $R$, called interaction range, if $(\Delta + k^2) \phi(x; k) = 0$ for $x > R$

$\therefore$ Lattice simulation for $H(p; k)$ is possible, if $R < L/2$

$\Diamond$ We consider $I = 2$ two-pion below inelastic threshold. $A_1^+$ projection is applied for S-wave in center of mass frame. Exp tails are assumed to be tiny and ignored.

\[
\phi(x; k) := \langle 0 | \Phi(x, t) | \pi^+ \pi^+, E_k \rangle e^{E_k t},
\]

\[
\Phi(x, t) := \sum_r \pi^+ (R_{A_1^+} [x] + r, t) \pi^+ (r, t),
\]

$R_{A_1^+} [x]$ : projector onto $A_1^+$ cubic group, $E_k = 2 \sqrt{m^2/\pi + k^2}$

\[
\Delta \phi(x; k) := \sum_{i=1}^3 (\phi(x + \hat{i}; k) + \phi(x - \hat{i}; k) - 2\phi(x; k)) \text{: Laplacian on lattices}
\]

\[
H(p; k) := -\int_{-\infty}^{\infty} d^3x \ e^{-i p \cdot x} (\Delta + k^2) \phi(x; k)
\]

\[
= -\sum_{|x| < R} e^{-i p \cdot x} (\Delta + k^2) \phi(x; k), \text{ if } (\Delta + k^2) \phi(x; k) = 0 \text{ for } x > R
\]

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[Formulation(continued)]

- Once \( H(p; k) \) at on-shell \( p = k \) is obtained, we can extract scattering phase shift \( \delta(k) \) and scattering length \( a_0 \), as in Lüscher formula

\[
\diamond \text{ NB. } H(k; k) \text{ appears in Lüscher’s formalism, though } H(k; k) \text{ is removed in the final form of Lüscher formula} \\
\rightarrow \text{ Our claim is ”} H(k; k) \text{ also keeps scattering info.”}
\]

\[
H(k; k) = \frac{4\pi}{k} e^{i\delta(k)} \sin \delta(k)
\]

\[
a_0 = \tan \delta(k)/k + O(k^2)
\]

[Quick derivation on Lüscher formula]

\[
\phi(x; k) \xrightarrow{x>R} v_{00} G(x; k), \quad G(x; k) : \text{solution of } (\Delta + k^2) \phi(x; k) = 0
\]

\[
= C_{00} e^{i\delta(k)} \frac{\sin(kx + \delta(k))}{kx} + (l \geq 4 \text{ terms}), \quad v_{00}, C_{00} : \text{constants}
\]

Expanding \( G(x; k) \) by \( j_l(kx) \) and \( n_0(kx) \) and comparing their coefficients leads to

\[
C_{00} H(k; k) = v_{00}
\]

\[
k \cot \delta(k) C_{00} H(k; k) = 4\pi v_{00} g_{00}(k)
\]

Taking a ratio of the above two equations gives Lüscher formula,

\[
k \cot \delta(k) = 4\pi g_{00}(k)
\]

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3 Set up of simulation

We use $I = 2 \pi \pi$ system in quenched lattice QCD as a test bed

- Iwasaki gauge action at $\beta = 2.334(a^{-1} = 1.207[\text{GeV}])$ CP-PACS(2001,2005)
- Valence Clover quark action with $C_{SW} = 1.398$

- Four random $Z(2)$ sources avoiding Fierz contamination and Wall sources for comparison + Coulomb gauge fixing
- Periodic boundary condition in space, Dirichlet boundary condition in time

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<th>Lattice</th>
<th>$\kappa_{\text{val}}$</th>
<th>$m_\pi$ [GeV]</th>
<th>$N_{\text{config}}$</th>
<th>$N_{\text{src}}$</th>
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<td>0.86</td>
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<td>$24^3 \times 96$</td>
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4 Result

[Check of plateau of temporal correlators]

- Effective masses of one-pion $m_{\text{eff}}(\pi)$ and $I = 2$ two-pion $E_{k_{\text{eff}}}^\pi(\pi\pi)$ have plateau in $t = [12, 44]$

→ No source dependence is observed for our case (fake plateau can appear for two baryons T.T.Takahashi and Y.K.Enyo(2009); HAL(Iritani et al.(2016));...)

◊ $k_t^2$ is determined from $E_k(\pi\pi) = 2\sqrt{m_{\pi}^2 + k_t^2}$, $k_t^2 \neq p^2 := (2\pi/L)^2 n^2$
[Check of plateau of BS wave functions $\phi(x; k)$]

- Ratio of wave functions $\phi(x; k)/\phi(x_{\text{ref}}; k)$ have plateau in $t = [32, 44]$

  ♦ Larger $t$ is required for wave functions, but still under control

  cf. temporal correlators have plateau in $t = [12, 44]$

  ♦ No source dependence is observed

\[
\phi(x; k) = \text{const} \times \text{prop}_{4\text{pt}}(x, t)e^{E_k t}
\]

\[
\phi(x; k)/\phi(x_{\text{ref}}; k) = \text{prop}_{4\text{pt}}(x, t)/\text{prop}_{4\text{pt}}(x_{\text{ref}}, t)
\]
Check of sufficient condition: $(\Delta + k^2)\phi(x; k) = 0, \ R < x < L/2$

- We confirm $R \sim 10$, which is consistent with the result by CP-PACS(2005).
  → The sufficient condition is satisfied within our statistical errors.
- $k_s^2$ can be inversely obtained by $k_s^2 = -\Delta \phi(x; k)/\phi(x; k)$, which is more precise than $k_t^2 = E_k^2/4 - m^2$ CP-PACS(2005)

Reference point $x_{\text{ref}} = (12, 7, 2)$ is chosen to minimize $l = 4$ contribution:
$\phi(x; k) = (l = 0 \text{ term}) + (l = 4 \text{ term}) + \ldots$

Strictly speaking, there must be exp tail, which is below our statistical error.
[Comparison of scattering length $a_0$]

- $a_0$ is evaluated by $H(k_t; k_t)$ inside the interaction range or by Lüscher’s formula outside the interaction range

- Both results agree well
- No source dependence is observed

$$a_0 / m_\pi = \tan \delta(k) / (k m_\pi) + O(k^2)$$

$$\tan \delta(k) = \frac{\sin(k x_{\text{ref}})}{4 \pi x_{\text{ref}} \phi(x_{\text{ref}}; k) / H_L(k; k) - \cos(k x_{\text{ref}})}$$

or

$$= 1 / ((4 \pi / k) g_{00}(k)) : \text{Lüscher’s formula}$$

24$^3 \times 64$, $\beta = 2.334$, $C_{SW} = 1.398$

$$\kappa_{\text{val}} = 0.13400$$
Not only on-shell amplitude $H(k; k)$ but also $H(p; k)$ can be estimated:

- $H(p; k)$ can be supplemental input to theoretical models of hadrons.
- Effective range $r_{\text{eff}}$ can be extracted from $H(p; k)$.
- NB. $H(p; k) / H(k; k)$ is available below $4\pi$ threshold, although there is no true inelastic threshold in quenched QCD (quenched artificial inelastic effects may appear).

\[
H_{L}(p; k) = - \sum_{x \in L^3} j_0(px)(\Delta + k^2)\phi(x; k)
\]

\[
H(p; k) / H(k; k) = H_{L}(p; k) / H_{L}(k; k)
\]

\[
k \cot \delta(k) = a_0^{-1} + r_{\text{eff}}k^2 + O(k^4)
\]

\[
r_{\text{eff}} \sim -\frac{2k^2H' + \sin^2\delta(k)}{2k \sin \delta(k) \cos \delta(k)}
\]

\[
H' := \frac{\partial^2 p^2 H(p; k)|_{p^2=|k|^2}}{H(k; k)}
\]
[Chiral extrapolation of $a_0$]

$a_0$ is extrapolated to the physical point using a ChPT motivated formula, ignoring chiral log due to our large pion masses

- Our result is consistent with ChPT(2001) and CP-PACS(2005)

- For definite comparison, more realistic data ($N_f = 2+1$ full QCD on the physical point) is needed

\[
\frac{a_0}{m_\pi} = A + Bm_\pi^2 + Cm_\pi^4 + O(m_\pi^2 \log m_\pi^2)
\]

\[
k_t^2 = \frac{E_k^2}{4} - m_\pi^2
\]

\[
k_s^2 = -\Delta \phi(x; k)/\phi(x; k)
\]
[Chiral extrapolation of $r_{\text{eff}}$]

$r_{\text{eff}}$ is extrapolated to the physical point using a ChPT motivated formula, ignoring chiral log due to our large pion masses

- Our result is consistent with CP-PACS(2005) at the simulation point, but underestimates ChPT(2001) and NPLQCD(2012) at the physical point

◊ For definite comparison, more realistic data ($N_f = 2+1$ full QCD on the physical point) is needed

\[
m_\pi r_{\text{eff}} = \frac{A}{m_\pi^2} + B + O(\log m_\pi^2)
\]
[Remark: scattering amplitude in momentum space $H(p; k)$]

- $H(p; k)$ can also be calculated using LSZ reduction formula in momentum space, instead of laplacian $\Delta$. cf. J.Carbonell and V.A.Karmanov (2016)

♦ Care is needed. If we change the integration range of $H(p; k)$ from $\infty$ to the interaction range $R$, a surface term appears.

\[
H(p; k) \overset{\text{def}}{=} - \int_{-\infty}^{\infty} d^3 x e^{-i p \cdot x} (\Delta + k^2) \phi(x; k)
\]
\[
= - \int_{-R}^{R} d^3 x e^{-i p \cdot x} (\Delta + k^2) \phi(x; k)
\]
\[
\downarrow \text{partial integration}
\]
\[
= (p^2 - k^2) \int_{-R}^{R} d^3 x e^{-i p \cdot x} \phi(x; k)
\]
\[
+ [\text{surface term}]_{-R}^{R}
\]

NB. on the lattice, $p^2 \rightarrow \tilde{p}^2$

\[
\tilde{p}_i := \frac{2}{a} \sin \frac{a p_i}{2}, \quad p_i = (2\pi/L) n_i
\]

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24^3 \times 64, \beta = 2.334, C_{SW} = 1.398
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\kappa_{\text{val}} = 0.13400
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p = (1,1,1)
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H_L(p;k_t)
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\begin{array}{c}
(\tilde{p}^2 - k_t^2) \phi(p;k_t)
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Summary

We evaluate a scattering length $a_0$ of $I = 2 \pi \pi$ system in the quenched lattice QCD as a test bed. We utilize Bethe-Salpeter wave function not only ”outside” the interaction range but also ”inside” the interaction range

- Consistency is checked
  Our result using the scattering amplitude ”inside” the interaction range agrees with the value of standard Lüscher’s finite volume method using data ”outside” the interaction range

- Additional output is obtained
  A half-off-shell scattering amplitude $H(p; k)$ is estimated by lattice QCD, which can be an additional input to theoretical models of hadrons, as a supplement to experiments
[Future work]
Apply our strategy to

- More realistic case \((N_f = 2 + 1\) full QCD on the physical point)
- More complicated system (other 2-body system with not only light quarks but also heavy quarks, and hopefully 3-body system)