$K\pi$ scattering with partial wave mixing & isovector excited meson spectroscopy

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Motivations & Overview

- study low-lying meson spectrum using lattice QCD:

- use $2 \rightarrow 2$ Lüscher formalism to calculate hadron-hadron scattering amplitudes
  - $P$-wave $K\pi$ scattering: $K^*(892)$ resonance parameters
  - $S$-wave $K\pi$ scattering: $K^*_0(800)/\kappa$ resonance parameters
  - include partial wave mixing for $\ell \leq 2$

- qualitative spectrum extraction with large operator bases
  - single- and multi-hadron interpolating operators in large volumes
  - identify $\bar{q}q$-dominated states, mixed states, etc.
Finite Volume Spectra

Scattering process: eg.

$I = 1 \quad \pi \pi \rightarrow \pi \pi$

∞-volume

\[ p = \frac{2\pi}{L} d \]

Finite volume

Momentum quantised \rightarrow No continuum of scattering states
Lüscher: Relationship between finite volume spectrum and infinite volume scattering matrix

- Quantisation condition:

\[
\det[\tilde{K}^{-1} - B] = 0
\]

- For each \(E_{\text{cm}}\) in spectrum, determinant gives single relation to entire scattering matrix

  \[
  \Rightarrow \text{Exactly solvable for single channel, single partial wave}
  \]

  \[
  \Rightarrow \ell \text{ mixing/coupled decay channels requires parameterisation of } \tilde{K}
  \]

  \[
  \text{and a fit (determinant residual method)}
  \]

Lüscher Quantisation

Infinite volume physics from LQCD

Lüscher: Relationship between finite volume spectrum and infinite volume scattering matrix

- Quantisation condition:

\[
\tilde{K}_\ell^{-1} = \left( \frac{q_{cm}}{m_\pi} \right)^{2\ell+1} \cot \delta \ell \quad \text{det}[(\tilde{K}^{-1} - B)] = 0
\]

- For each \( E_{cm} \) in spectrum, determinant gives single relation to entire scattering matrix
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Lüscher quantisation

Infinite volume physics from LQCD

- Quantisation condition:

\[ \tilde{K}_\ell^{-1} = \left( \frac{q_{cm}}{m_\pi} \right)^{2\ell+1} \cot \delta_\ell \text{det} \left[ \tilde{K}^{-1} - B \right] = 0 \]

- For each \( E_{cm} \) in spectrum, determinant gives single relation to entire scattering matrix

\[ \Rightarrow \text{Exactly solvable for single channel, single partial wave} \]
\[ \Rightarrow \ell \text{ mixing/coupled decay channels requires parameterisation of } \tilde{K} \text{ and a fit (determinant residual method)} \]

Lüscher Quantisation
Infinite volume physics from LQCD

Lüscher: Relationship between finite volume spectrum and infinite volume scattering matrix

- Quantisation condition:

\[
\tilde{K}^{-1}_\ell = \left( \frac{q_{\text{cm}}}{m_\pi} \right)^{2\ell+1} \cot \delta_\ell \det[\tilde{K}^{-1} - B] = 0
\]

box matrix: known function of \((E_{\text{cm}}, L)\)

- For given \(P\), \((\tilde{K}^{-1} - B)\) block diagonal in little group irrep, simplifies determinant calculation

⇒ All infinite volume physics in \(\tilde{K}^{-1}\), finite volume in \(B\)
⇒ \(B\) describes how partial waves fit into cubic volume
⇒ Software available containing \(B\) elements up to \(\ell = 6\)

$K\pi$ energies in finite volume

- ensemble: $32^3 \times 256$, $m_\pi \approx 230$ MeV, $m_\pi L \sim 4.4$
- 13 single-hadron ($K$) and 33 two-hadron ($K\pi$) interpolating operators
- all-to-all propagation using stochastic LapH method

Decay of $K^*(892)$

- large $32^3 \times 256$ anisotropic lattice, $m_\pi \approx 230$ MeV
- included $\ell = 0, 1, 2$ partial waves
- fit forms

$$ (\tilde{K}^{-1})_{11} = \frac{6\pi E_{\text{cm}}}{g^2 m_\pi} \left( \frac{m_{K^*}^2}{m_\pi^2} - \frac{E_{\text{cm}}^2}{m_\pi^2} \right) \quad (\tilde{K}^{-1})_{22} = -\frac{1}{m_\pi^5 a_2} $$

$$ (\tilde{K}^{-1})_{\text{lin}}^{00} = a_1 + b_1 E_{\text{cm}}, \quad (\tilde{K}^{-1})_{\text{quad}}^{00} = a_q + b_q E_{\text{cm}}^2, $$

$$ (\tilde{K}^{-1})_{\text{ERE}}^{00} = -\frac{1}{m_\pi a_0} + \frac{m_\pi r_0}{2} \frac{q_{\text{cm}}^2}{m_\pi^2}, \quad (\tilde{K}^{-1})_{\text{BW}}^{00} $$

- results

$$ \frac{m_{K^*}}{m_\pi} = 3.808(18), \quad g = 5.33(20), \quad m_\pi a_0 = -0.353(25), $$

$$ m_\pi^5 a_2 = -0.0013(68), \quad \chi^2/\text{dof} = 1.42 $$

Decay of $K^*(892)$

- plots of $P$-wave and $S$-wave phase shift ($\tilde{E} = (E_{cm} - m_K)/m_\pi$)
- $\kappa(\ell = 0)$ fit: Breit-Wigner or effective range range
Decay of $K^*(892)$

- summary of lattice calculations of $K^*(892)$ resonance parameters
- phenomenological values shown as asterisks
Excited meson spectroscopy

- As $m^\text{lat}_\pi \rightarrow m^\text{phys}_\pi$, fewer resonances lie below 3- & 4-particle thresholds
  ⇒ three particle quantisation condition in development (see talks by Mai (Mon) & Sharpe (Today))

- For now, want to get a qualitative look at the excited meson spectrum
  ⇒ Using large bases of single- and two-hadron interpolating operators, extract excited spectrum
  ⇒ Identify $\bar{q}q$-like states by operator overlaps

- Using anisotropic $N_f = 2 + 1$ ensembles with clover fermions:
  ⇒ $(32^3|230)$: 412 configs $32^3 \times 256$, $m_\pi \approx 230$ MeV, $m_\pi L \sim 4.4$
  ⇒ $(24^3|390)$: 551 configs $24^3 \times 128$, $m_\pi \approx 390$ MeV, $m_\pi L \sim 5.7$
$I = 1, S = 0$ meson spectroscopy (*preliminary*)

eg. $T_{1u}^+$ with $m_\pi \sim 230$ MeV

- 73 (9 SH + 64 MH) interpolating operators: full spectrum
\( I = 1, S = 0 \) meson spectroscopy (\textit{preliminary})

eg. \( T_{1u}^+ \) with \( m_\pi \sim 230 \text{ MeV} \)

- (9) SH operators only: experiment vs \( \bar{q}q \)

<table>
<thead>
<tr>
<th>( E )</th>
<th>( 2m_K )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho(770) )</td>
<td>( \rho(1450) )</td>
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<td>( \rho(1570) )</td>
<td>( \rho(1690) )</td>
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<td>( \rho(1700) )</td>
<td>( \rho(1900) )</td>
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<tr>
<td>( \rho_s(1990) )</td>
<td>( \rho(2150) )</td>
</tr>
</tbody>
</table>

\( \text{lattice } q\bar{q} \text{ state} \)
\( \text{experimental mass} \)
\( \text{experimental width} \)
\( I = 1, S = 0 \) meson spectroscopy (\textit{preliminary})

eg. \( T_{1u}^+ \) with \( m_\pi \sim 230 \text{ MeV} \)

- full operator basis: experiment vs \( \bar{q}q \)
Conclusions - Scattering

- meson-meson scattering at a mature stage
  - moving towards physical point results - large volumes required


- successful results depend on time-slice to time-slice quark propagators needed for temporal correlators involving two-hadron operators
  - Stochastic LapH method (minimal volume scaling)

- box matrix formulation/software handles partial wave mixing & coupled channels
Conclusions - Spectroscopy

- goal: *qualitative* description of resonant spectrum

- high computational cost for large operator bases
  - Stochastic LapH method

- $\bar{q}q$ states straightforward but many interesting states not well described
  ⇒ hybrids
  ⇒ molecular states
  ⇒ ...

- effective Hamiltonian models to further explore content of finite volume QCD spectrum
$S$-wave $K\pi$ amplitude: $K^*_0(800)$

\[
\frac{m_{K^*}}{m_{\pi}} = 3.808(18), \quad g = 5.33(20), \quad m_{\pi} a_0 = -0.353(25), \\
m_{\pi}^5 a_2 = -0.0013(68), \quad \chi^2/\text{dof} = 1.42
\]

- based on LO ERE, $m_{\pi} a_0 < 0$ suggests virtual bound state
- however, NLO parameters give $1 - 2r_0/a_0 = -8.9(2.4)$ which must be $> 0$ for a (real or virtual) bound state
- zeros of $q_{cm} \cot \delta_0 - iq_{cm}$: $m_R/m_{\pi} = 4.66(13) - 0.87(18)i$
  - consistent with BW fit
- better energy resolution & careful analytic continuation required

$I = 1, \ S = 0$ meson spectroscopy (preliminary)
$I = 1$, $S = 0$, $A_{1u}^+$ spectrum (preliminary)

$m_\pi \sim 390$ MeV
$I = 1, S = 0, A_{1u}^+$ spectrum (preliminary)

$m_\pi \sim 230$ MeV
$I = 1, S = 0, A_{2u}^+$ spectrum (preliminary)

$m_\pi \sim 390$ MeV
$I = 1, S = 0, A_{2u}^+$ spectrum (preliminary)

$m_\pi \sim 390$ MeV

$\rho_3(1690)$, $\rho_3(1990)$

Experiment

Lattice

$m_\pi \sim 390$ MeV

$E \sim \frac{2m_K}{E}$

Experimental mass

Experimental width

Lattice $q\bar{q}$ state

Lattice mixed state

Experimental mass

Experimental width
$I = 1, S = 0, A_{2u}^+$ spectrum (preliminary)

$m_\pi \sim 230$ MeV
$I = 1, S = 0, A_{2u}^+$ spectrum (preliminary)

$E = \frac{E}{2m_K}$

$m_\pi \sim 230$ MeV

**Experiment**

- $\rho_3(1690)$
- $\rho_3(1990)$

**Lattice**

- Lattice $q\bar{q}$ state
- Lattice mixed state
- Experimental mass
- Experimental width

$\rho_3(1690)$
$\rho_3(1990)$
$I = 1, S = 0, E_u^+$ spectrum (preliminary)

$m_\pi \sim 390$ MeV
$I = 1, S = 0, E_u^+$ spectrum (preliminary)

$m_\pi \sim 230$ MeV
$I = 1, S = 0, T_{1u}^+$ spectrum (preliminary)

$m_\pi \sim 390$ MeV
\(I = 1, S = 0, T_{1u}^+\) spectrum (preliminary)

\[ m_\pi \sim 390 \text{ MeV} \]

Experiment

Lattice

\( m_\pi \sim 390 \text{ MeV} \)
$I = 1, S = 0, T_{1^u}^+$ spectrum (preliminary)

$m_\pi \sim 230$ MeV
$I = 1, S = 0, T_{1_u}^{+}$ spectrum (preliminary)
\( I = 1, S = 0, T_{2u}^+ \) spectrum \((\text{preliminary})\)

\[
m_\pi \sim 390 \text{ MeV}
\]
$I = 1, S = 0, T_{2u}^+$ spectrum (preliminary)

$m_\pi \sim 390$ MeV

$E \sim \frac{2m_K}{\rho_3(1690)}$  

Lattice

Experimental mass

Experimental width

lattice $q\bar{q}$ state

experimental mass

experimental width
$I = 1, S = 0, T_{2u}^+$ spectrum (preliminary)

$m_\pi \sim 230$ MeV
$I = 1, S = 0, T_{2u}^+ \text{ spectrum (preliminary)}$

Experiment

Lattice

$m_\pi \sim 230 \text{ MeV}$

$m_\pi$}

E

Experiment Lattice

lattice $q\bar{q}$ state

lattice mixed state

experimental mass

experimental width

$m_\pi \sim 230 \text{ MeV}$

$E$

$m_{\pi} \sim 230 \text{ MeV}$
Extracting finite volume spectra

- Signal of interest: deviation of finite-volume two hadron levels from non-interacting counterparts
  ⇒ Extract energy difference from
  \[
  R_n(t) = \frac{\tilde{C}_n(t)}{C_\pi(d_\pi^2, t) C_K(d_K^2, t)} \to A_n e^{-\Delta E_n t}
  \]

- Reconstruct:
  \[
  a_t E_n = a_t \Delta E_n + \sqrt{a_t^2 m_\pi^2 + \left(\frac{2\pi a_s}{\xi L}\right)^2 d_\pi^2} + \sqrt{a_t^2 m_K^2 + \left(\frac{2\pi a_s}{\xi L}\right)^2 d_K^2}.
  \]

- Where $\Delta E_n$ is small, these ratio fits generally have smaller excited state contamination than direct fits to $\tilde{C}_n(t)$
## Ratio fits

<table>
<thead>
<tr>
<th>$E_{cm}/m_\pi$</th>
<th>$A_1(4)$, $E_1$ ratio</th>
<th>single-exp.</th>
<th>two-exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.6</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3.4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$E_{cm}/m_\pi$</th>
<th>$E(3)$, $E_1$ ratio</th>
<th>single-exp.</th>
<th>two-exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4.0</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Each row corresponds to the three fits for a single level specified in the left column as ‘$\Lambda(d^2)$, $E_n$', denoting the $n$th level in finite volume irrep $\Lambda$ with total momentum $d^2$. 

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Ruairí Brett  
Lattice 2018  
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Decay of $\rho(770)$

- initially applied to $P$-wave $I = 1 \rho \to \pi\pi$ system
- now have included $\ell = 1, 3, 5$ partial waves
- large $32^3 \times 256$ anisotropic lattice, $m_\pi \approx 230$ MeV
- fit forms (first ever inclusion of $\ell = 5$ in lattice QCD):

$$
(\tilde{K}^{-1})_{11} = \frac{6\pi E_{cm}}{g^2 m_\pi} \left( \frac{m_\rho^2}{m_\pi^2} - \frac{E_{cm}^2}{m_\pi^2} \right)
$$

$$
(\tilde{K}^{-1})_{33} = \frac{1}{m_\pi^7 a_3}, \quad (\tilde{K}^{-1})_{55} = \frac{1}{m_\pi^{11} a_5}
$$

- results

$$
\frac{m_\rho}{m_\pi} = 3.349(25), \quad g = 5.97(27), \quad m_\pi^7 a_3 = -0.00021(100),
$$

$$
\frac{m_\pi^{11} a_5}{m_\pi^{11} a_5} = -0.00006(24), \quad \chi^2/\text{dof} = 1.15
$$

[J Bulava, B Fahy, B Hörz, K J Juge, C Morningstar, CH Wong; NPB 910, 842 (2016)]
[C Morningstar, J Bulava, B Singha, RB, J Fallica, A Hanlon, B Hörz; NPB 924, 477 (2017)]
Decay of $\rho(770)$

- $\ell = 1, 3, 5$ phase shifts

[Image of a graph showing phase shifts vs. $E_{cm}/m_\pi$]

[J Fallica, PhD Thesis (2017)]
Decay of $\Delta(1232)$

- included $\ell = 1$ wave only (for now)
- large $48^3 \times 128$ isotropic lattice, $m_\pi \approx 280$ MeV, $a \sim 0.076$ fm
- Breit-Wigner fit gives $g_{\Delta N\pi} = 19.0(4.7)$ in agreement with experiment $\sim 16.9$

[CW Andersen, J Bulava, B Hörz, C Morningstar; PRD 97, 014506 (2018)]